TECHNISCHE UNIVERSITÄT DRESDEN

THE RETURN OF THE SMOOTH DIGRAPHS (MODULO PP-CONSTRUCTABILITY)

Florian Starke

Definitions

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

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not h1: $f(x, x) \approx x$, $f(f(x, y), z) \approx f(x, f(y, z))$

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h1-condition: finite set of h1-identities

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$f(x, x, y) \approx f(y, x, x) \approx f(y, y, y)$

$\tilde{f}\colon A \to A$ **satisfies** $f(x) \approx f(y)$ if $\tilde{f}(a) = \tilde{f}(b)$ for all $a, b \in A$.

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$\tilde{f}\colon A \to A$ **satisfies** $f(x) \approx f(y)$ if $\tilde{f}(a) = \tilde{f}(b)$ for all $a, b \in A$.

A set of functions **satisfies** a h1-condition Σ if it contains functions that satisfy Σ.

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

$$\begin{array}{c} \mathfrak{A} \\ | \\ \mathfrak{B} \end{array} \quad :\text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array}$$

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Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

Central definition of this Talk

$$\begin{array}{c} \mathfrak{A} \\ | \\ \mathfrak{B} \end{array} \quad :\text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array}$$

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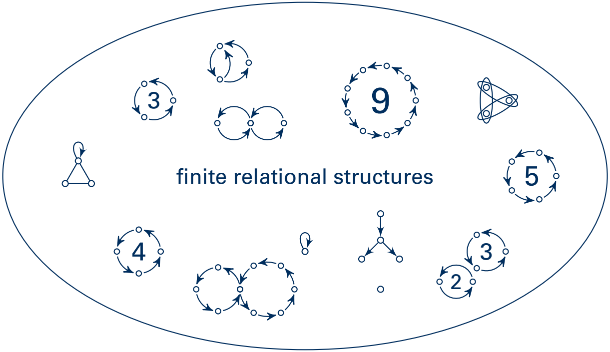
Poset-Introduction

Goal: understand finite structures ordered by pp-constructability



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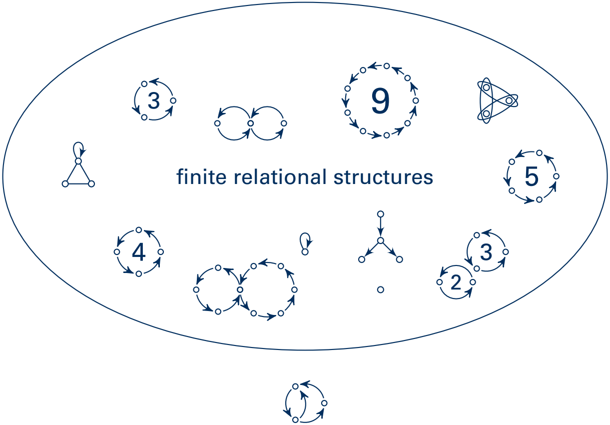

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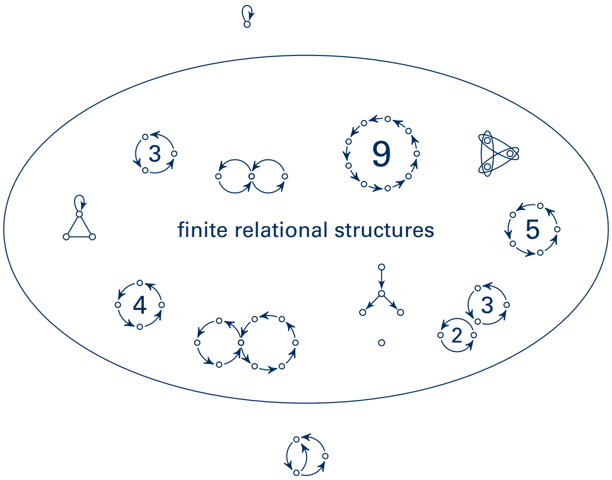

Poset-Introduction

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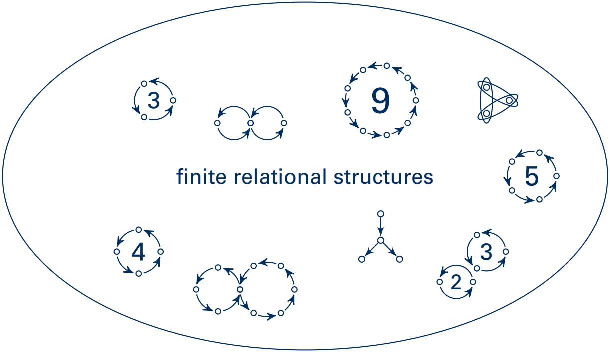
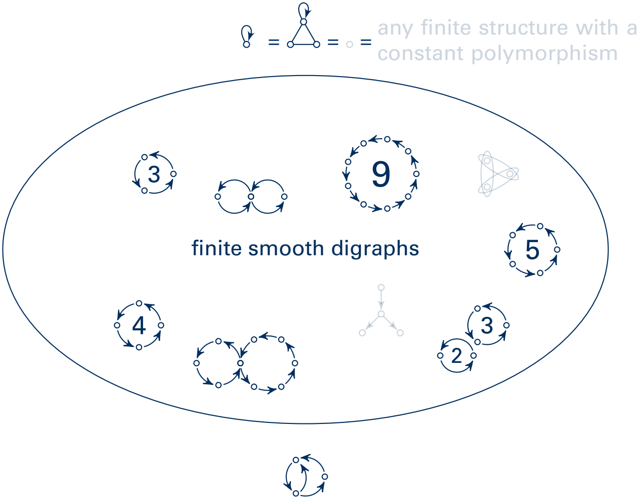
any finite structure with a constant polymorphism



