CSPs with finite duality closed under primitive positive constructions

Florian Starke

joint work with Manuel Bodirsky

CSPs

Definition

$\mathbb{T}$ a relational structure

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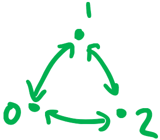

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$$\mathrm{CSP}(\mathbb{P}_2) = \{\mathbb{G} \mid \text{no directed path of length 2 in } \mathbb{G}\}$$

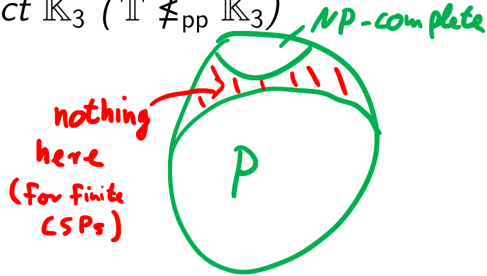
Dichotomy

$\mathbb{T}$ a finite relational structure

Theorem (Bulatov17,Zhuk17)

The following are equivalent

1. $\mathbb{T}$ *has a Siggers polymorphism and* CSP($\mathbb{T}$) *is in P*
2. $\mathbb{T}$ *cannot pp-construct* $\mathbb{K}_3$ *(*$\mathbb{T} \not\leq_{\mathrm{pp}} \mathbb{K}_3$*)*



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has NP-complete CSP

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Definition later

Datalog

Theorem

The following are equivalent

1. $\mathrm{CSP}(\mathbb{T})$ *is solved by a Datalog program*
2. $\mathbb{T}$ *cannot pp-construct* $\mathbb{D}_{3\mathrm{LIN}\,p}$ *for any prime p*

$$\mathbb{D}_{3\mathrm{LIN}\,p} := (\{0, \ldots, p-1\}, R_{0000}, R_{1000}, R_{2000}, \ldots)$$

$$R_{abcd} := \{(x, y, z) \mid ax + ay + cz = d\}$$

linear equations over $\mathbb{Z}_p$
with 3 variables

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Lets come up with a new theorem!

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$\mathbb{T}$ has *finite duality* if there is a finite set $D(\mathbb{T})$ of finite structures such that for all $\mathbb{I}$

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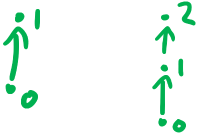

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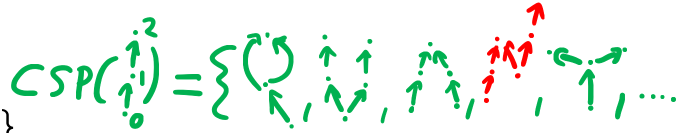

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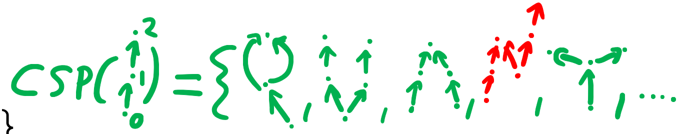

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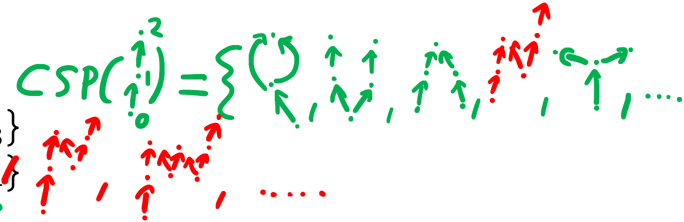

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$$\mathrm{FD} := \{\mathrm{CSP}(\mathbb{T}) \mid \mathbb{T} \text{ has finite duality}\}$$

Finite Duality

Theorem (Atserias05)

$$\mathrm{FD} = \mathrm{FO}$$

Finite Duality

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$$\begin{array}{c} \mathrm{NL} \\ | \\ \mathrm{L} \\ | \\ \mathrm{FO} = AC_0 \end{array}$$

Finite Duality

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Can we get a theorem for FD similar to the one for Datalog?

GOAL: find structures $\mathbb{A}_1, \mathbb{A}_2, \ldots$ such that
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homomorphic equivalence

If $\mathbb{T} \to \mathbb{S}$ and $\mathbb{S} \to \mathbb{T}$, then $\mathbb{T} =_{\mathrm{pp}} \mathbb{S}$.

