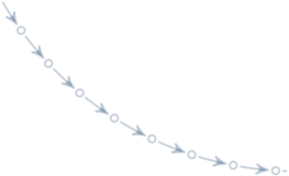
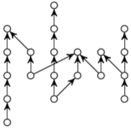
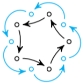


Digraphs modulo primitive positive constructability

PhD Defense of
Florian Starke





18.04.2024

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Introduction – Complexity Theory

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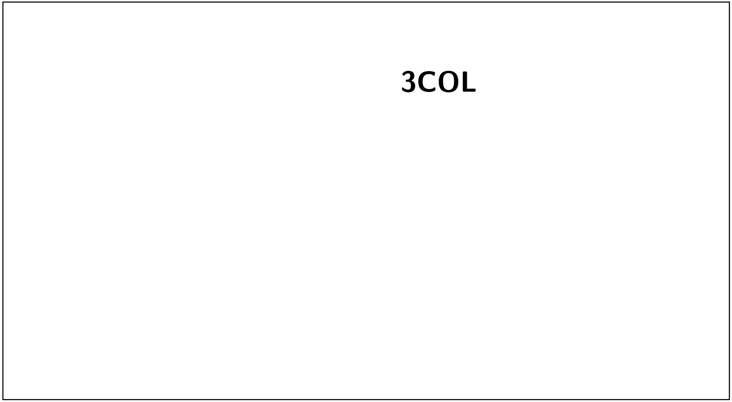

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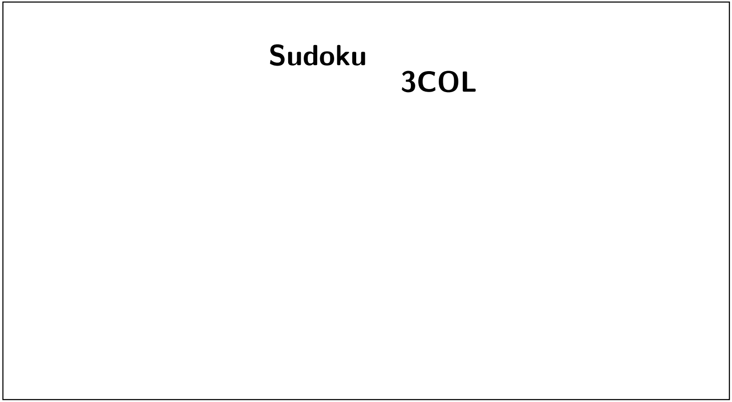

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Chess

Sudoku

3COL

HornSAT

DReach

UReach

2COL

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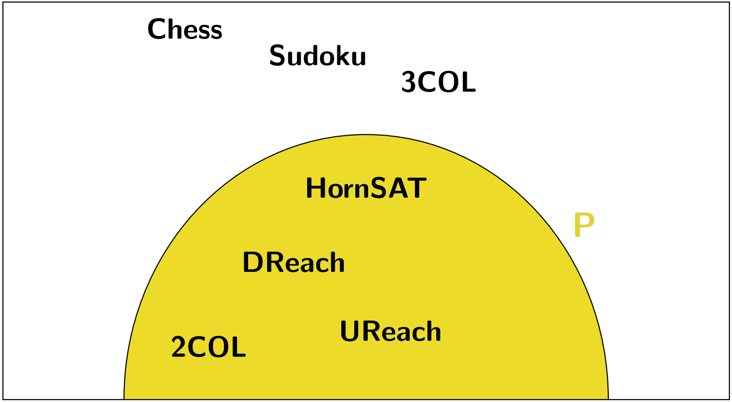

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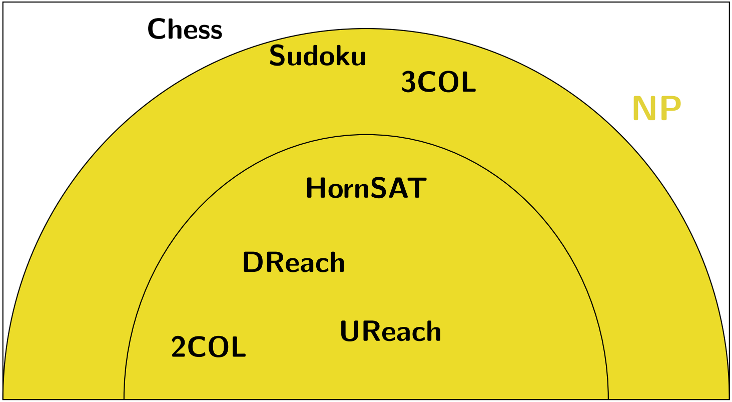

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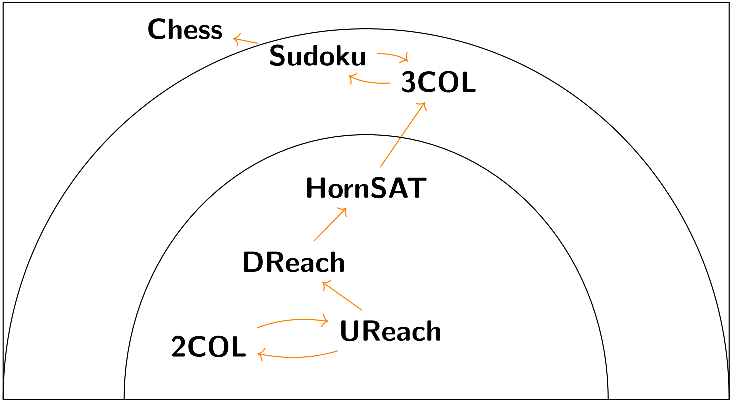

