Digraphs

modulo primitive positive constructability

(this time not only cycles)

Florian Starke

joint work with Manuel Bodirsky

Define the Order

homomorphic equivalence

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homomorphic equivalence

$$G \rightleftarrows H \Rightarrow G = H$$

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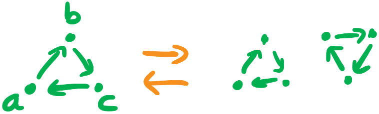

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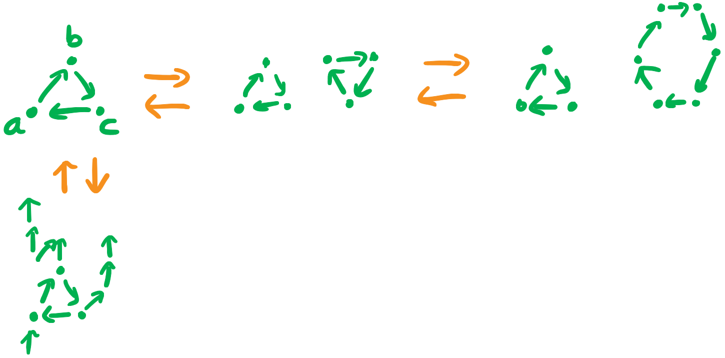

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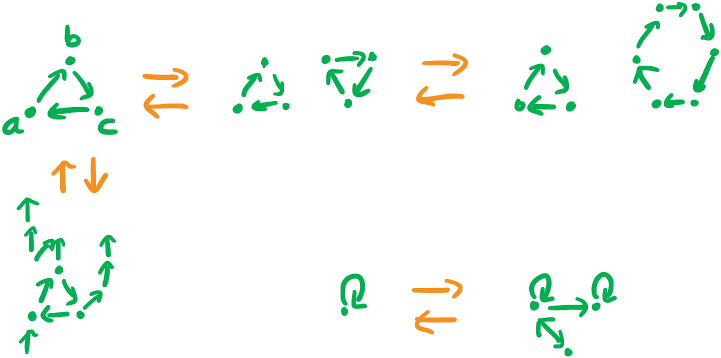

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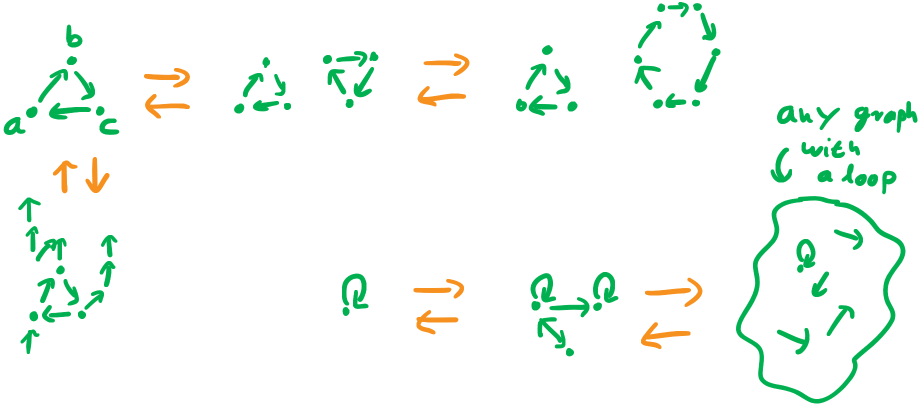

Define the Order

pp power

$G \rightleftarrows H \Rightarrow G = H$

Define the Order

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$$G \rightleftarrows H \Rightarrow G = H$$

$$G \to \boxed{P, x, y} \to H \Rightarrow G \leq H$$

Define the Order

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Digraph

n-tuples from vertices of P

$$G \to \boxed{P, x, y} \to H \Rightarrow G \le H$$

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Digraph

n-tuples from vertices of P

$G \rightarrow \boxed{P, x, y} \rightarrow H \Rightarrow G \leq H$

$$V(H) = \{(v_1, \ldots, v_n) \mid v_i \in V(G)\}$$

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$$V(H) = \{(v_1, \ldots, v_n) \mid v_i \in V(G)\}$$

$$\begin{tikzcd} (u_1, \ldots, u_n) \\ (v_1, \ldots, v_n) \arrow[u, bend left] \end{tikzcd}$$

iff

Define the Order

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Digraph

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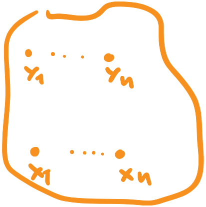
$G \to \boxed{P, x, y} \to H \Rightarrow G \leq H$

$V(H) = \{(v_1, \dots, v_n) \mid v_i \in V(G)\}$

$(v_1, \dots, v_n) \to (u_1, \dots, u_n)$ iff

P



$\xrightarrow[x_i \mapsto v_i]{y_i \mapsto u_i} G$

Define the Order

pp power

$$\boxed{G \rightleftarrows H \Rightarrow G = H}$$

Digraph

n-tuples from vertices of P

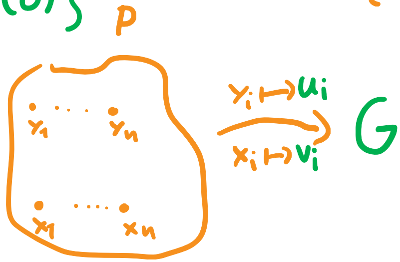
$$G \to \boxed{P, x, y} \to H \Rightarrow G \le H$$

transitive

$$V(H) = \{(v_1, \dots, v_n) \mid v_i \in V(G)\}$$

$$(v_1, \dots, v_n) \to (u_1, \dots, u_n)$$

iff



Goal:

understand poset of finite digraphs

Define the Order

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Digraph

n-tuples from vertices of P

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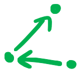
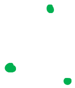


$$\to \boxed{x_1 = y_1} \to$$



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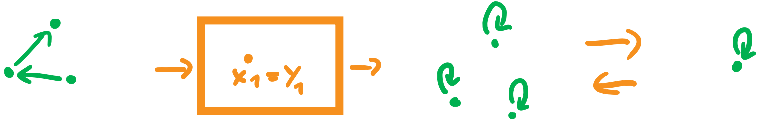

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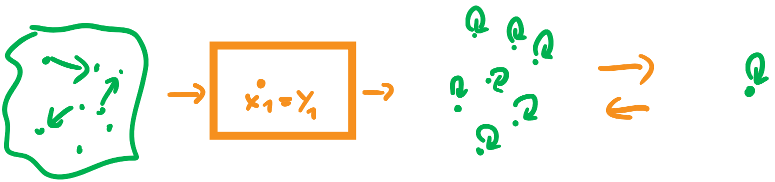

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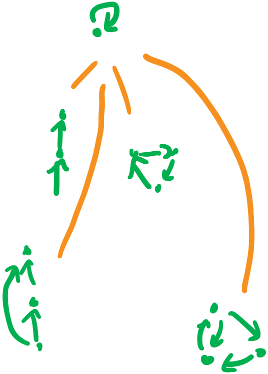

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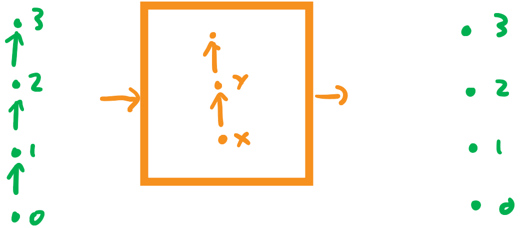

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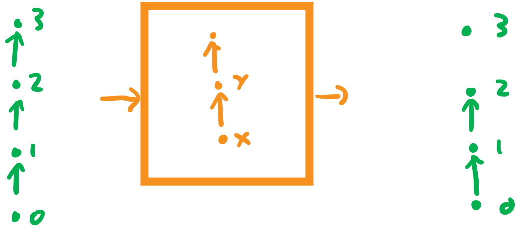

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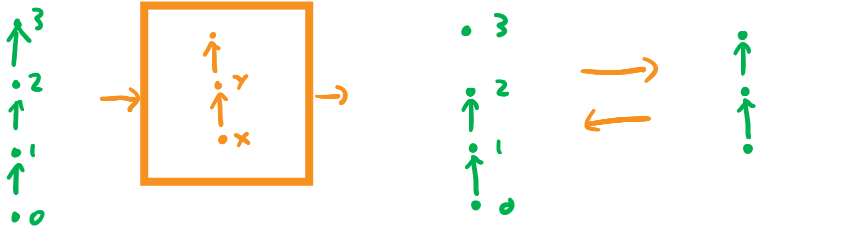

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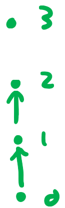
$G \rightleftarrows H \Rightarrow G = H$