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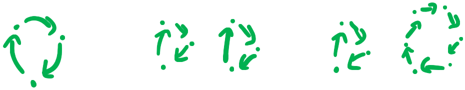

$$\mathbb{C}_3 \leftrightarrow \mathbb{C}_{3,3} \leftrightarrow \mathbb{C}_{3,6}$$

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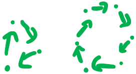
Example   

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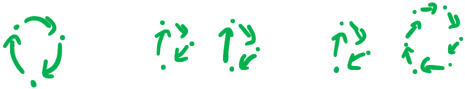

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Example

CSP( ) = { G | every cycle in G has a net length divisible by 3 }

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T and S can have different signatures

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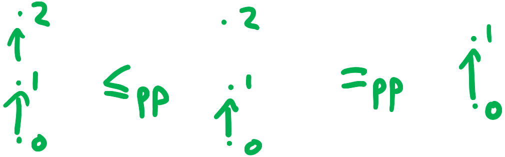

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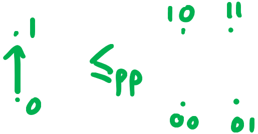

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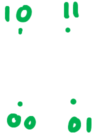

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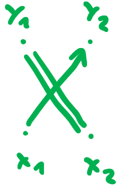

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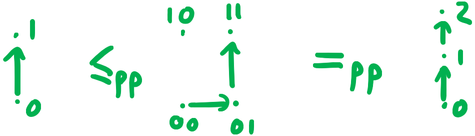

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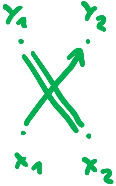

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$\Rightarrow$ FD is not closed under pp-constructions

NEW GOAL: find structures $\mathbb{A}_1, \mathbb{A}_2, \ldots$ such that
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1. CSP($\mathbb{T}$) is in PP(FD)
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$\mathrm{FD} \subset \mathrm{L}$

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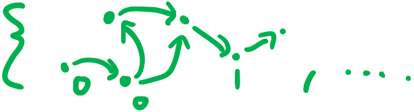

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$\Vert$

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can solve exactly the CSPs with tree duality

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PP(FD) $\subseteq$ PP(L) = L

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does not have tree duality

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$\Longrightarrow$ no problem in PP(FD) can pp-construct $\mathbb{C}_p$ for any p

Fun Fact CSP($\mathbb{C}_2$) = 2Colorability $\in$ L

Conjecture

The following are equivalent

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Conjecture

The following are equivalent

1. CSP($\mathbb{T}$) is in PP(FD)
2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots$

What to do now?

Conjecture

The following are equivalent

1. CSP($\mathbb{T}$) is in PP(FD)
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What to do now?
Find another equivalent statement.

Dusl Programms

Dusl Programms

Definition

a *Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(\overline{x}) \dashv \exists \overline{y} : R_1(\overline{z_{11}}) \wedge \ldots \wedge S_1(\overline{z_{21}}) \wedge \ldots$

$S_1, \ldots \in \tau$, $R, R_1, \ldots \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

Datalog programm for $\mathrm{CSP}(\mathbb{P}_2)$

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Datalog programm for CSP($\mathbb{P}_2$)

$$G \dashv \exists Y_1, Y_2, Y_3 : Y_1 \to Y_2 \wedge Y_2 \to Y_3$$

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Datalog programm for $\mathrm{CSP}(\mathbb{P}_2)$

$G \dashv$

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Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

Datalog programm for CSP($\mathbb{T}$) $\in$ FD

$G \dashv \mathbb{F}$ for any $\mathbb{F} \in D(\mathbb{T})$

Dusl Programms

Definition

a *linear* *Datalog Program* consists of two signatures τ, σ, and a finite set of rules
Rules: $R(\overline{x}) \dashv \exists \overline{y} : R_1(\overline{z_1}) \wedge S_1(\overline{z_{21}}) \wedge \ldots$
$S_1, \ldots \in \tau$, $R, R_1 \in \sigma$
Input: a finite structure with signature τ
Output: can the program derive $G \in \sigma$

---

linear Datalog programm for $\mathrm{CSP}(\mathbb{C}_2)$

Dusl Programms

Definition

a *linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(\overline{x}) \dashv \exists \overline{y} : R_1(\overline{z_1}) \wedge S_1(\overline{z_{21}}) \wedge \dots$