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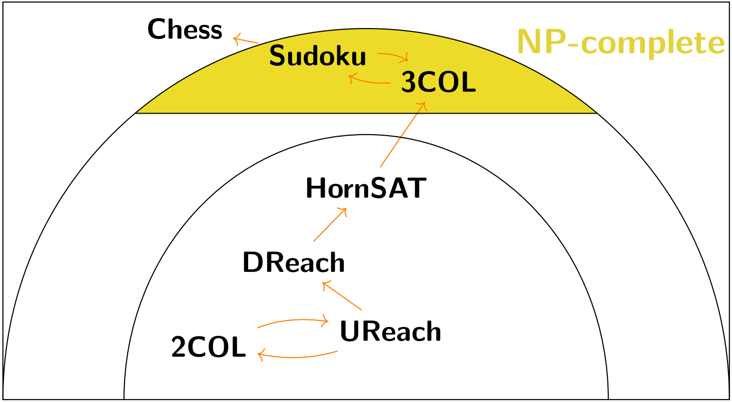

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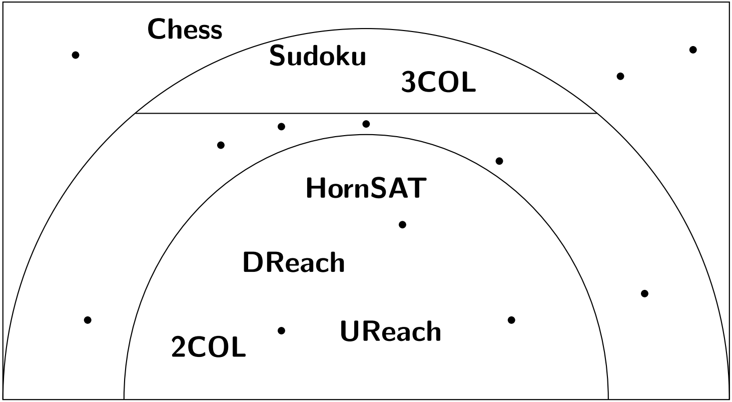

Introduction – Complexity Theory

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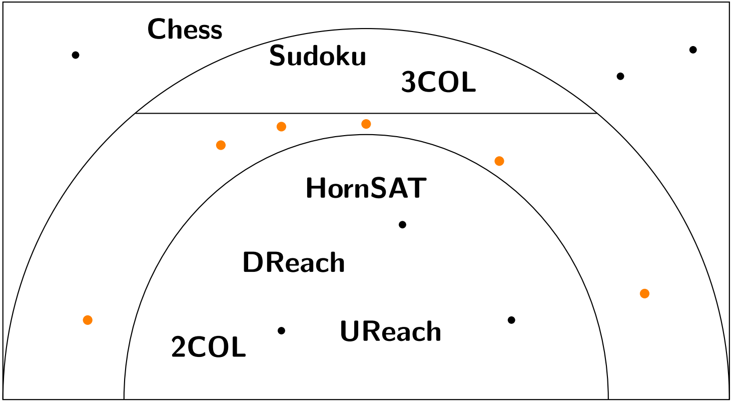
Theorem (Ladner 1975): P-NP intermediate problems exist (assuming $P \neq NP$).



Introduction – Constraint Satisfaction Problems

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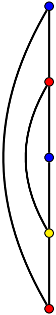

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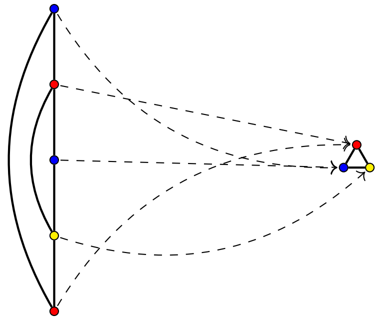

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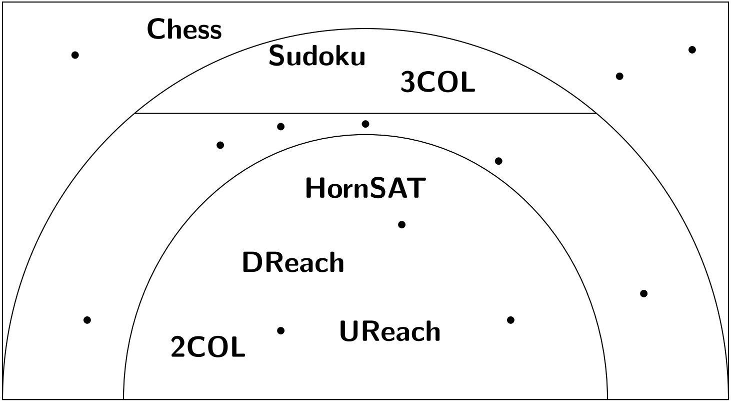