$G \rightarrow \boxed{P, x, y} \rightarrow H \Rightarrow G \leq H$

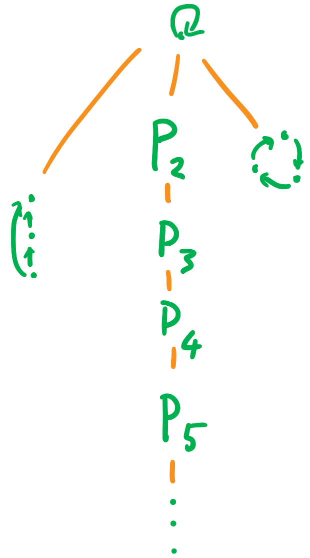
$P_4 \leq$



$= P_3$

Define the Order

pp power



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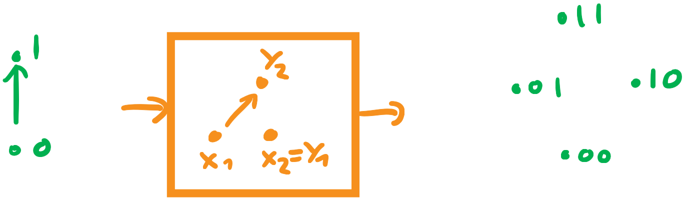

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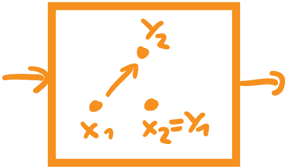

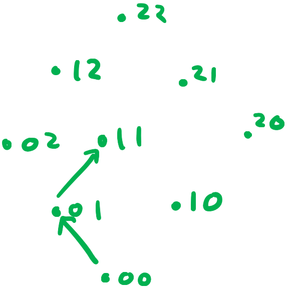

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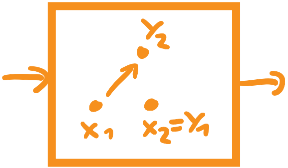

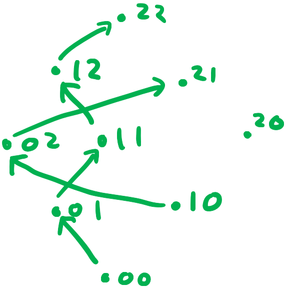

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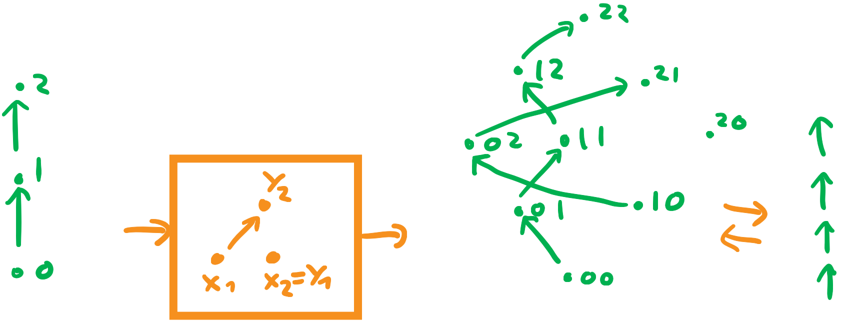

Define the Order

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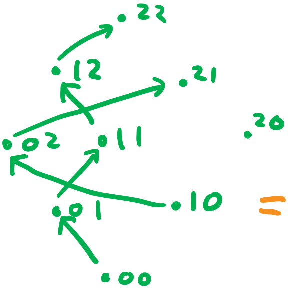
$G \rightleftarrows H \Rightarrow G = H$

$G \to \boxed{P, x, y} \to H \Rightarrow G \leq H$

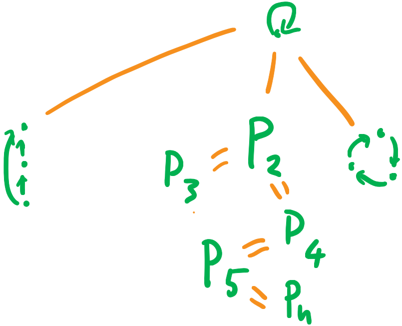
$P_3 \leq$



$= P_5$

Define the Order

pp power



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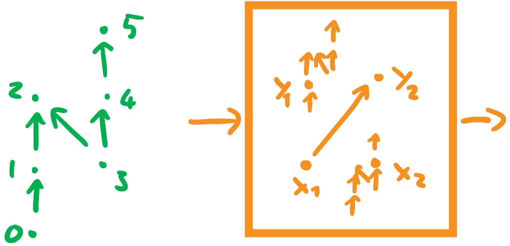

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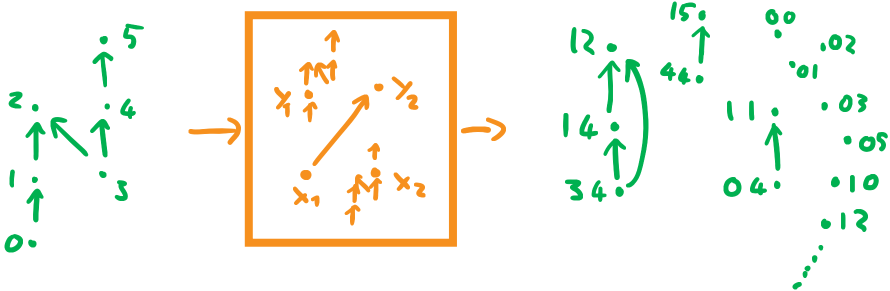

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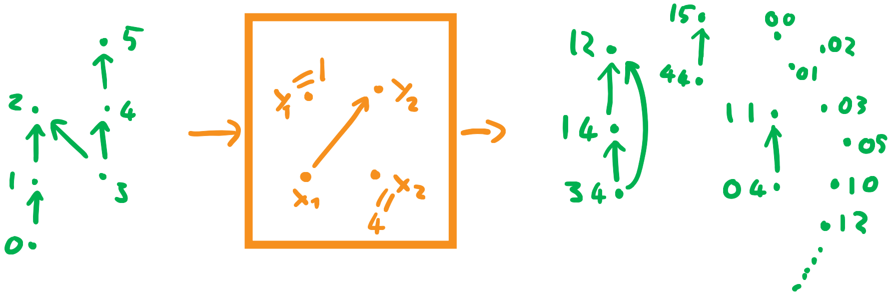


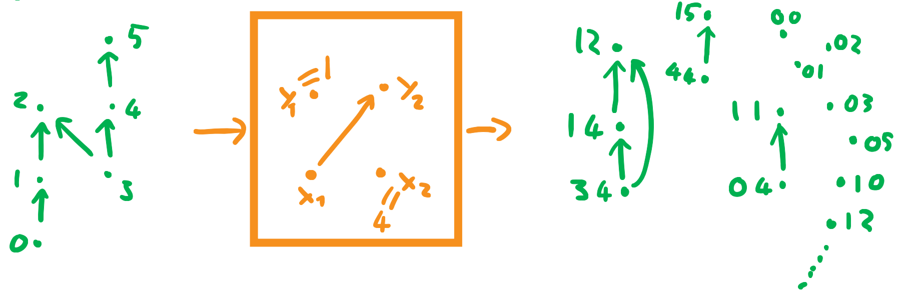
Define the Order

pp power

Core (End = Emb)

$G \rightleftarrows H \Rightarrow G = H$

$G \to \boxed{P_{1,x,y}} \to H \Rightarrow G \leq H$



Define the Order

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Core (End = Emb)

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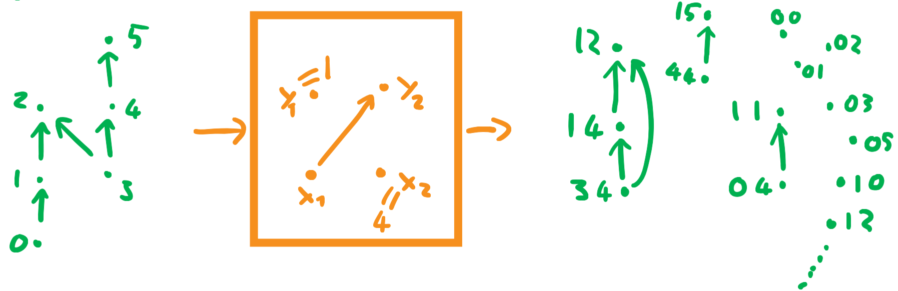
$G \to \boxed{P_{1,x,y}} \to H \Rightarrow G \leq H$

in cores
we can also
use constants



Define the Order

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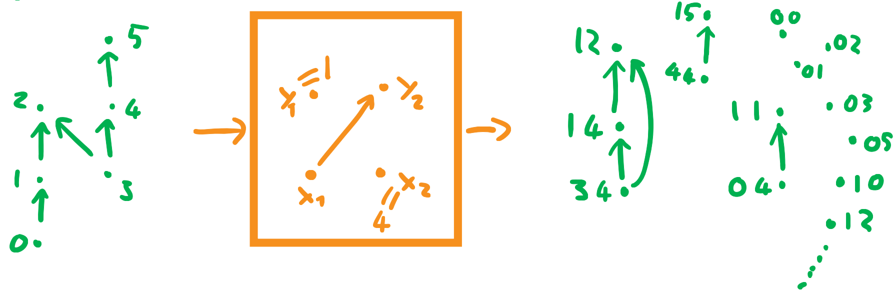
Core (End = Emb)

