TECHNISCHE UNIVERSITÄT DRESDEN

Weakest Non-trivial Finite Structures

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joint work with Manuel Bodirsky

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PP-Constructability Poset

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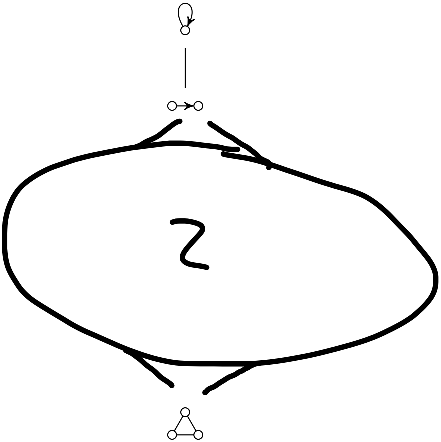
Theorem (Barto, Opršal, Pinsker 2015)

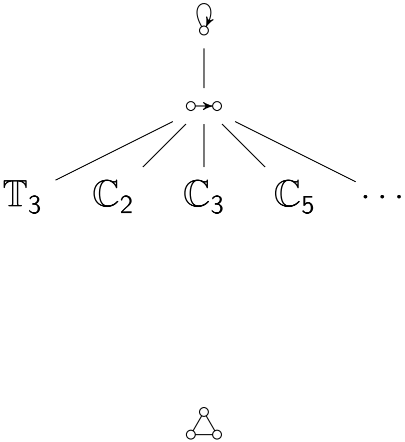
Let $\mathbb{A}$ and $\mathbb{B}$ be a finite structures. TFAE

1. $\mathbb{A}$ *can pp-construct* $\mathbb{B}$ ($\mathbb{A} \leq_{\mathrm{pp}} \mathbb{B}$)
2. $\mathbb{B}$ *is homomorphically equivalent to a pp-power of* $\mathbb{A}$
3. $\mathrm{Pol}(\mathbb{A})$ *has a minion homomorphism to* $\mathrm{Pol}(\mathbb{B})$

PP-Constructability Poset



PP-Constructability Poset (for Digraphs)



Weakest Non-trivial Finite Digraphs

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Malt: quasi Maltsev, $m(x, y, y) = m(x, x, x) = m(y, y, x)$
Σ_p: p-cyclic, $f(x_1, x_2, \ldots, x_p) = f(x_2, \ldots, x_p, x_1)$

Theorem

Let $\mathbb{G}$ be a finite digraph. TFAE

1. $\circ\!\to\!\circ \leq_{\mathrm{pp}} \mathbb{G}$
2. $\mathbb{G}$ *has* Malt *and* Σ_p *polymorphisms for all* p
3. $\mathbb{G}$ *is homomorphically equivalent to a directed path or a loop*

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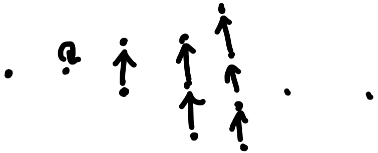


Weakest Finite Structures

Weakest Finite Structures

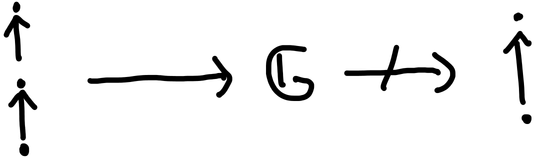
Theorem

Let $\mathbb{A}$ be a finite structure. TFAE

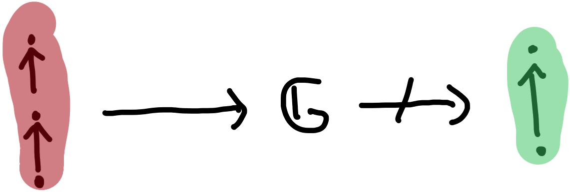
1.  $\leq_{\text{pp}} \mathbb{A}$
2. $\mathbb{A}$ *has a constant polymorphism*
3. $\mathbb{A}$ *is homomorphically equivalent to a one-element structure*

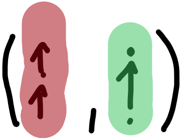
Duality Pairs (Nešetřil, Tardif 1998)

Duality Pairs



Duality Pairs



 is a duality pair

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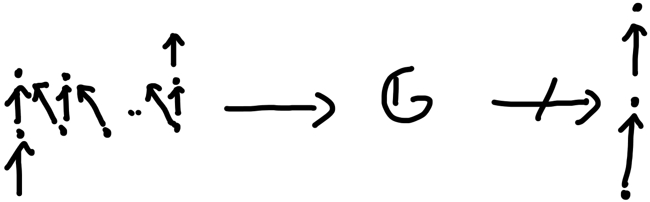