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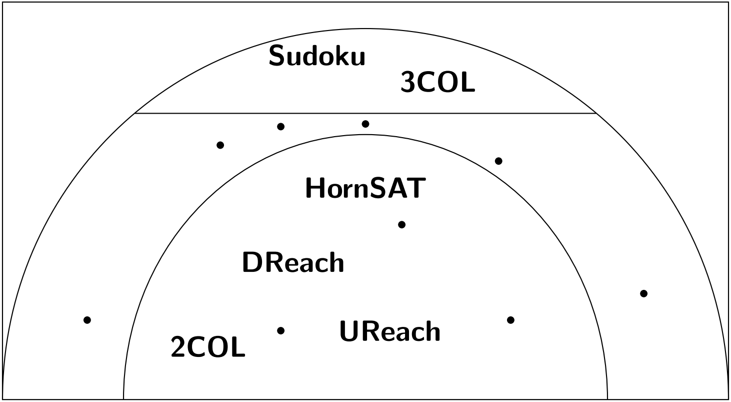

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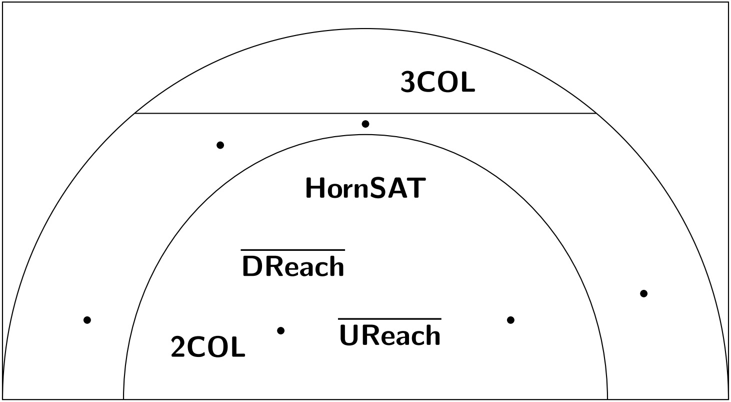

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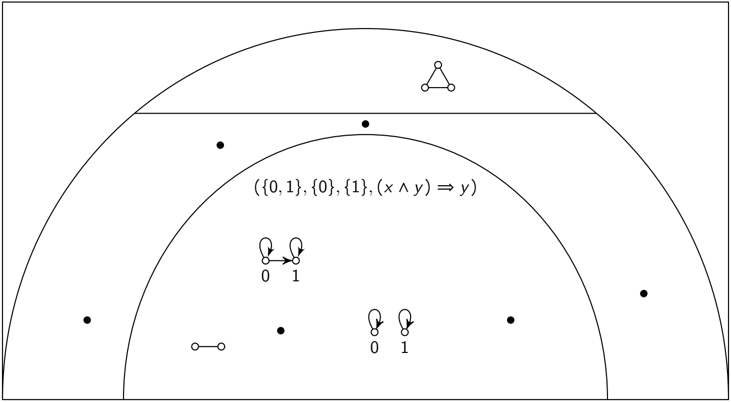

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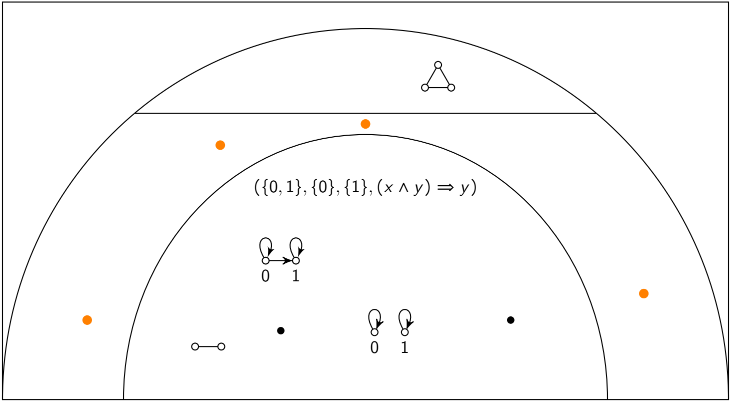
Conjecture (Feder, Vardi 1999): P-NP intermediate CSPs do not exist.



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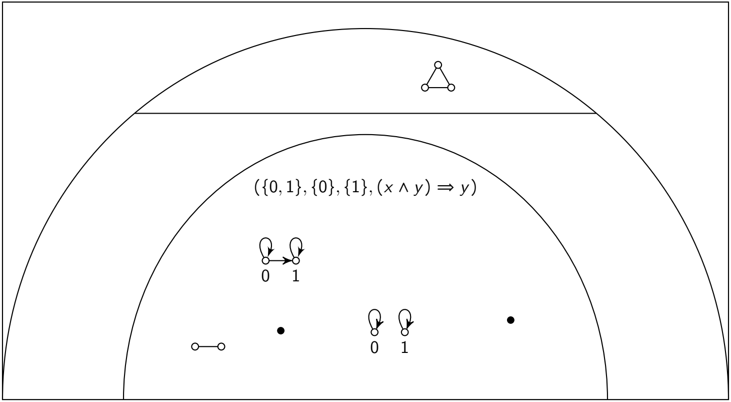
For all finite $\mathbb{A}$: $\mathrm{CSP}(\mathbb{A})$ is in NP.
Theorem (Bulatov, Zhuk 2017): P-NP intermediate CSPs do not exist.