$\forall G\ \exists!\ \text{Core } G' : G \equiv G'$

$G \rightleftarrows H \Rightarrow G \equiv H$

$G \rightarrow \boxed{P_{1,x,y}} \rightarrow H \Rightarrow G \leq H$

in cores we can also use constants



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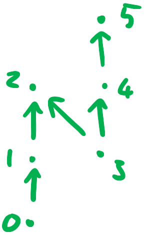

$\leq \quad T_3$

Define the Order

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in cores

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$G \rightleftarrows H \Rightarrow G = H$

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core

G

$u \neq v$

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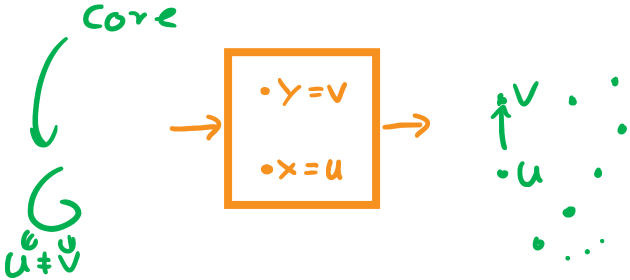

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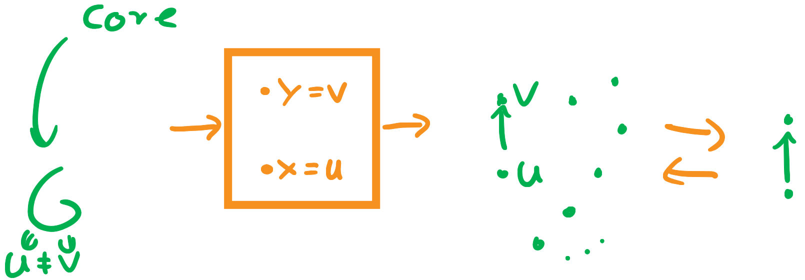

Define the Order

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we can also
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core

$$G \leq P_2$$

$u \neq v$

Define the Order

pp power

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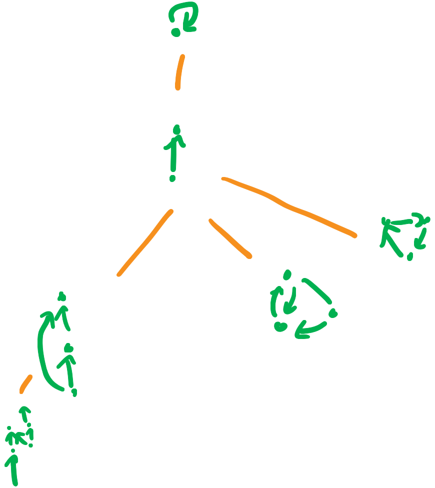
$$G \to \boxed{P, x, y} \to H \Rightarrow G \leq H$$



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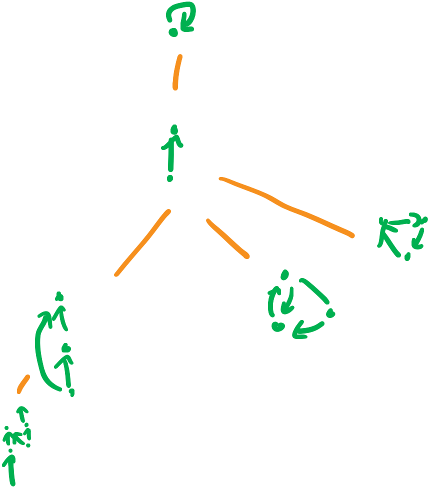

Lemma $\forall G$ with

$|Core(G)| \geq 2$

$G \leq P_2 \leq C_1$

Define the Order

pp power

$G \rightleftarrows H \Rightarrow G = H$

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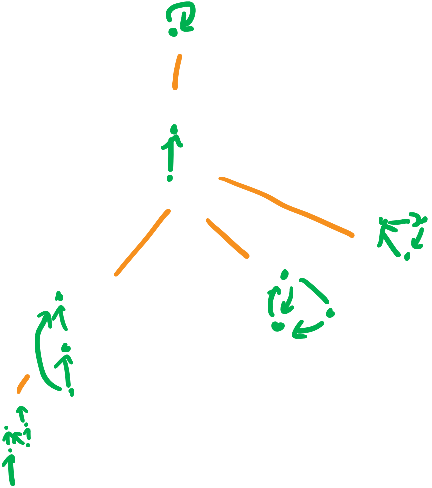


Lemma $\forall G$ with $|Core(G)| \geq 2$

$G \leq P_2 < C_1$

every pp power of C_1 has just one element

Define the Order using Minor(h1)-Conditions

Define the Order using Minor(h1)-Conditions

Polymorphisms

$f \in \mathrm{Pol}(G)$ if

$f: G^n \to G$ s.t.

$$\begin{array}{ccccccccc} f(& u_1, & u_2, & \ldots, & u_n &) = & & u \\ & \uparrow & \uparrow & \cdots & \uparrow & & \Rightarrow & \uparrow \\ f(& v_1, & v_2, & \ldots, & v_n &) = & & v \end{array}$$

Define the Order using Minor(h1)-Conditions

$(u,v) \mapsto \min(u,v) \in \mathrm{Pol}\Big($  $\Big)$

Define the Order using Minor(h1)-Conditions

$$(u,v) \mapsto \min(u,v) \in \mathrm{Pol}\Big($$ 

$$\Big)$$

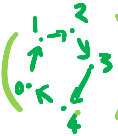
$$(u,v,w) \mapsto u - v + w \in \mathrm{Pol}\Big($$ 

$$\Big)$$

Define the Order using Minor(h1)-Conditions

$(u,v) \longmapsto \min(u,v) \in \mathrm{Pol}\Big($  $\Big)$

$(u,v,w) \longmapsto u-v+w \in \mathrm{Pol}\Big($  $\Big)$

$(u,v,w) \longmapsto 2\cdot(u+v+w) \in \mathrm{Pol}\Big($  $\Big)$

Define the Order using Minor(h1)-Conditions

$\models f(a,b) = f(b,a)$

$(u,v) \longmapsto \min(u,v) \in \mathrm{Pol}($  $)$

$(u,v,w) \longmapsto u - v + w \in \mathrm{Pol}($  $)$

$(u,v,w) \longmapsto 2\cdot(u+v+w) \in \mathrm{Pol}($  $)$

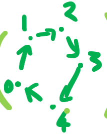
Define the Order using Minor(h1)-Conditions

$\models f(a,b) = f(b,a)$

$(u,v) \mapsto \min(u,v) \in \mathrm{Pol}\big($  $\big)$

$\models m(a,a,b) = m(b,a,a) = m(b,b,b)$

$(u,v,w) \mapsto u - v + w \in \mathrm{Pol}\big($  $\big)$

$(u,v,w) \mapsto 2\cdot(u+v+w) \in \mathrm{Pol}\big($  $\big)$

Define the Order using Minor(h1)-Conditions

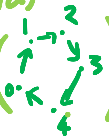
$\models f(a,b)=f(b,a)$

$(u,v)\mapsto \min(u,v) \in \mathrm{Pol}\Big($  $\Big)$

$\models m(a,a,b)=m(b,a,a)=m(b,b,b)$

$(u,v,w)\mapsto u-v+w \in \mathrm{Pol}\Big($  $\Big)$

$\models f(a,b,c)=f(b,c,a)$

$(u,v,w)\mapsto 2\cdot(u+v+w) \in \mathrm{Pol}\Big($  $\Big)$

Define the Order using Minor(h1)-Conditions

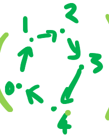
$$\models \Sigma_2$$

$$(u,v) \mapsto \min(u,v) \in \mathrm{Pol}\Big($$  $$\Big)$$

$$\models m(a,a,b) = m(b,a,a) = m(b,b,b)$$

$$(u,v,w) \mapsto u - v + w \in \mathrm{Pol}\Big($$  $$\Big)$$

$$\models \Sigma_3$$

$$(u,v,w) \mapsto 2 \cdot (u+v+w) \in \mathrm{Pol}\Big($$  $$\Big)$$

Define the <u>Order</u> using Minor(h1)-Conditions

$\models \Sigma_2$

$(u,v) \mapsto \min(u,v) \in \mathrm{Pol}($  $)$

$\models \Sigma_M \leftarrow$ maltsev

$(u,v,w) \mapsto u - v + w \in \mathrm{Pol}($  $)$

$\models \Sigma_3$

$(u,v,w) \mapsto 2\cdot(u+v+w) \in \mathrm{Pol}($  $)$

Define the Order using Minor(h1)-Conditions

$G \leq H$

Define the Order using Minor(h1)-Conditions

$$G \leq H$$

$$\Pi$$

$$\Sigma_2$$

has kommutative Polymorphism

Define the Order using Minor(h1)-Conditions

$$\begin{array}{ccc} G & \leq & H \\ \rotatebox{-90}{$\models$} & & \rotatebox{-90}{$\models$} \\ \Sigma_2 & \Rightarrow & \Sigma_2 \end{array}$$

has kommutative Polymorphism

Define the Order using Minor(h1)-Conditions

$$
\begin{array}{ccc}
G & \leq & H \\
\pi & & \pi \\
\Sigma & \Rightarrow & \Sigma
\end{array}
$$

any h1-condition

$$
\left(
\begin{array}{ll}
f(\ldots) = g(\ldots) & \\
\quad \shortparallel & \\
f(\ldots) & g(\ldots) = g(\ldots)
\end{array}
\right)
$$