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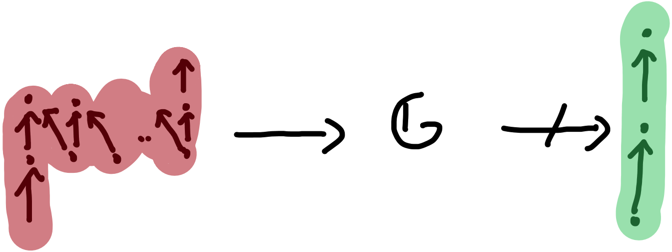
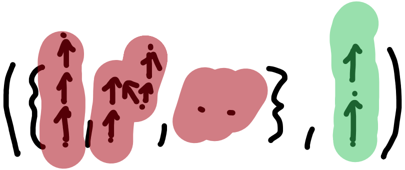
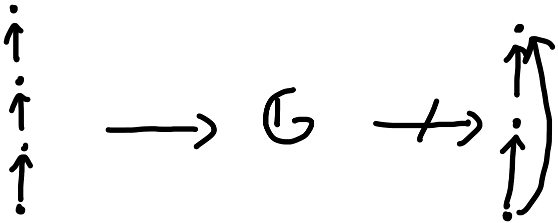


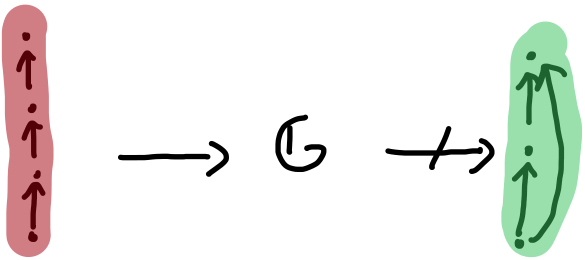


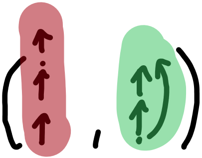
is a duality pair

Duality Pairs



Duality Pairs



 is a duality pair

Tree Duality

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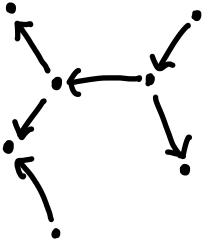

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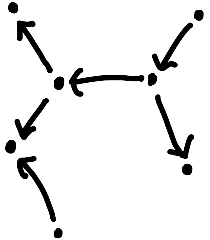

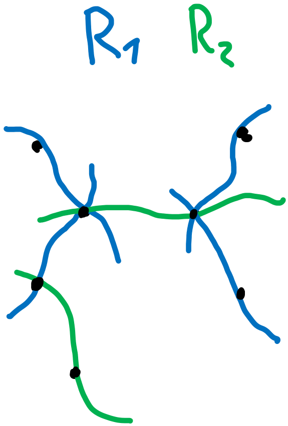

Tree Duality

TS(n): totally symmetric, $f(x_1, \ldots, x_n)$ only depends on $\{x_1, \ldots, x_n\}$.

Theorem (Feder, Vardi 93)

Let $\mathbb{A}$ be a finite structure. TFAE

1. HornSat $\leq_{pp} \mathbb{A}$
2. $\mathbb{A}$ *has* TS(n) *polymorphisms of all* n
3. $\mathbb{A}$ *has tree duality*

Caterpillar Duality

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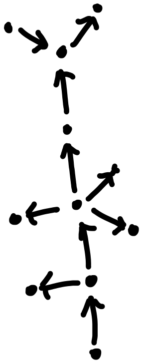
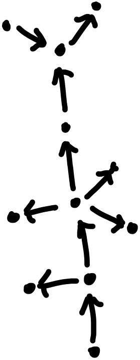
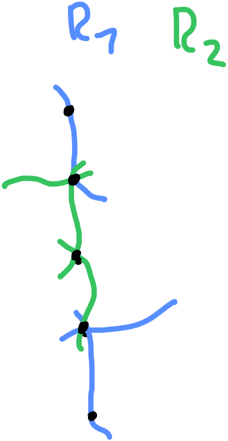


Caterpillar Duality





Caterpillar Duality

k-ABS: k-absorbing block symmetric
$f(x_{1,1}, \ldots, x_{1,k}, \ \ldots \ , x_{k,1}, \ldots, x_{k,k})$ only depends on the set of w.r.t. $\subseteq$ **minimal** sets in

$$\{\{x_{1,1}, \ldots, x_{1,k}\}, \ldots, \{x_{k,1}, \ldots, x_{k,k}\}\}$$

Theorem (Carvalho, Dalmau, Krokhin 2010)