Introduction – PP-Construction Poset

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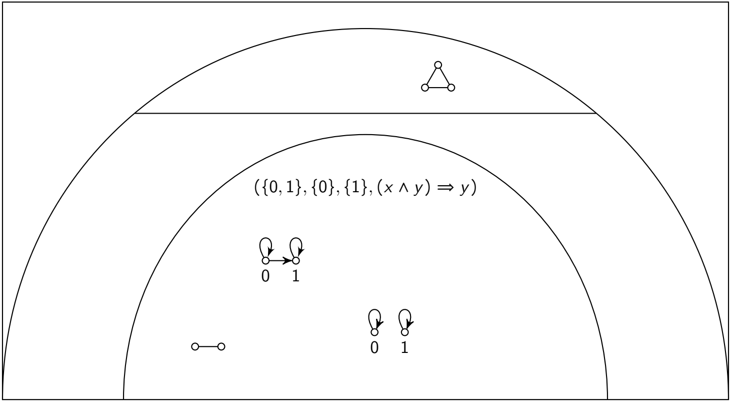
$$\mathbb{A} \leq_{\text{pp}} \mathbb{B} \qquad \Longrightarrow \qquad \mathrm{CSP}(\mathbb{B}) \leq_{\text{log-space}} \mathrm{CSP}(\mathbb{A})$$



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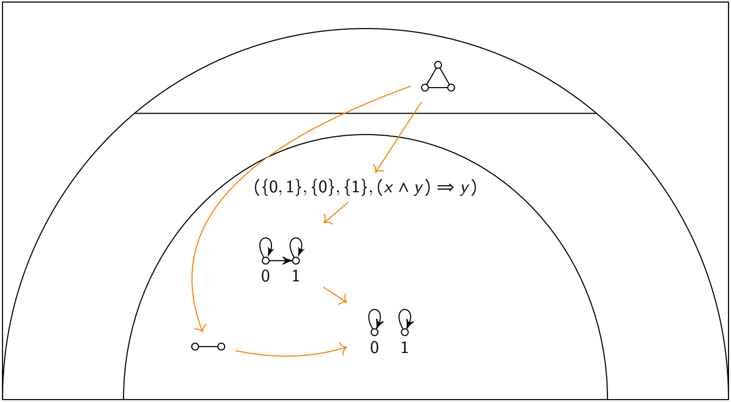

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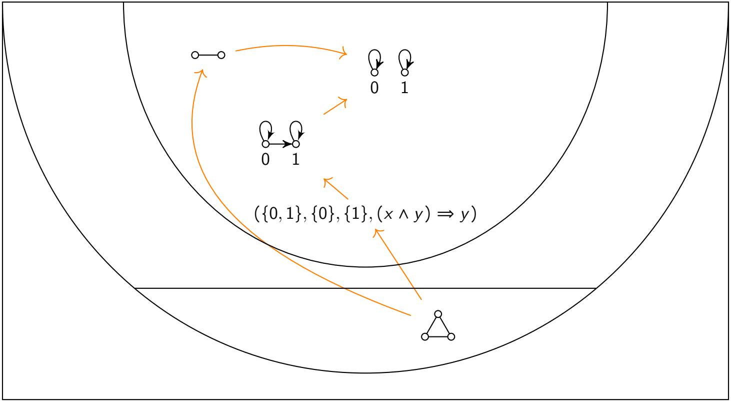
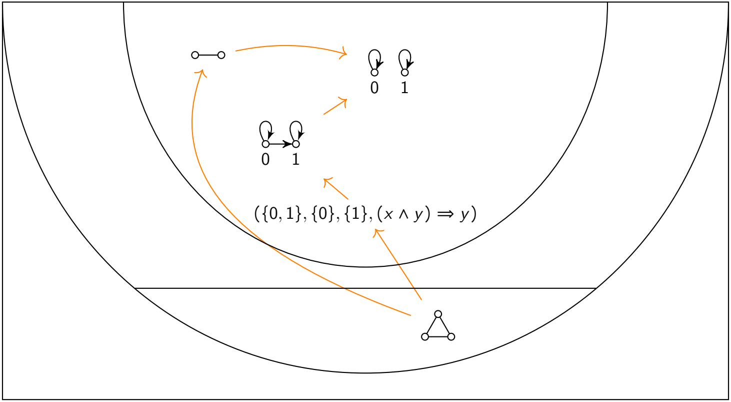


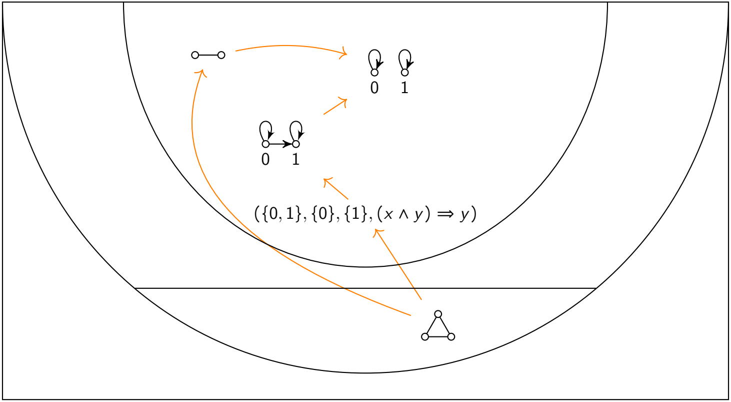
Introduction – PP-Construction Poset

Goal: Understand the complexity of CSPs within P.



Introduction – PP-Construction Poset

Goal: Understand the poset of finite structures ordered by $\leq_{\text{pp}}$.