Define the Order using Minor(h1)-Conditions

$$G \leq H$$

$$\Pi \qquad\qquad \Pi$$

$$\Sigma \quad \Longrightarrow \quad \Sigma$$

any h1-condition

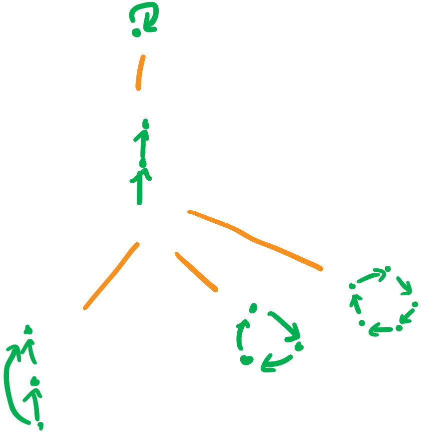
$$\left(\begin{array}{ll} f(\ldots) = g(\ldots) & \\ \quad \shortparallel & \\ f(\ldots) & g(\ldots) = g(\ldots) \end{array} \right)$$

In cores we can add the identity

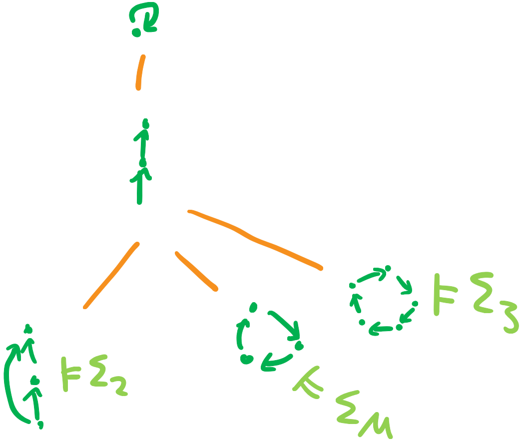
$$f(a, \ldots, a) = a$$

without changing the satisfyability of the condition

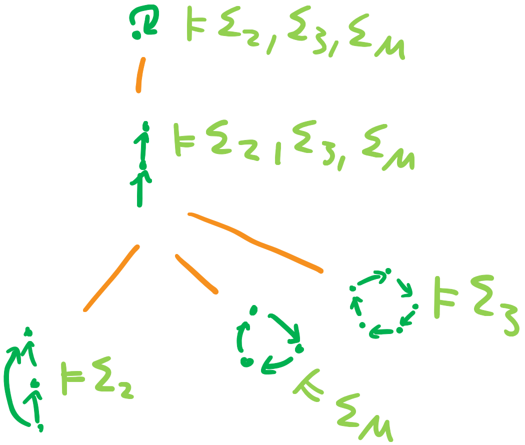
Observations



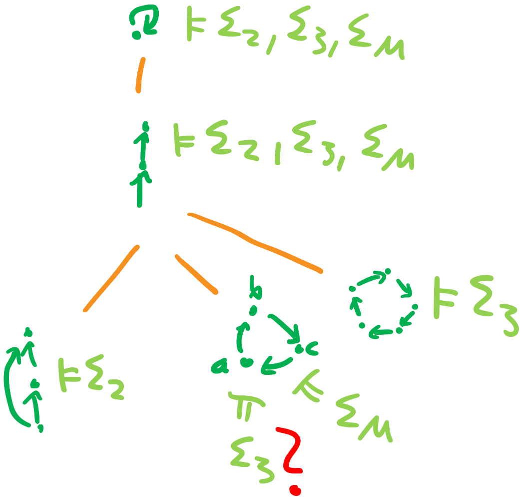
Observations



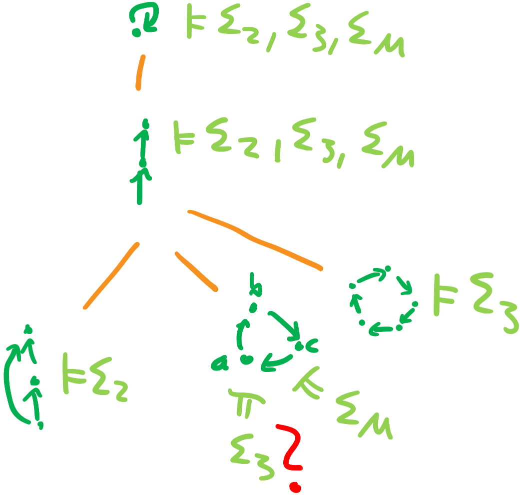
Observations



Observations

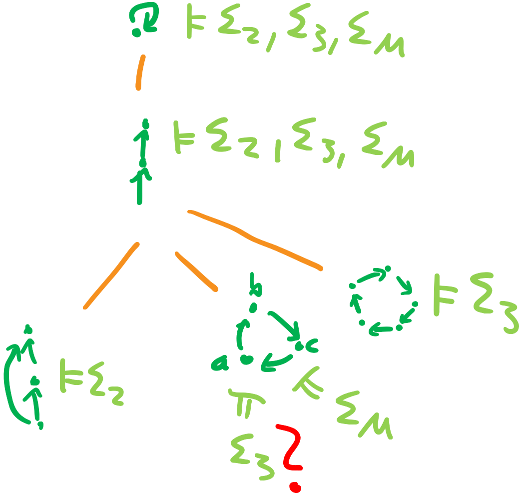


Observations

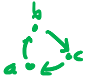


$$f(a, b, c) = V$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$f(b, c, a) = V$$

Observations

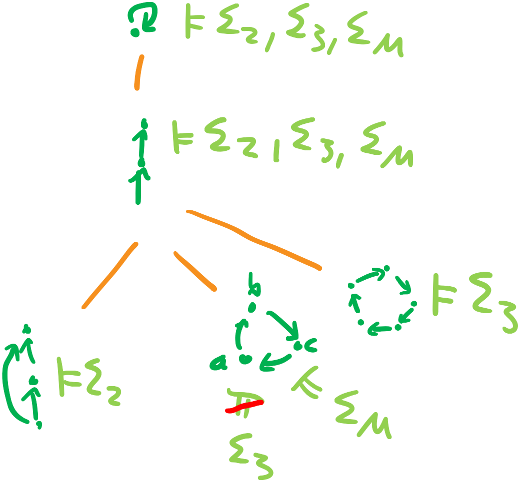


$$\begin{array}{c}
f(a,b,c) = V \\
\downarrow\ \downarrow\ \downarrow\ \quad \downarrow \\
f(b,c,a) = V
\end{array}$$

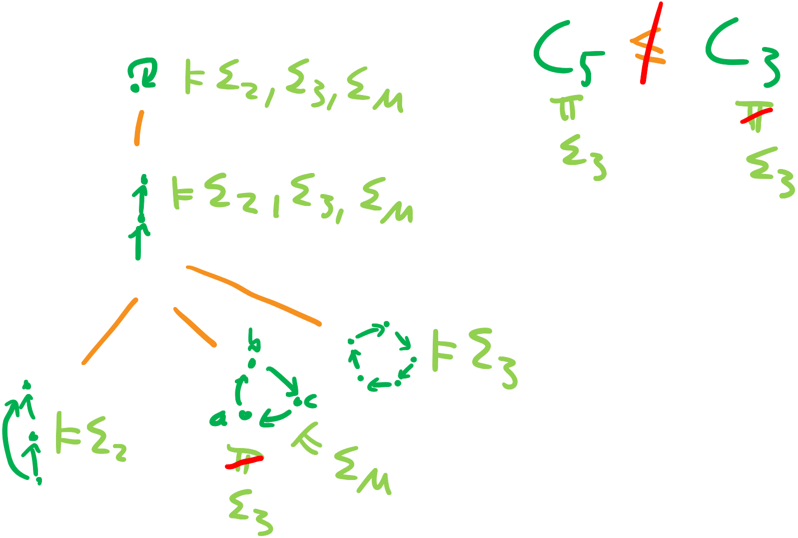
V in 

↯

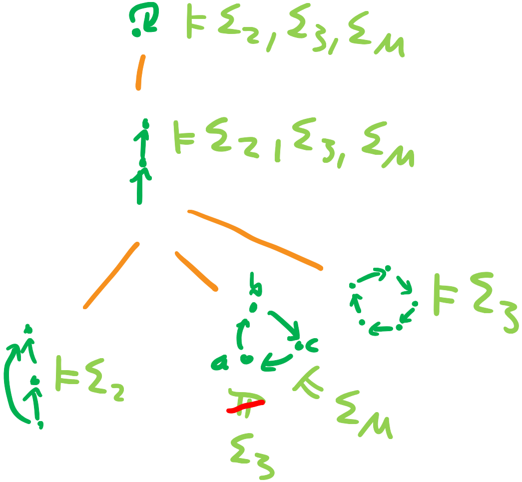
Observations



Observations



Observations



$C_5 \not\leq C_3$

$\Pi\Sigma_3$ $\quad$ $\cancel{\Pi}\Sigma_3$

Lets find some separaitions

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
0				
$0 \to 1 \to 2$				
C_2				
C_3				
C_5				

