CSPs with finite duality closed under primitive positive constructions

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joint work with Manuel Bodirsky

CSPs

Definition

$\mathbb{T}$ a relational structure

$$\mathrm{CSP}(\mathbb{T}) := \{\mathbb{I} \mid \mathbb{I} \text{ finite structure such that } \mathbb{I} \to \mathbb{T}\}$$

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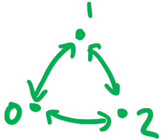

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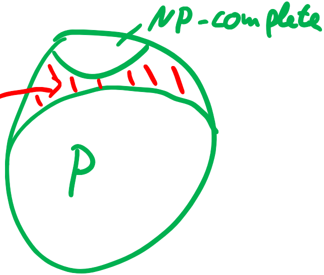
$$\mathrm{CSP}(\mathbb{P}_2) = \{\mathbb{G} \mid \text{no directed path of length 2 in } \mathbb{G}\}$$

Dichotomy



$\mathbb{T}$ a finite relational structure

Theorem (Bulatov17,Zhuk17)

The following are equivalent

1. $\mathbb{T}$ *has a Siggers polymorphism and* CSP($\mathbb{T}$) *is in P*
2. $\mathbb{T}$ *cannot pp-construct* $\mathbb{K}_3$ *(*$\mathbb{T} \not\leq_{\text{pp}} \mathbb{K}_3$*)*

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$\Rightarrow$ use $\leq_{\mathrm{pp}}$ to study complexity classes in P

Datalog

Theorem

The following are equivalent

1. $\mathrm{CSP}(\mathbb{T})$ *is solved by a Datalog program*
2. $\mathbb{T}$ *cannot pp-construct* $\mathbb{D}_{3\mathrm{LIN}\,p}$ *for any prime* p

$$\mathbb{D}_{3\mathrm{LIN}\,p} := (\{0, \ldots, p-1\}, R_{0000}, R_{1000}, R_{2000}, \ldots)$$
$$R_{abcd} := \{(x, y, z) \mid ax + ay + cz = d\}$$

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Lets come up with a new theorem!

Finite Duality

Definition

$\mathbb{T}$ has *finite duality* if there is a finite set $D(\mathbb{T})$ of finite structures such that for all $\mathbb{I}$

$$\mathbb{I} \not\to \mathbb{T} \Leftrightarrow (\exists \mathbb{F} \in D(\mathbb{T}) : \mathbb{F} \to \mathbb{I})$$

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$$\text{FD} := \{\text{CSP}(\mathbb{T}) \mid \mathbb{T} \text{ has finite duality}\}$$

Finite Duality

Theorem (Atserias05)

FD = FO

Finite Duality

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NL
|
L
|
FO

Finite Duality

Theorem (Atserias05)

$$\text{FD} = \text{FO}$$

$$\begin{array}{c}\text{NL}\\ |\\ \text{L}\\ |\\ \text{FO}\end{array}$$

Can we get a theorem for FD similar to the one for Datalog?

GOAL: find structures $\mathbb{A}_1, \mathbb{A}_2, \ldots$ such that
The following are equivalent

1. CSP($\mathbb{T}$) is in FD
2. $\mathbb{T}$ cannot pp-construct $\mathbb{A}_1, \mathbb{A}_2, \ldots$

Define pp-construction

homomorphic equivalence

If $\mathbb{T} \to \mathbb{S}$ and $\mathbb{S} \to \mathbb{T}$, then $\mathbb{T} =_{\text{pp}} \mathbb{S}$.

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Example

$$\mathbb{C}_3 \leftrightarrow \mathbb{C}_{3,3} \leftrightarrow \mathbb{C}_{3,6}$$

any graph with a loop $\leftrightarrow \mathbb{C}_1$

Note that: $\mathbb{T} \leftrightarrow \mathbb{S}$ implies $\mathrm{CSP}(\mathbb{T}) = \mathrm{CSP}(\mathbb{S})$

Define pp-construction

pp-power

If $S = T^n$ and every relation $R^{(K)}$ of $\mathbb{S}$ is (as a nk-ary relation) pp-definable in $\mathbb{T}$, then $\mathbb{T} \leq_{pp} \mathbb{S}$.