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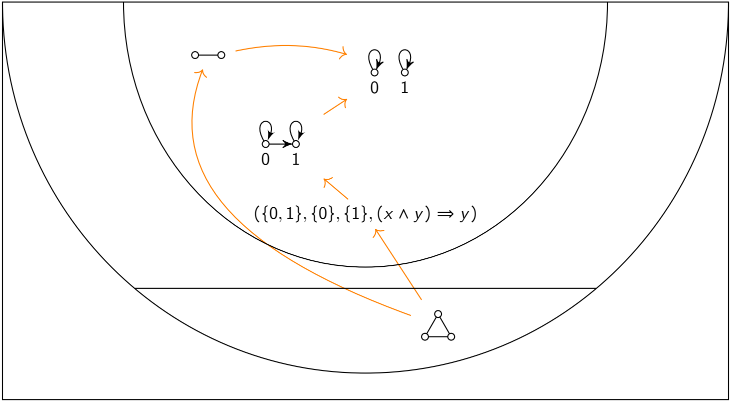
Theorem (Bulín, Delic, Jackson, Niven 2015): For all structures $\mathbb{A}$ there exists a digraph $\mathbb{G}$: $\text{CSP}(\mathbb{A}) \equiv_{\text{log-space}} \text{CSP}(\mathbb{G})$.



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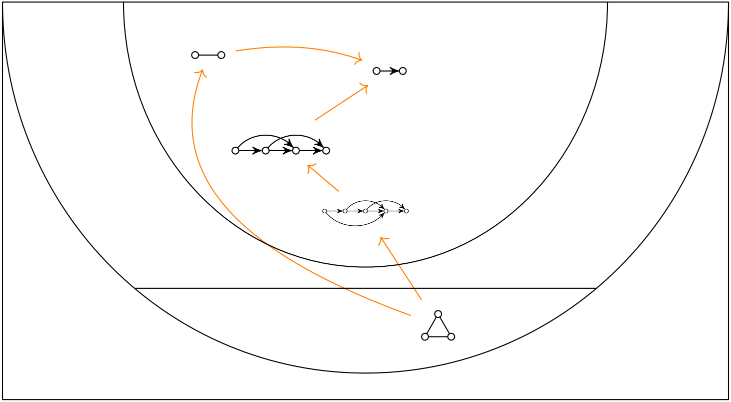

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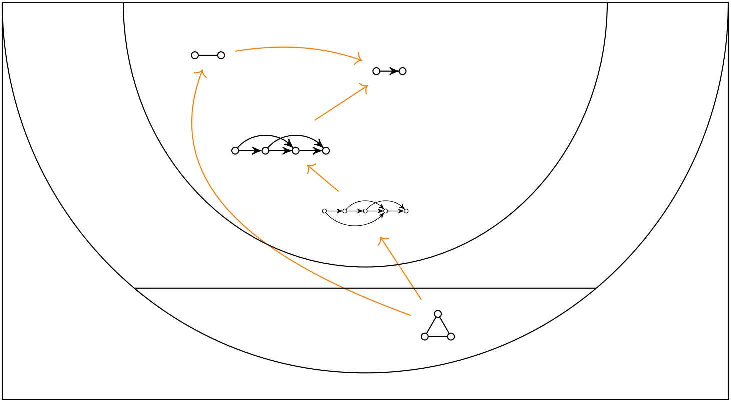

Overview – Thesis

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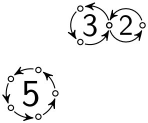
Smooth Digraphs



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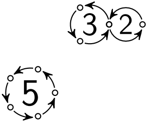
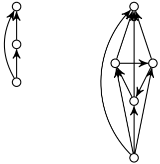


Tournaments



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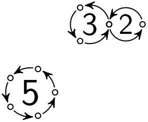

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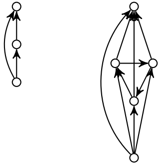
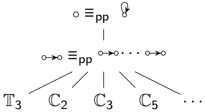


Digraphs at the top of $\mathfrak{P}_{\text{Digraphs}}$



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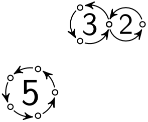

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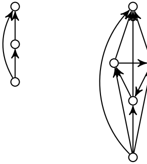

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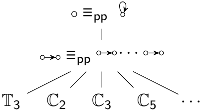
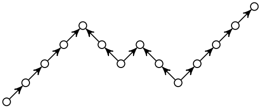


Orientations of Paths



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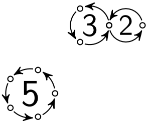

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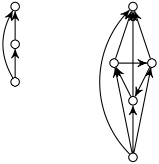

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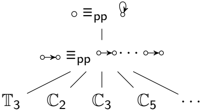

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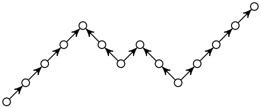
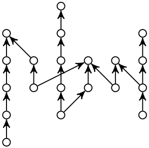


Orientations of Trees



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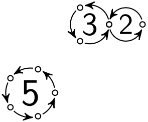

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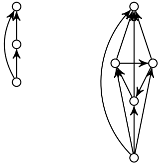

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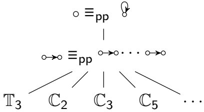

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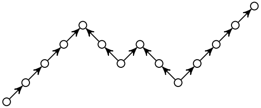

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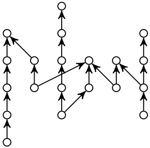
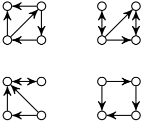


Digraphs with at most 4 vertices



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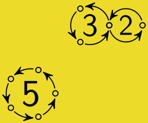

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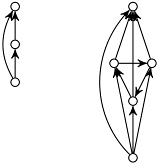

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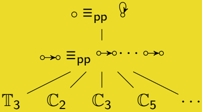