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	$(0,0) \mapsto 0$			
	min			
C_2				
C_3				
C_5				

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	$(0,0)\mapsto 0$			
	min			
C_2	$f(0,1)=v$ $f(1,0)=v$			
C_3				
C_5				

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	$(0,0) \mapsto 0$			
	min			
C_2	$f(0,1) = v$ $f(1,0) = v$			
C_3	$(u,v) \mapsto 2\cdot(u+v)$			
C_5	$(u,v) \mapsto 3\cdot(u+v)$			

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	$(0,0) \mapsto 0$			
	min			
C_2	$f(0,1) = v$ $f(1,0) = v$			
C_3	$(u,v) \mapsto 2 \cdot (u+v)$ $2 \cdot 2 \equiv 1 \pmod 3$			
C_5	$(u,v) \mapsto 3 \cdot (u+v)$ $3 \cdot 2 \equiv 1 \pmod 5$			

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓			
	✓			
C_2	✗			
C_3	✓			
C_5	✓			

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	$(0,0,0) \mapsto 0$		
	✓	min		
C_2	✗			
C_3	✓			
C_5	✓			

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	$(0,0,0) \mapsto 0$		
	✓	min		
C_2	✗			
C_3	✓	$f(1,2,0)=v$ $f(0,1,2)=v$		
C_5	✓			

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	$(0,0,0) \mapsto 0$		
	✓	min		
C_2	✗	$(u,v,w) \mapsto \overset{=3^{-1}}{1} \cdot (u+v+w)$		
C_3	✓	$F(1,2,0) = v$ $F(0,1,2) = v$		
C_5	✓	$(u,v,w) \mapsto \overset{=3^{-1}}{2} \cdot (u+v+w)$		

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	✓		
$0 \to 1 \to 2$	✓	✓		
C_2	✗	✓		
C_3	✓	✗		
C_5	✓	✓		

Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	✓	✓	
	✓	✓	✓	
C_2	✗	✓	✓	
C_3	✓	✗	✓	
C_5	✓	✓	✗	

Observations

	Σ_2	Σ_3	Σ_5	Σ_m
$\bullet_0 \circlearrowleft$	✓	✓	✓	$(0,0,0) \mapsto 0$
$0 \to 1 \to 2$, $0 \to 2$	✓	✓	✓	$m(2\ 2\ 1) = v$ $\uparrow\ \uparrow\ \uparrow\qquad\uparrow$ $m(1\ 0\ 0) = v$
C_2	✗	✓	✓	$(u,v,w) \mapsto u-v+w$
C_3	✓	✗	✓	$(u,v,w) \mapsto u-v+w$
C_5	✓	✓	✗	$(u,v,w) \mapsto u-v+w$

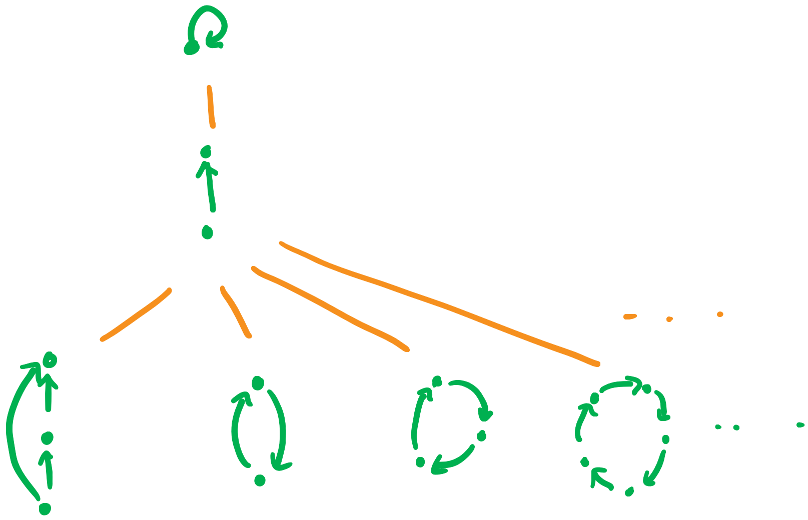
Observations

	Σ_2	Σ_3	Σ_5	Σ_n
$0 \circlearrowleft$	✓	✓	✓	✓
$0 \to 1 \to 2$	✓	✓	✓	✗
C_2	✗	✓	✓	✓
C_3	✓	✗	✓	✓
C_5	✓	✓	✗	✓

