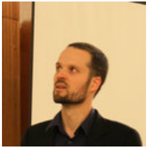
The Smallest Hard Trees

Florian Starke

joint work with Manuel Bodirsky, Jakub Bulín, and Michael Wernthaler



CSPs

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The CSP of G is the following problem:

finite digraphs

Input: H

Output: $\exists$ homomorphism $H \to G$?

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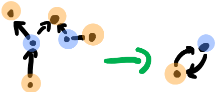

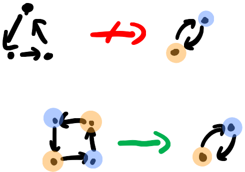

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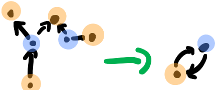
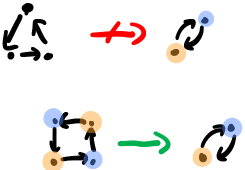
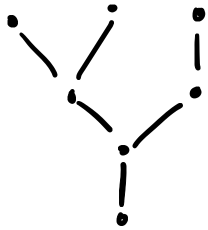




$H \to$  iff H contains no odd cycle

Trees



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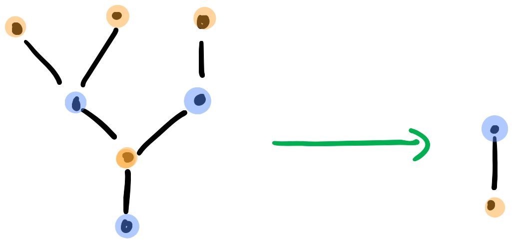

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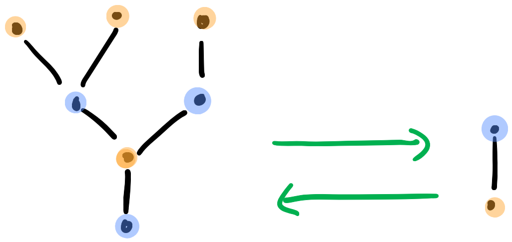

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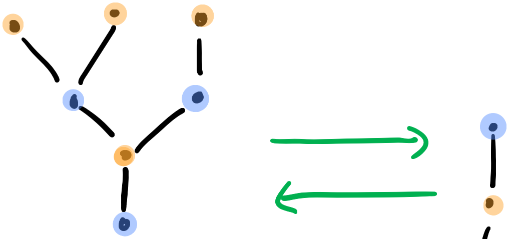
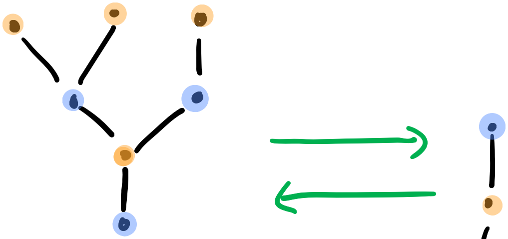


CSP( ) = CSP( )

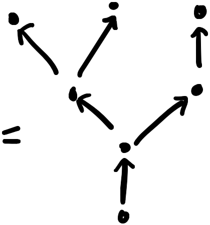
Trees



$\mathrm{CSP}(\quad) = \mathrm{CSP}(\quad)$

Undirected edges
are uninteresting

Trees

$T =$ 

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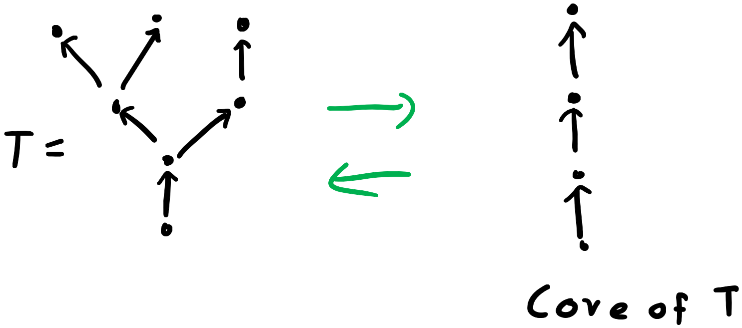

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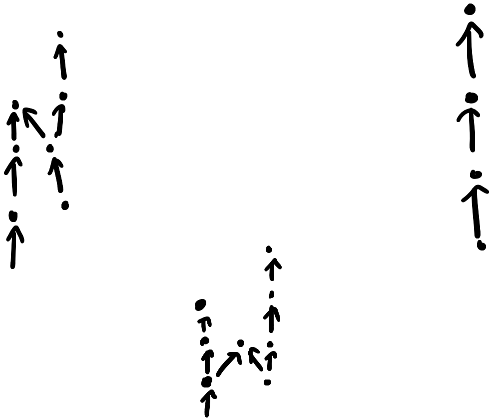

History of NP-Hard Trees

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Goal:

assuming $P \neq NP$

Find the smallest NP-hard Tree.