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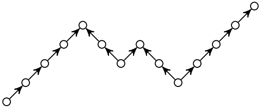

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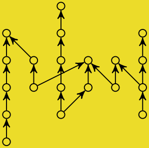
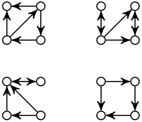


Digraphs with at most 4 vertices



PP-Constructions – Example 1

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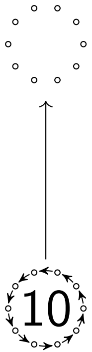

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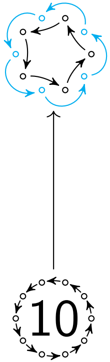

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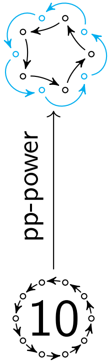

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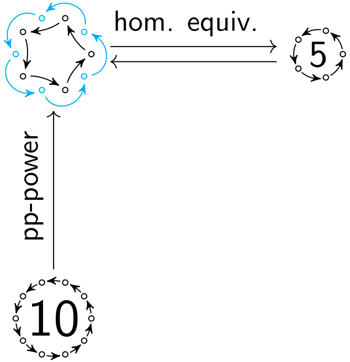

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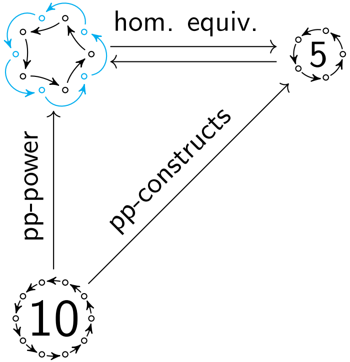

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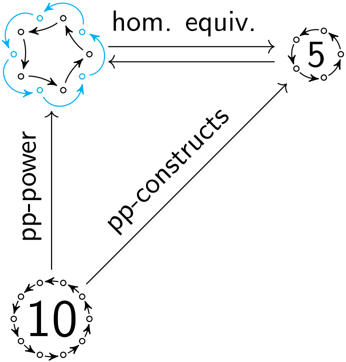

$$\mathbb{C}_{10} \leq_{\mathrm{pp}} \mathbb{C}_5$$

PP-Constructions – Example 2

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$$\Phi_E\begin{pmatrix} x_1, x_2, x_3, \\ y_1, y_2, y_3 \end{pmatrix} = x_1 \to y_3$$
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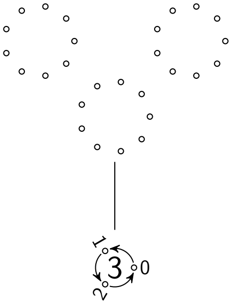

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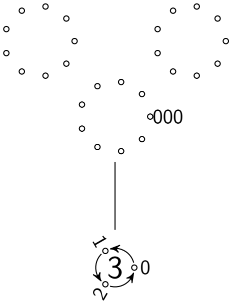

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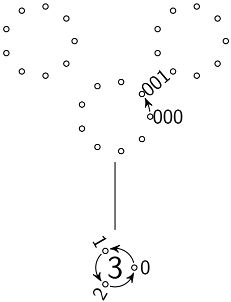

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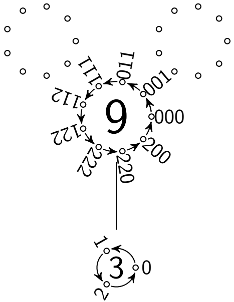

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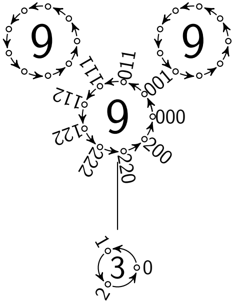

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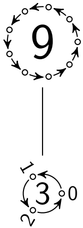

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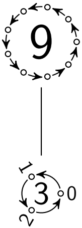

$$\mathbb{C}_3 \leq_{\mathrm{pp}} \mathbb{C}_9$$

PP-Constructions – Poset

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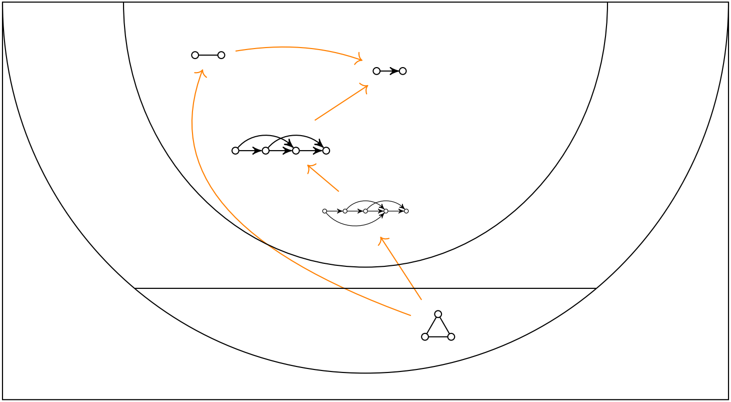

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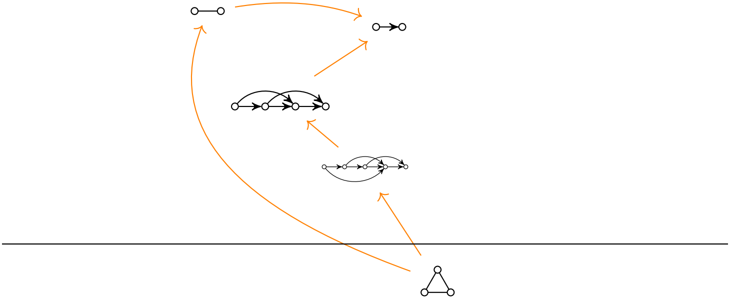