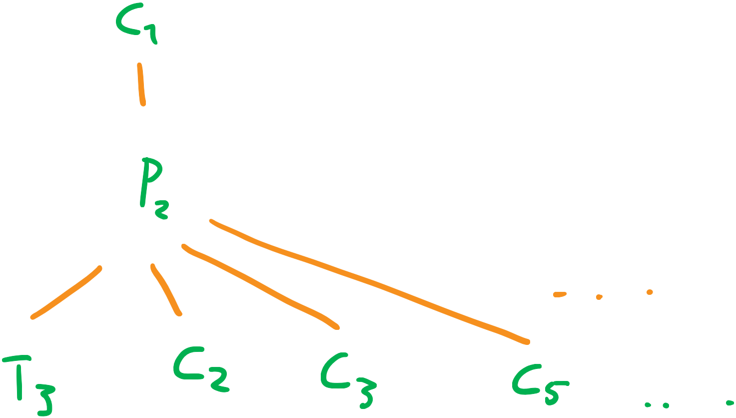
Observations

	Σ_2	Σ_3	Σ_5	Σ_n
	✓	✓	✓	✓
	✓	✓	✓	✗
C_2	✗	✓	✓	✓
C_3	✓	✗	✓	✓
C_5	✓	✓	✗	✓

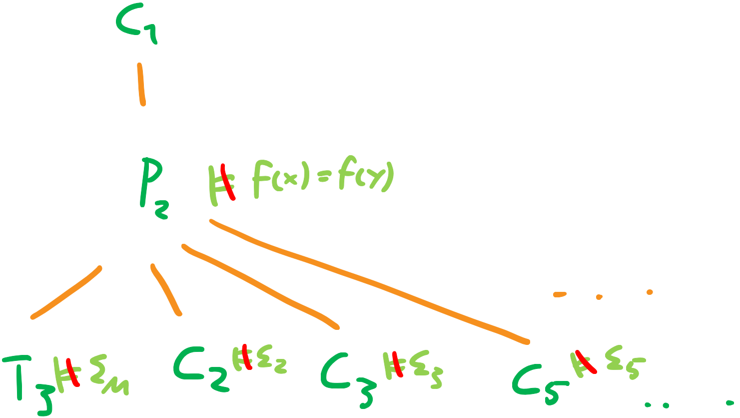
New Results



New Results

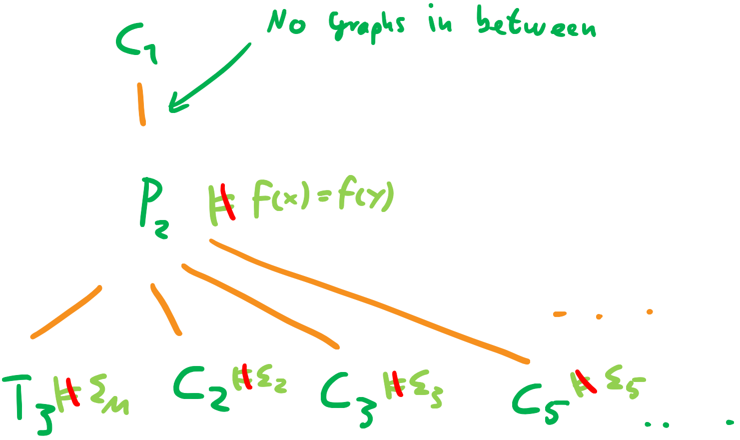


New Results

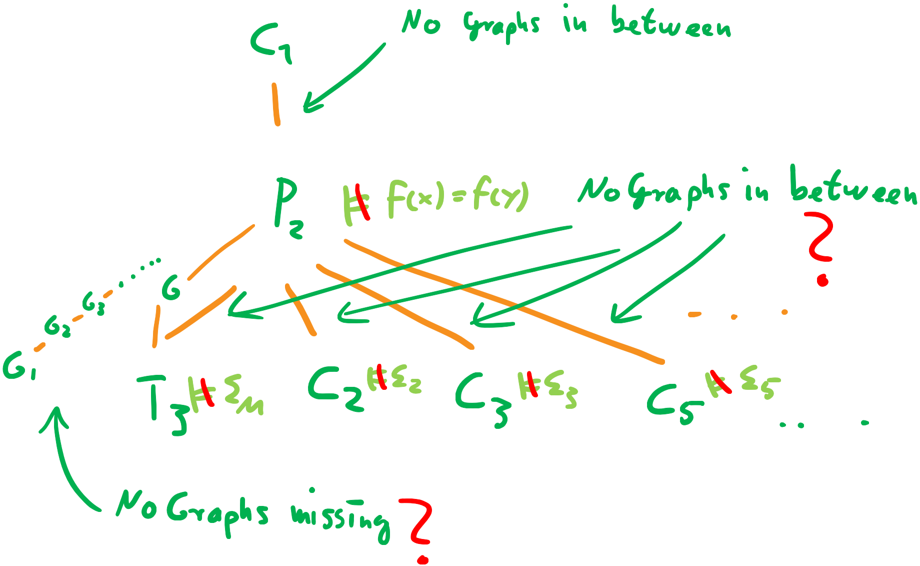


C_1
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$

New Results

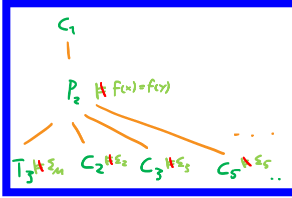

No graphs in between
C_1
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_4$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$

New Results



No Graphs in between
C_7
$P_2 \nvDash f(x)=f(y)$
No Graphs in between
G
G_1
G_2
G_3
$T_3 \nvDash \Sigma_M$
$C_2 \nvDash \Sigma_2$
$C_3 \nvDash \Sigma_3$
$C_5 \nvDash \Sigma_5$
No Graphs missing?

New Results



I]

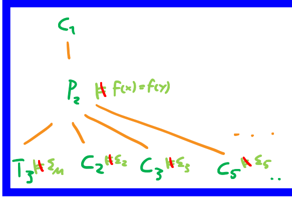
$$G \not\models \Sigma_m \implies G \leq T_3$$

$$G \not\models \Sigma_p \implies G \leq C_p$$

II] $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$$\implies P_2 \leq G$$

New Results



I]

$$G \not\models \Sigma_m \implies G \leq T_3$$

$$G \not\models \Sigma_p \implies G \leq C_p \quad \checkmark$$

Cycles Paper B+V+S

II]

$$G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \dots$$

$$\implies P_2 \leq G$$

New Results

$\boxed{\text{II}}$ $G \models \Sigma_{m}, \Sigma_{2}, \Sigma_{3}, \Sigma_{5}, \ldots$

$\Longrightarrow P_2 \leq G$

New Results

$\boxed{\text{II}}$ $\underbrace{G}_{\text{core}} \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

$m \in Pol(G)$ $\quad m(a,b,b) = m(b,b,a) = a$

New Results

$\boxed{\text{II}}$ $\underline{G \models \Sigma_m}, \Sigma_2, \Sigma_3, \Sigma_5, \dots$

$$\implies P_2 \leq G$$

$m \in Pol(G)$ $\quad m(a,b,b) = m(b,b,a) = a$

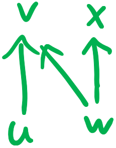
$$\begin{tikzcd}
v & x \\
u \arrow[u] & w \arrow[ul] \arrow[u]
\end{tikzcd}$$

New Results

$\boxed{\text{II}}$ $\underline{G \models \Sigma_m}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$$\Longrightarrow \quad P_2 \leq G$$

$m \in Pol(G) \qquad m(a,b,b) = m(b,b,a) = a$



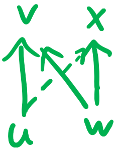
$$\begin{array}{ccccccc} m(& v, & v, & x &) = & x \\ & \uparrow & \uparrow & \uparrow & & \uparrow \\ m(& u, & w, & w &) = & u \end{array}$$

New Results

$\boxed{\mathrm{II}}$ $\underline{G \models \Sigma_m}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

$m \in \mathrm{Pol}(G)$ $\quad m(a,b,b) = m(b,b,a) = a$



$$\begin{array}{cccccc} m(& v, & v, & x &) = & x \\ & \uparrow & \uparrow & \uparrow & & \uparrow \\ m(& u, & w, & w &) = & u \end{array}$$

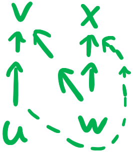
1-rectangular

New Results

II] $\underline{G \models \Sigma_m}, \Sigma_2, \Sigma_3, \Sigma_5, \dots$

$\Longrightarrow P_2 \leq G$

$m \in Pol(G)$ $\quad m(a,b,b) = m(b,b,a) = a$



$$\begin{array}{ccccccccc} m(& v, & v & , & x &) & = & x \\ & \uparrow & \uparrow & & \uparrow & & & \uparrow \\ & \uparrow & \uparrow & & \uparrow & & & \uparrow \\ m(& u, & w & , & w &) & = & u \end{array}$$

2-rectangular

New Results

$\boxed{\text{II}}$ $\underline{G \models \Sigma_n}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

k-rect. $\forall k$

Lemma

$G \models \Sigma_n \Longrightarrow G$ is totally rectangular

core

New Results

$\boxed{\text{II}}$ $\underline{G \models \Sigma_m}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies P_2 \leq G$

k-rect. $\forall k$

<u>Lemma</u>

core $G \models \Sigma_m \implies G$ is totally rectangular

<u>Lemma</u>

union of directed cycles

G is totally rectangular

$\implies G \rightleftarrows P_n$ or $G \rightleftarrows C$

New Results

$\boxed{\text{II}}$ $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies P_2 \leq G$

$G \models \Sigma_m \implies G = P_n$ or $G = C$

core

union of directed cycles

New Results

$\boxed{\text{II}}$ $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

union of directed cycles

$G \models \Sigma_m \Longrightarrow G = P_n$ or $G = C$

core

C_n is in C

$P_2 \leq G$

New Results

$\boxed{\text{II}}$ $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies P_2 \le G$

union of directed cycles

core

$G \models \Sigma_m \implies G = P_n$ or $G = C$

C_n is in C

$P_2 \le G$

$n = 1$

$C =$

New Results

$\boxed{\text{II}}$ $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

union of directed cycles

$G \models \Sigma_m \Longrightarrow G = P_n$ or $G = C$

core

$P_2 \leq G$

C_n is in C

$n = 1$

$C = \Omega$

$n = p \cdot k$

$C \models \Sigma_p$

New Results

II $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies P_2 \leq G$

union of directed cycles

core

$G \models \Sigma_m \implies G = P_n$ or $G = C$

$P_2 \leq G$

C_n is in C

$n = 1$

$C =$ 

$n = p \cdot k$

$C \models \Sigma_p$

$f(0, k, 2k, \ldots, n-k) = v$

$f(k, 2k, 3k, \ldots, 0) = v$

New Results

$\boxed{\text{II}}$ $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Longrightarrow P_2 \leq G$

union of directed cycles

core

$G \models \Sigma_n \Longrightarrow G = P_n$ or $G = C$

$P_2 \leq G$

C_n is in C

$n = 1$

$C = \Omega$

$n = p \cdot k$

$C \models \Sigma_p$

$f(0, k, 2k, \ldots, n-k) = v$

$f(k, 2k, 3k, \ldots, 0) = v$

$\Longrightarrow C_k$ is in C

New Results

II $G \models \Sigma_n, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies P_2 \leq G$

core

$G \models \Sigma_n \implies G = P_n$ or $G = C$

union of directed cycles

$P_2 \leq G$

C_n is in C

$n = 1$

$C = \circlearrowleft$

$n = p \cdot k$ $\quad f(0, k, 2k, \ldots, n-k) = v$

$C \models \Sigma_p$ $\quad f(k, 2k, 3k, \ldots, 0) = v$

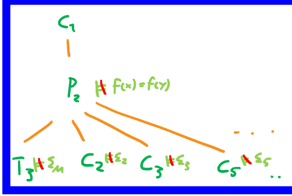
$\implies C_k$ is in C

$k = 1$

$C = \circlearrowleft$

$k = q \cdot \ell$ . . . . .