$S_1, \dots \in \tau,\ R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

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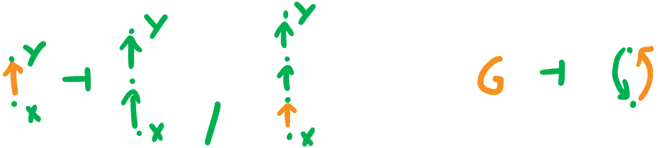
linear Datalog programm for CSP$(\mathbb{C}_2)$



Dusl Programms

Definition

a *unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(x) \dashv \exists \overline{y} : R_1(z) \land S_1(\overline{z_{11}}) \land \ldots$

$S_1, \ldots \in \tau$, $R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

---

unary linear Datalog programm for CSP($\mathbb{O}$)

Dusl Programms

Definition

a *unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules
Rules: $R(x) \dashv \exists \overline{y} : R_1(z) \wedge S_1(\overline{z_{11}}) \wedge \dots$
$S_1, \dots \in \tau,\ R, R_1 \in \sigma$
Input: a finite structure with signature τ
Output: can the program derive $G \in \sigma$

---

unary linear Datalog programm for CSP($\mathbb{O}$)

$1(\overset{x}{\cdot}) \dashv 1(\overset{x}{\cdot}),\quad \begin{array}{l} \bullet\, x \\ \uparrow \\ \bullet\, 1 \end{array}$ $\qquad G \dashv \begin{array}{l} \bullet\, 0 \\ \uparrow \\ \bullet\, 1 \end{array}$

Dusl Programms

Definition

a *symmetric* *unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(x) \dashv \exists \overline{y} : R_1(z) \wedge S_1(\overline{z_{11}}) \wedge \ldots$

$R_1(z) \dashv \exists \overline{y'} : R(x) \wedge S_1(\overline{z_{11}}) \wedge \ldots$

$S_1, \ldots \in \tau$, $R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

Dusl programm for $\mathrm{CSP}(\mathbb{P}_3)$

Dusl Programms

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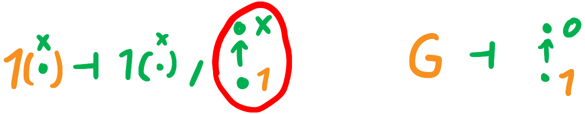
$S_1, \ldots \in \tau,\ R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

---

was not symmetric, $1(y) \dashv$  is missing



Dusl Programms

Definition

a *symmetric unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

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Dusl programm for $\mathrm{CSP}(\mathbb{P}_3)$

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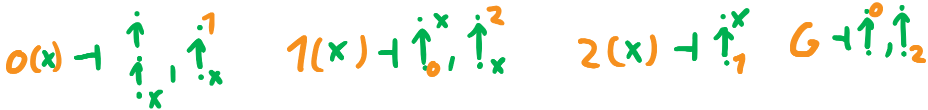
Dusl programm for $\mathrm{CSP}(\mathbb{P}_3)$



Conjecture

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1. CSP($\mathbb{T}$) is in PP(FD)
2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots$
3. CSP($\mathbb{T}$) is solved by some Dusl program

What we know

1. CSP($\mathbb{T}$) is in PP(FD)

if $L \neq NL$

2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \overbrace{\mathbb{C}_2, \mathbb{C}_3, \ldots}^{\text{Solved by AC}}$

3. CSP($\mathbb{T}$) is solved by some Dusl program

What we know

1. CSP($\mathbb{T}$) is in PP(FD)
2. $\mathbb{T}$ cannot pp-construct $\underbrace{\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots}$
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What we know

1. CSP($\mathbb{T}$) is in PP(FD)

solved by AC

if $L \neq NL$

2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots$

if $L \neq NL$

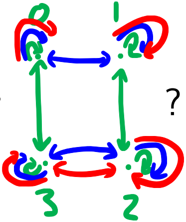
if Dusl is closed under $\leq_{pp}$

3. CSP($\mathbb{T}$) is solved by some Dusl program

Open Questions

1. are dusl programms closed under pp constructions?
2. Is $\mathbb{N}_{123}$ in PP(FD)?

3. Is there a Dusl program for



?

with constants