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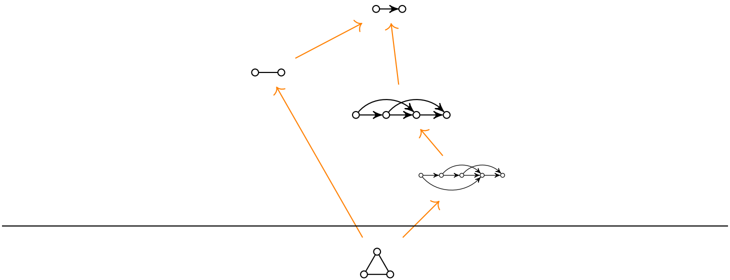

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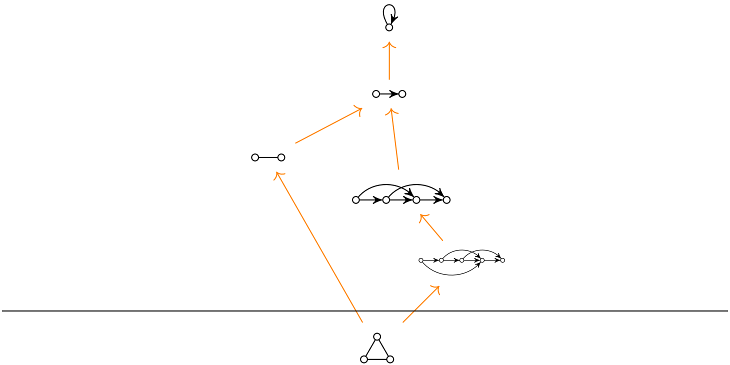

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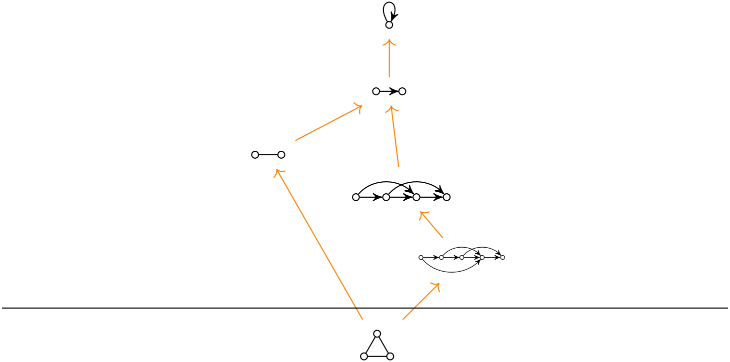
For all structures $\mathbb{A}$:  $\leq_{\mathsf{pp}}$ $\mathbb{A}$ $\leq_{\mathsf{pp}}$ 



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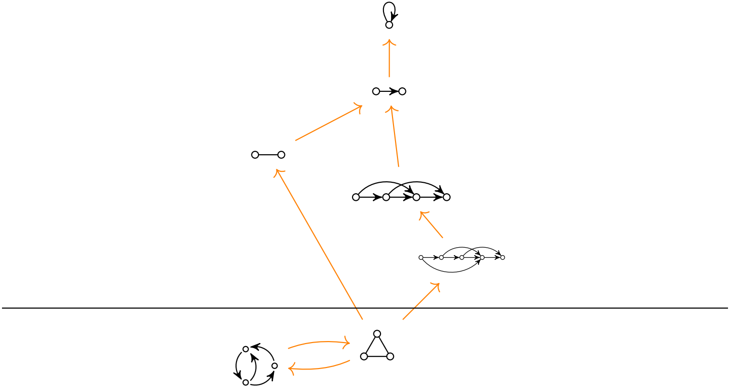

Smooth Digraphs

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Theorem (Barto, Kozik, Niven 2009): Let $\mathbb{G}$ be a smooth digraph. Then exactly one of the following is true:

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$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{C}_2 \arrow[dr, dash] & & \mathbb{C}_3 \arrow[dl, dash] \\
 & \mathbb{C}_6 &
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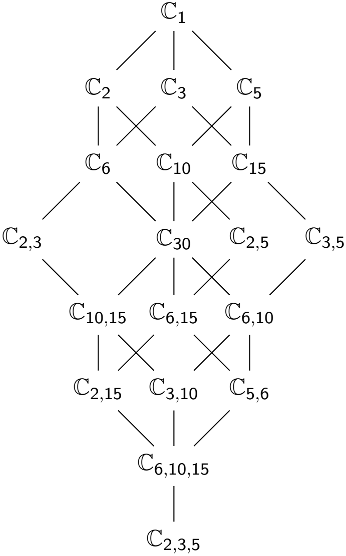

Exploring the Upper Levels of $\mathfrak{P}_{\mathsf{Digraphs}}$

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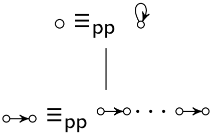

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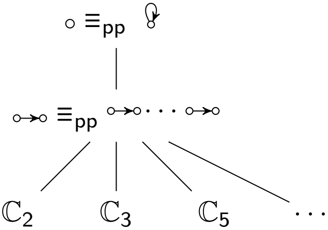