New Results



I]

$$G \not\models \Sigma_m \implies G \leq T_3$$

$$G \not\models \Sigma_p \implies G \leq C_p$$

✓ Cycles Paper B+V+S

II] $G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$$\implies P_2 \leq G$$

✓ total rectangularity

New

New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

New Results

I) $G \not\models \Sigma_n \implies G \leq T_3$

Recall

core

$G \models \Sigma_n \implies G$ is totally rectangular

$\Downarrow$

$G = P_n$ or $G = C$

New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

$\underset{\text{core}}{G} \models \Sigma_n \implies G$ is totally rectangular

$\Downarrow$

$G = \underset{\underset{\Sigma_n}{\Pi}}{P_n}$ or $G = \underset{\underset{\Sigma_n}{\Pi}}{C}$

$\Longleftarrow$

New Results

I] $G \nvDash \Sigma_n \implies G \leq T_3$

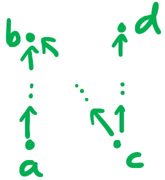
$G \nvDash \Sigma_n \implies G$ is not totally rectangular

core

New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

core $G \not\models \Sigma_n \implies G$ is **not** totally rectangular

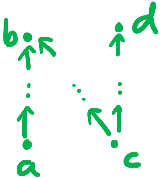
$\implies$  in G

New Results

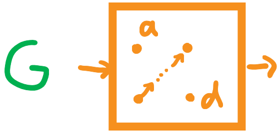
I] $G \not\models \Sigma_n \implies G \leq T_3$

$G \not\models \Sigma_n \implies G$ is not totally rectangular

core

$\implies$ 

in G



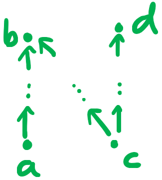
New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

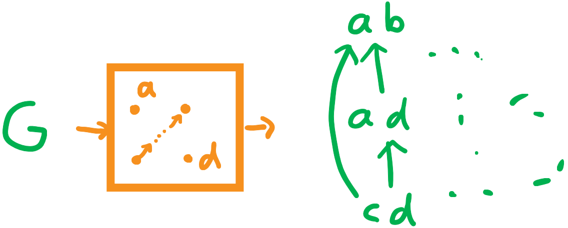
$G \not\models \Sigma_n \implies$ G is not totally rectangular

core

$\implies$



in G



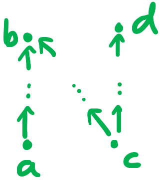
New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

$G \not\models \Sigma_n \implies$ G is ~~not~~ totally rectangular

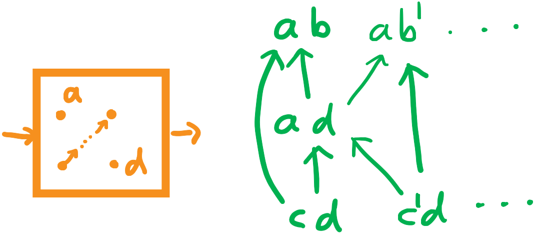
core

$\implies$



in G

G



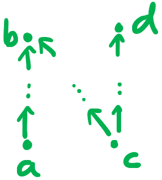
New Results

I] $G \not\models \Sigma_n \implies G \leq T_3$

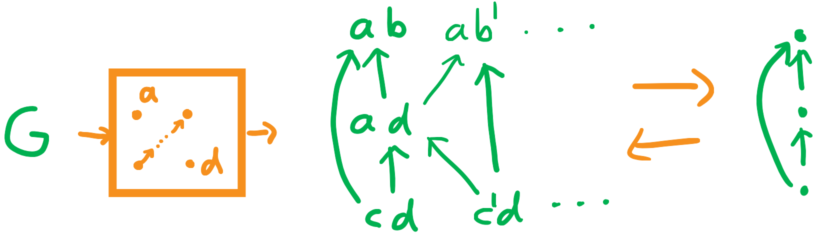
$G \not\models \Sigma_n \implies G$ is **not** totally rectangular

core

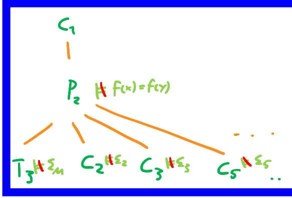
$\implies$



in G



New Results



I]

$$G \not\models \Sigma_m \implies G \le T_3$$ ✓ New

$$G \not\models \Sigma_p \implies G \le C_p$$ ✓ Cycles Paper B+V+S

II]

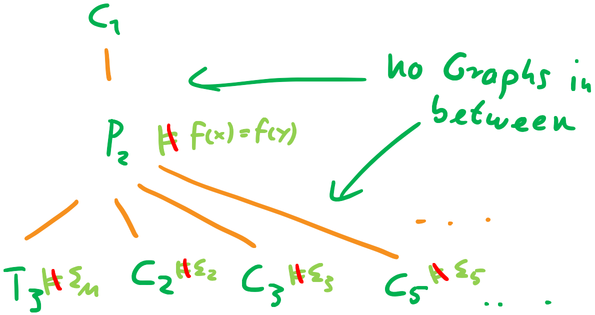
$$G \models \Sigma_m, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$$

$$\implies P_2 \le G$$

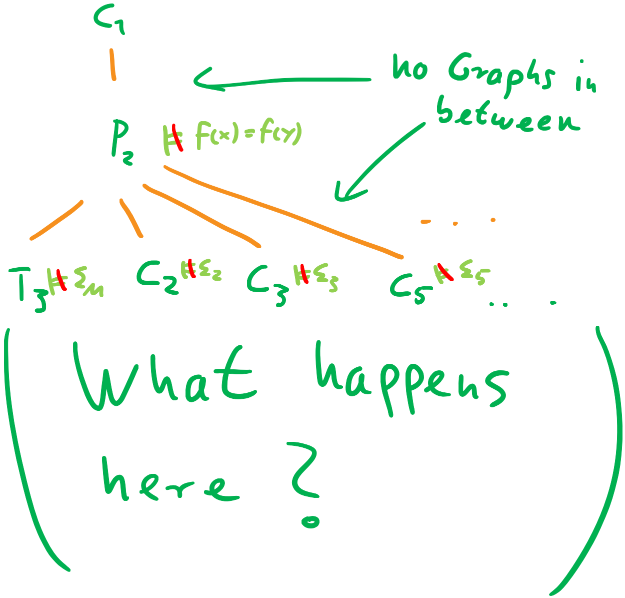
✓ total rectangularity

New

New Results

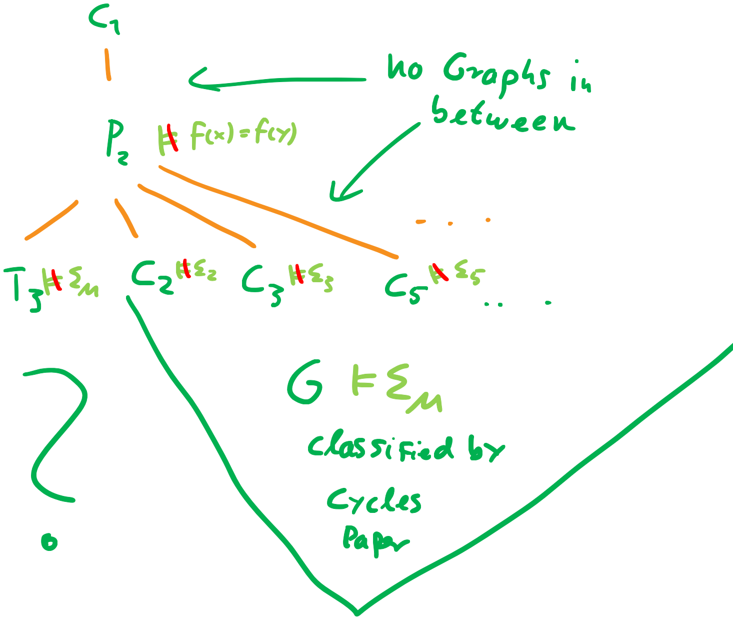

C_1
$P_2 \not\models f(x)=f(y)$
no Graphs in between
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$

New Results

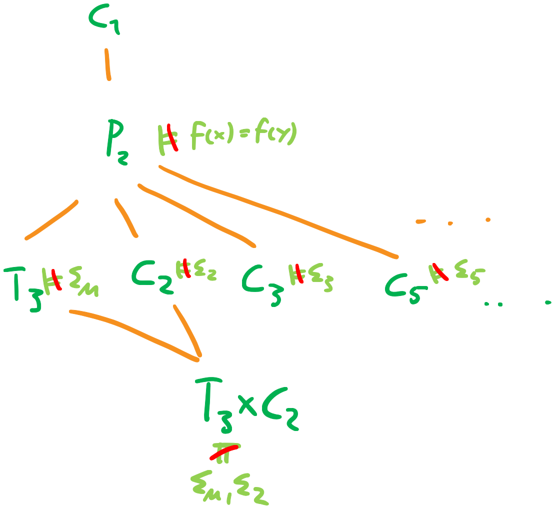


C_1
$P_2 \not\models f(x)=f(y)$
no Graphs in between
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
What happens here?