Let $\mathbb{A}$ be a finite structure. TFAE

1. st-Con $\leq_{\mathrm{pp}} \mathbb{A}$
2. $\mathbb{A}$ *has k-ABS polymorphisms for all k*
3. $\mathbb{A}$ *has caterpillar duality*
4. $\mathbb{A}$ *is homomorphically equivalent to a structure $\mathbb{A}'$ with lattice polymorphisms $\wedge, \vee$*

Unfolded Caterpillar Duality

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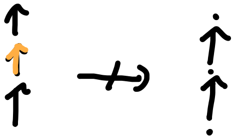

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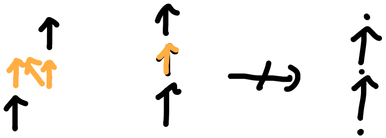

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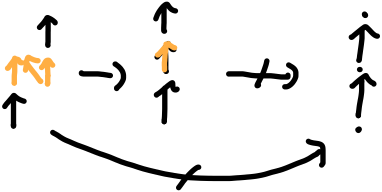

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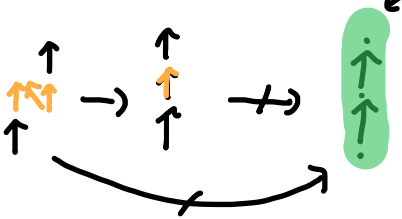
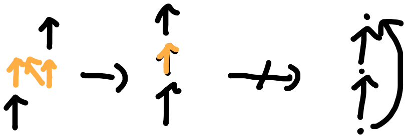


has unfolded caterpillar duality

Unfolded Caterpillar Duality



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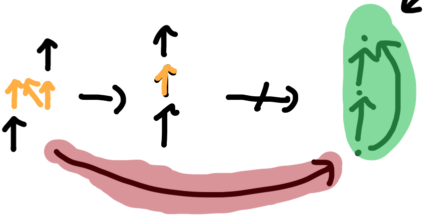

does not have unfolded CP duality

Unfolded Caterpillar Duality

Theorem (Bodirsky, Starke 2024)

Let $\mathbb{A}$ be a finite structure. TFAE

1. $\circ\!\!\rightarrow\!\!\circ \leq_{\mathrm{pp}} \mathbb{A}$
2. $\mathbb{A}$ *has* Malt *and* k-ABS *polymorphism for all* k
3. $\mathbb{A}$ *has unfolded caterpillar duality*
4. $\mathbb{A}$ *is homomorphically equivalent to a structure* $\mathbb{A}'$ *with lattice polymorphisms* $\wedge, \vee$ *and a* Malt *polymorphism*

Unfolded Caterpillar Duality

Theorem (Bodirsky, Starke 2024)

Let $\mathbb{A}$ be a finite structure. TFAE

1. $\circ\!\!\rightarrow\!\!\circ \leq_{\mathrm{pp}} \mathbb{A}$
2. $\mathbb{A}$ *has* Malt *and* k-ABS *polymorphism for all* k

2' $\mathbb{A}$ *has generalised minority polymorphisms of all odd arities and* $\mathrm{TS}(n)$ *polymorphisms for all* n *(Vucaj, Zhuk 2024)*

3. $\mathbb{A}$ *has unfolded caterpillar duality*
4. $\mathbb{A}$ *is homomorphically equivalent to a structure* $\mathbb{A}'$ *with lattice polymorphisms* $\wedge, \vee$ *and a* Malt *polymorphism*

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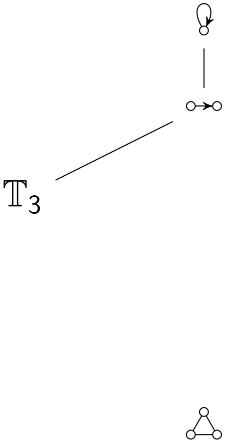

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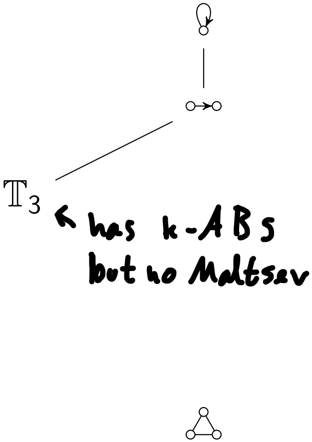

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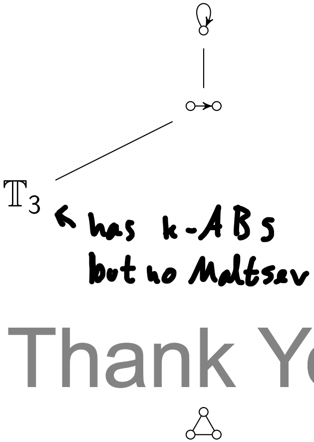

Thank You