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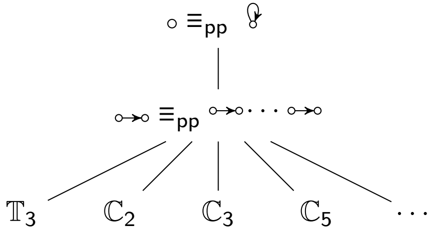

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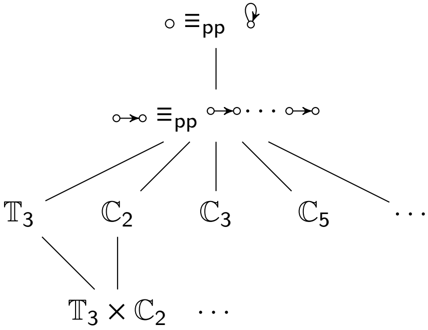

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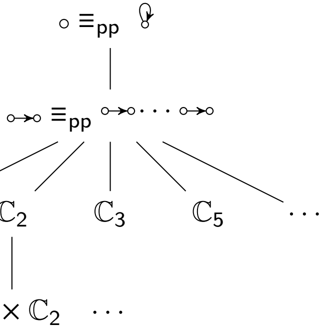

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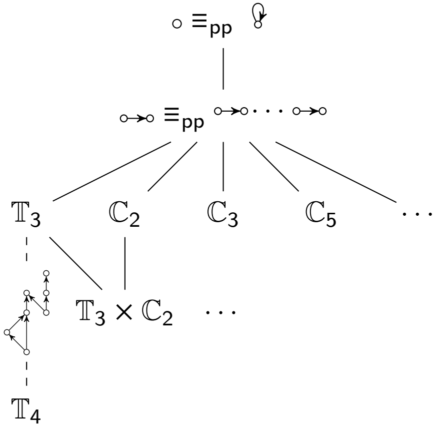

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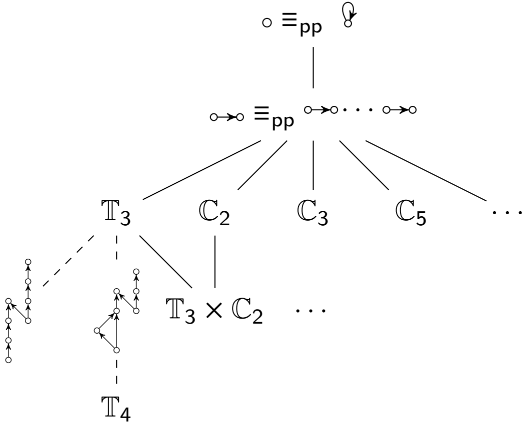

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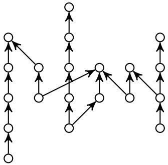

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Hell, Nešetřil, and Zhu	1996	45
Barto, Kozik, Maróti, and Niven	2009	39
Fischer	2015	30
Tatarko	2019	26

Smallest Hard Trees

n	trees	core trees	total time
10	24635	36	13 ms
11	108968	85	33 ms
12	492180	226	84 ms
13	2266502	578	236 ms
14	10598452	1569	657 ms
15	50235931	4243	2.0 s
16	240872654	11848	5.7 s
17	1166732814	33104	16.6 s
18	5702001435	94221	49.3 s
19	28088787314	269455	2.5 min
20	139354922608	779268	7.4 min

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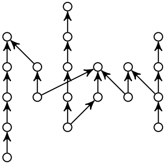

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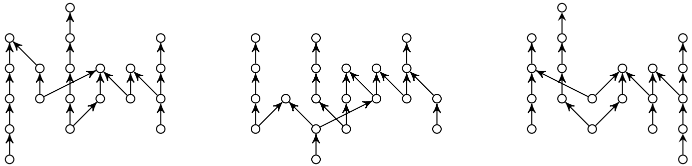

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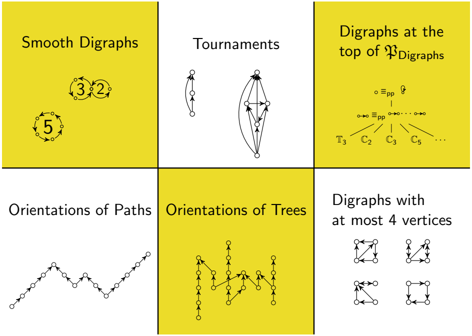

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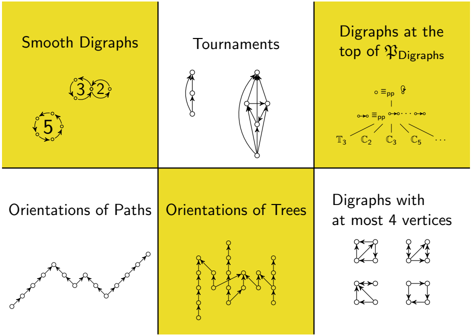

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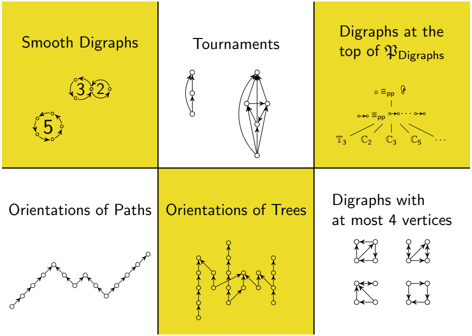

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Thank You!