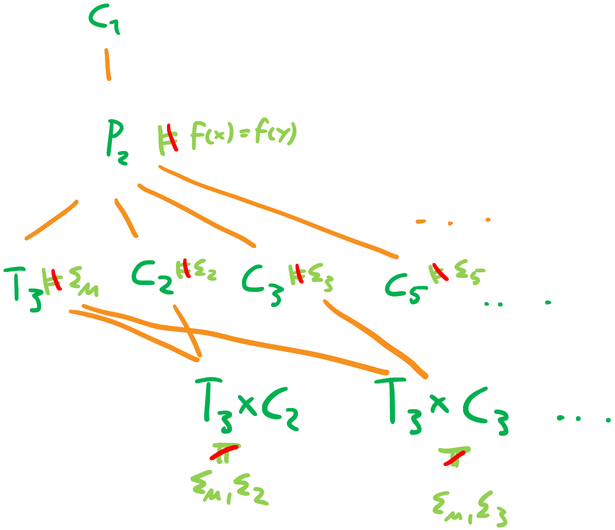
New Results


C_1
no Graphs in between
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
?
$G \models \Sigma_M$
classified by Cycles Paper

New Results

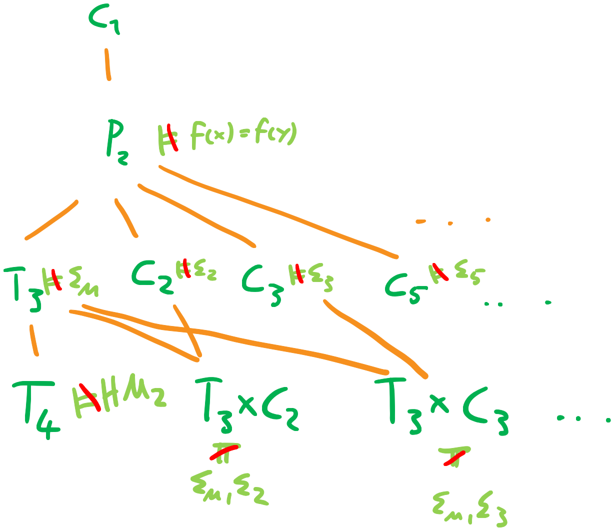

C_1
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
$T_3 \times C_2$
Σ_M, Σ_2

New Results



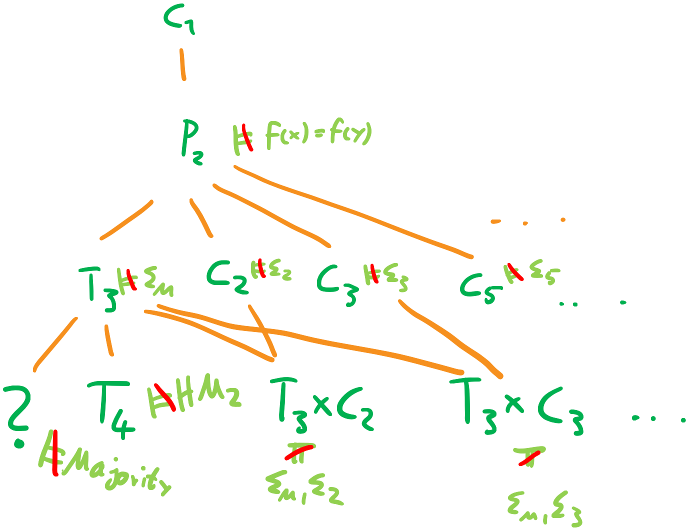
C_1
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
$T_3 \times C_2$
Σ_M, Σ_2
$T_3 \times C_3$
Σ_M, Σ_3

New Results



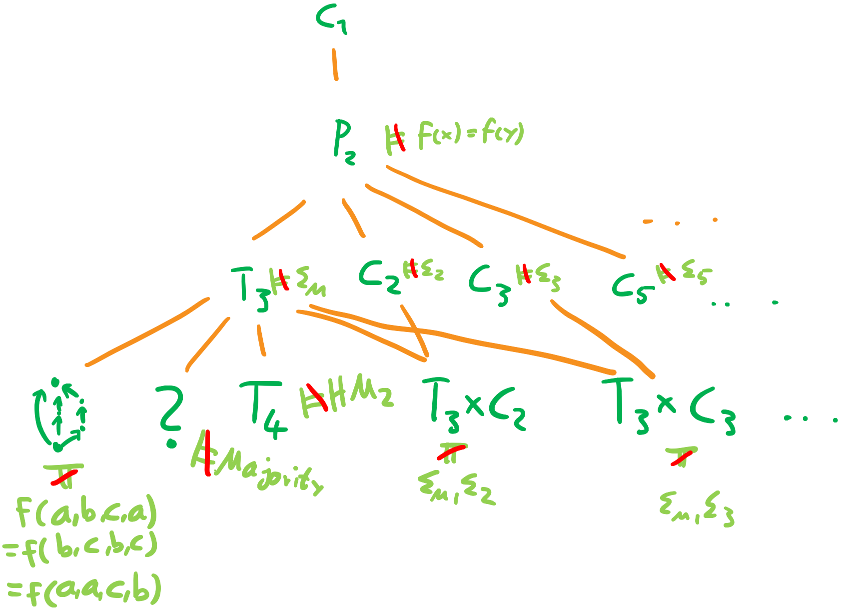
C_1
$P_2 \not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
$T_4 \not\models HM_2$
$T_3 \times C_2$
Σ_M, Σ_2
$T_3 \times C_3$
Σ_M, Σ_3

New Results



C_1
P_2 $\not\models f(x)=f(y)$
$T_3 \not\models \Sigma_M$
$C_2 \not\models \Sigma_2$
$C_3 \not\models \Sigma_3$
$C_5 \not\models \Sigma_5$
$?$ $\not\models$ Majority
$T_4 \not\models HM_2$
$T_3 \times C_2$
Σ_M, Σ_2
$T_3 \times C_3$
Σ_M, Σ_3

New Results



C1
P2
⊭ f(x)=f(y)
T3 ⊭ ΣM
C2 ⊭ Σ2
C3 ⊭ Σ3
C5 ⊭ Σ5
?
⊭ Majority
T4 ⊭ HM2
T3 × C2
ΣM, Σ2
T3 × C3
ΣM, Σ3
f(a,b,c,a)
=f(b,c,b,c)
=f(a,a,c,b)

Open Questions

Let $G \not\models \Sigma_n$ $(G \leq T_3)$

I] $G \not\models HM_2 \implies G \leq T_4$

$G \not\models \text{Majority} \implies G \leq ?$

$G \not\models \begin{array}{l} f(a,b,c,a) \\ =f(b,c,b,c) \\ =f(a,a,c,b) \end{array} \implies G \leq$ 

$G \not\models \Sigma_p \implies G \leq T_3 \times C_p$

Open Questions

Let $G \not\models \Sigma_n$ $\quad (G \leq T_3)$

I] $G \not\models HM_2 \Rightarrow G \leq T_4$

$G \not\models \text{Majority} \Rightarrow G \leq ?$

$G \not\models \begin{array}{l} f(a,b,c,a) \\ =f(b,c,b,c) \\ =f(a,a,c,b) \end{array} \Rightarrow G \leq$ 

$G \not\models \Sigma_p \Rightarrow G \leq T_3 \times C_p$

II] $G \models HM_2, \text{Majority}, \begin{array}{l} f(a,b,c,a) \\ =f(b,c,b,c) \\ =f(a,a,c,b) \end{array}, \Sigma_2, \Sigma_3, \Sigma_5, \dots$

$\Rightarrow T_3 \leq G$

Open Questions

Let $G \not\models \Sigma_n$ $\left(G \le T_3\right)$

I] $G \not\models HM_2 \Rightarrow G \le T_4$ ✓ ← New

$G \not\models \text{Majority} \Rightarrow G \le ?$

$G \not\models \begin{array}{l} f(a,b,c,a) \\ =f(b,c,b,c) \\ =f(a,a,c,b) \end{array} \Rightarrow G \le$ 

$G \not\models \Sigma_p \Rightarrow G \le T_3 \times C_p$ ✓ ← New

II] $G \models HM_2, \text{Majority}, \begin{array}{l} f(a,b,c,a) \\ =f(b,c,b,c) \\ =f(a,a,c,b) \end{array}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\Rightarrow T_3 \le G$

Open Questions

Let $G \not\models \Sigma_n$ $\left(G \leq T_3 \right)$

I] $G \not\models HM_2 \implies G \leq T_4$ ✓ ← New

$G \not\models \text{Majority} \implies G \leq ?$

$G \not\models \begin{array}{l} f(a,b,c,a) \\ = f(b,c,b,c) \\ = f(a,a,c,b) \end{array} \implies G \leq$ 

$G \not\models \Sigma_p \implies G \leq T_3 \times C_p$ ✓ ← New

II] $G \models HM_2, \text{Majority}, \begin{array}{l} f(a,b,c,a) \\ = f(b,c,b,c) \\ = f(a,a,c,b) \end{array}, \Sigma_2, \Sigma_3, \Sigma_5, \ldots$

$\implies T_3 \leq G$

Thank you

Paper:

Maximal Digraphs With Respect to Primitive Positive Constructibility

https://arxiv.org/abs/2103.08625