NP-hard Trees

TFAE (assuming $P \neq NP$)

- CSP(T) is NP-hard
- T can pp-construct 

$f(area) = f(rare)$

- T has no Siggers-polymorphism

NP-hard Trees

TFAE (assuming P≠NP)

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can be computed by testing

$$T^4 /_{area = rare} \longrightarrow T$$

contract tuples that should have the same image

NP-hard Trees

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$f(a,r,e,a) = f(r,a,r,e)$
- T has no Siggers-polymorphism

can be computed by testing

$$20^4 \to T^4/_{area=rare} \to T$$

$\parallel$

160000 :(

contract tuples that should have the same image

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- T has no Kearnes-Marković-McKenzie-Polymorphisms

$p(abb) = q(baa) = q(aab)$
$p(aba) = q(aba)$

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$p(abb) = q(baa) = q(aab)$

$p(aba) = q(aba)$

$2 \cdot 20^3 = 16000$

much better

but still not easy

How many Trees are there?

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n	trees
1	1
2	1
3	3
4	8
5	27
6	91
7	350
8	1376
9	5743
10	24635

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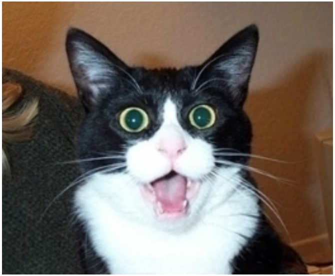
16	240872654
17	1166732814
18	5702001435
19	28088787314
20	139354922608 $\approx 10^{12}$



testing 100 trees per second

this would take 317 years

How many Trees are there?

n	trees	cores
1	1	1
2	1	1
3	3	1
4	8	1
5	27	1
6	91	2
7	350	3
8	1376	7
9	5743	15
10	24635	36
11	108968	85
12	492180	226
13	2266502	578
14	10598452	1569
15	50235931	4243
16	240872654	11848
17	1166732814	33104
18	5702001435	94221
19	28088787314	269455
20	139354922608	779268

testing 100 trees per second
this would take 2 hours

much better
but still not easy

Core Trees

Core Trees

Theorem *Let $\mathbb{T}$ be a finite tree. Then the following are equivalent.*

1. *$\mathbb{T}$ is a core;*
2. *$\mathrm{End}(\mathbb{T}) = \{\mathrm{id}_T\}$;*
3. *$\mathrm{AC}_{\mathbb{T}}(\mathbb{T})$ terminates such that the list for each vertex contains a single element.*