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Example

$\mathbb{P}_3 \leq_{pp} \mathbb{P}_2$

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OH NO! CSP($\mathbb{P}_2$) $\in$ FD, CSP($\mathbb{P}_3$) $\notin$ FD, and
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$\Rightarrow$ FD is not closed under pp-constructions

NEW GOAL: find structures $\mathbb{A}_1, \mathbb{A}_2, \dots$ such that
The following are equivalent

1. CSP($\mathbb{T}$) is in PP(FD)
2. $\mathbb{T}$ cannot pp-construct $\mathbb{A}_1, \mathbb{A}_2, \dots$

PP(FD) - First observations

$\mathrm{PP}(\mathrm{FD}) \subseteq \mathrm{PP}(\mathrm{L}) = \mathrm{L}$

PP(FD) - First observations

PP(FD) $\subseteq$ PP(L) = L

CSP($\mathbb{O}$) is NL-hard
$\Longrightarrow$ If L $\neq$ NL, then no problem in PP(FD) can pp-construct $\mathbb{O}$

any problem in FD can be solved by Arc Consistency
AC is closed under pp-constructions
AC cannot solve CSP($\mathbb{C}_p$) for any prime p
$\Longrightarrow$ no problem in PP(FD) can pp-construct $\mathbb{C}_p$ for any p

Conjecture

The following are equivalent

1. CSP($\mathbb{T}$) is in PP(FD)
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What to do now?

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What to do now?

Find another equivalent statement.

Dusl Programms

Definition

a *Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(\overline{x}) \dashv \exists \overline{y} : R_1(\overline{z_{11}}) \wedge \ldots \wedge S_1(\overline{z_{21}}) \wedge \ldots$

$S_1, \ldots \in \tau$, $R, R_1, \ldots \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

Datalog programm for $\mathrm{CSP}(\mathbb{P}_2)$

Dusl Programms

Definition

a *linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(\overline{x}) \dashv \exists \overline{y} : R_1(\overline{z_1}) \wedge S_1(\overline{z_{21}}) \wedge \dots$

$S_1, \dots \in \tau,\ R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

---

linear Datalog programm for CSP($\mathbb{C}_2$)

Dusl Programms

Definition

a *unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(x) \dashv \exists \overline{y} : R_1(z) \wedge S_1(\overline{z_{11}}) \wedge \ldots$

$S_1, \ldots \in \tau$, $R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

---

unary linear Datalog programm for CSP($\mathbb{O}$)

Dusl Programms

Definition

a *symmetric unary linear Datalog Program* consists of two signatures τ, σ, and a finite set of rules

Rules: $R(x) \dashv \exists \overline{y} : R_1(z) \wedge S_1(\overline{z_{11}}) \wedge \dots$

$R_1(z) \dashv \exists \overline{y'} : R(x) \wedge S_1(\overline{z_{11}}) \wedge \dots$

$S_1, \dots \in \tau$, $R, R_1 \in \sigma$

Input: a finite structure with signature τ

Output: can the program derive $G \in \sigma$

Dusl programm for $\mathrm{CSP}(\mathbb{P}_3)$

Conjecture

The following are equivalent

1. CSP($\mathbb{T}$) is in PP(FD)
2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots$
3. CSP($\mathbb{T}$) is solved by some DusI program

What we know

1. CSP($\mathbb{T}$) is in PP(FD)

2. $\mathbb{T}$ cannot pp-construct $\mathbb{O}, \mathbb{C}_2, \mathbb{C}_3, \ldots$

3. CSP($\mathbb{T}$) is solved by some Dusl program

What we know

dusl $\Rightarrow$ TS (AC)
dusl $\Rightarrow$ HM (if L $<$ NL)
PP(fd) $\Rightarrow$ NU (cite paper)
PP(fd) $\Rightarrow$ TS (if L $<$ NL)
PP(fd) $\Rightarrow$ HM (if L $<$ NL)
TS $\Rightarrow$ cannot pp-construct Cp for any p ($\Leftrightarrow$ for digraphs)
HM $\Leftrightarrow$ cannot pp-construct Ord
something more about symmetric datalog

Open Questions

1. are dusl programms closed under pp constructions?
2. Is $\mathbb{N}_{123}$ in PP(FD)?

3. Is there a Dusl program for ?