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easy

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easy   fun exercise

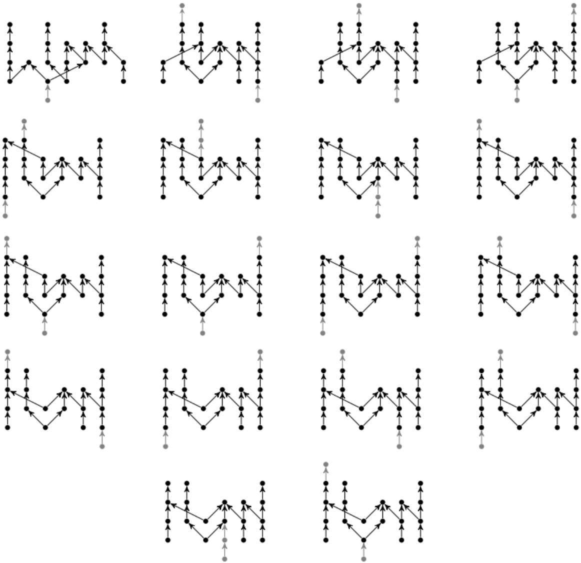
NP-hard Trees

foreach core tree T

  IF T has no KMM-polymorphisms **then**

    return T

NP-hard Trees



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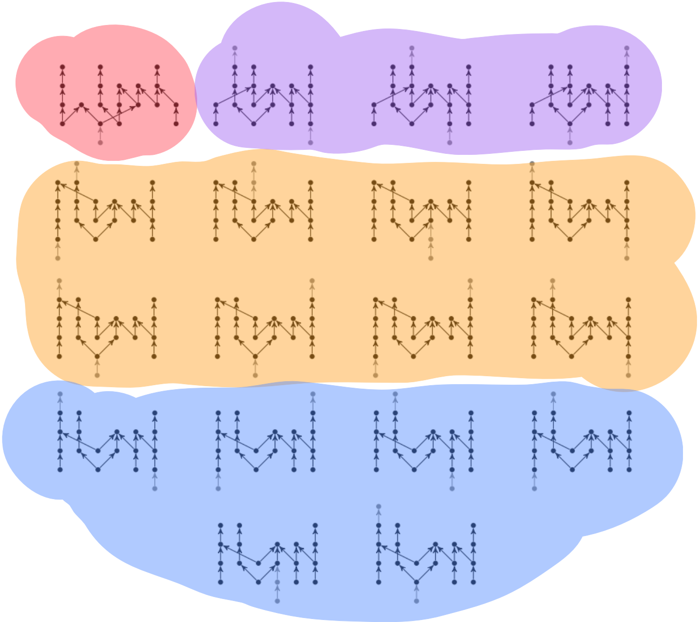

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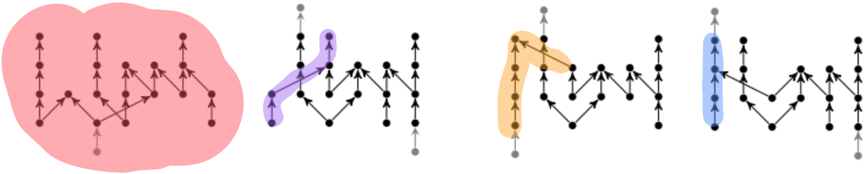

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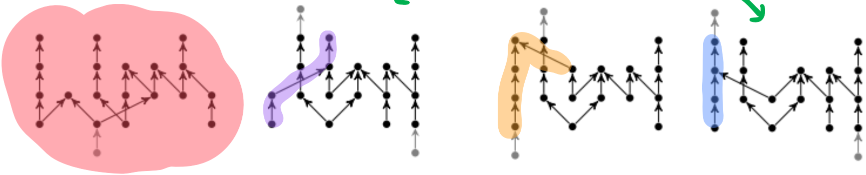
Present article	2022	20	Smallest tree

assuming P≠NP

NP-hard Trees

Hoppla



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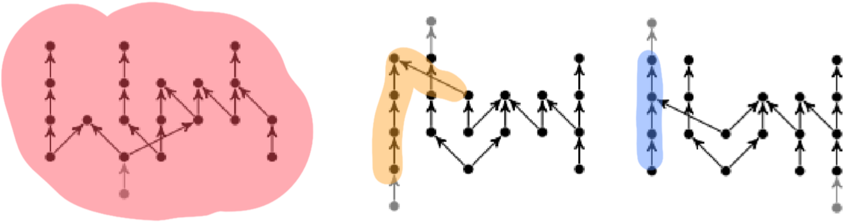

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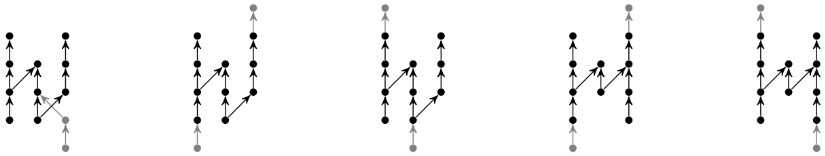
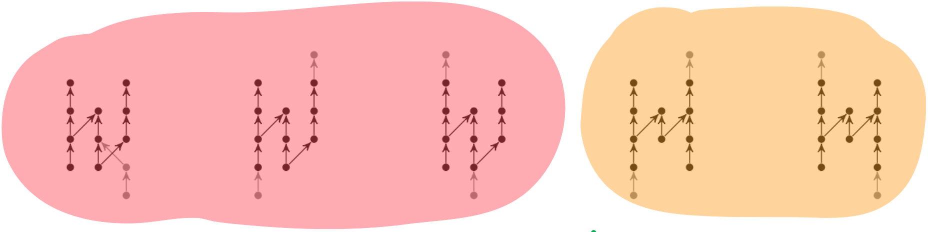


assuming $L \neq NL$

smallest NL-hard trees

(12 vertices)

Other smallest Trees



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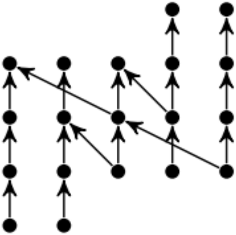

smallest tree not solved by AC

(19 vertices)

Open Problem

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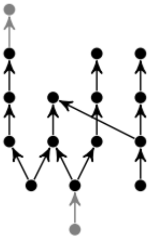

- in P
- NL-hard
- Smallest tree without majority polymorphism
- has Kearnes-Kiss-chain of length 5 (no KK-chain => P-hard)
- has no Jónsson-chain of length 1000 (J-chain => in NL)

(16 vertices)

Paper:

The Smallest Hard Trees

https://arxiv.org/abs/2205.07528