Poset-Introduction

goal: understand finite smooth digraphs ordered by pp-constructability



Poset … duction

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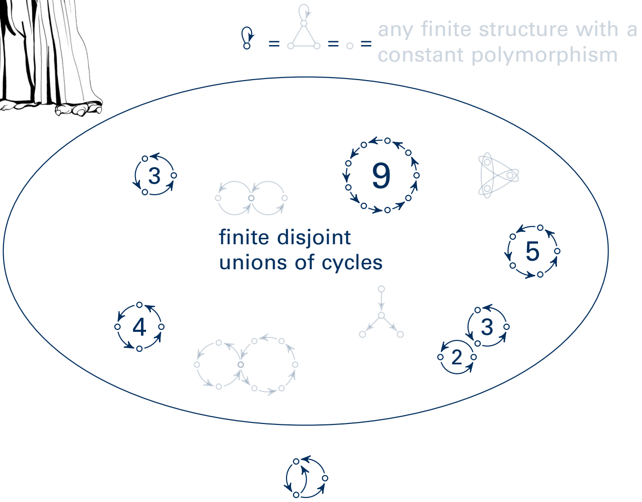

Cyclic loop conditions

Pol() $\not\models$ $f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0)$

Cyclic loop conditions

Pol() $\not\models \; f(0,1,2) \approx f(1,2,0)$

Cyclic loop conditions

$$\mathrm{Pol}(\vec{C}_3) \not\models f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0) \qquad \mathrm{Pol}(\vec{C}_2) \not\models f(x_0, x_1, x_2, x_3) \approx f(x_1, x_2, x_3, x_0)$$

Cyclic loop conditions

Pol(

) $\not\models f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0)$

Pol(

) $\not\models f(0,1,0,1) \approx f(1,0,1,0)$

Cyclic loop conditions

Pol () $\not\models \begin{array}{c} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array}$ Pol () $\not\models \begin{array}{c} f(x_0, x_1, x_2, x_3) \\ \approx \\ f(x_1, x_2, x_3, x_0) \end{array}$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \mathbin{\dot\cup} \mathfrak{C}_{20} \mathbin{\dot\cup} \mathfrak{C}_{15}) \not\models \begin{array}{c} f(x_0, x_1, y_0, y_1, y_2) \\ \approx \\ f(x_1, x_0, y_1, y_2, y_0) \end{array}$$

Cyclic loop conditions

Pol() $\not\models$ $f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0)$

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$$\mathrm{Pol}\,(\mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15}) \not\models \begin{array}{c} f(0', 10', 0, 4, 2) \\ \approx \; \downarrow_{10} \; \downarrow_{10} \; \downarrow_{10} \; \downarrow_{10} \; \downarrow_{10} \\ f(10', 0', 4, 2, 0) \end{array}$$

Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0)$

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Cyclic loop conditions

Pol () $\not\models \Sigma_3$

Pol () $\not\models$ $f(x_0, x_1, x_2, x_3) \approx f(x_1, x_2, x_3, x_0)$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models f(x_0, x_1, y_0, y_1, y_2) \approx f(x_1, x_0, y_1, y_2, y_0)$$

Cyclic loop conditions

Pol () $\not\models \Sigma_3$

Pol () $\not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

Cyclic loop conditions

Pol $\Bigg($ 

$\Bigg) \not\models \Sigma_3$

Pol $\Bigg($ 

$\Bigg) \not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

$\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{b_1,\dots,b_n}$ **iff** **for all** $\mathfrak{C}_{a_1},\dots,\mathfrak{C}_{a_n} \hookrightarrow \mathfrak{A}$ **exists** $\mathfrak{C}_a \hookrightarrow \mathfrak{A}$ **such that** a **divides** $\mathrm{lcm}(a_1 \dot{-} b_1, \dots, a_n \dot{-} b_n)$

Order

$$\mathrm{Pol}(\mathfrak{A}) \models \Sigma$$
$$\Uparrow$$
$$\mathrm{Pol}(\mathfrak{B}) \models \Sigma$$

$$\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}$$
$$|$$
$$\mathfrak{B}$$

Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array}$$

$$\begin{array}{c} \mathfrak{C}_6 \mathbin{\dot\cup} \mathfrak{C}_{20} \mathbin{\dot\cup} \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array}$$

Order

$$\begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma\end{array} \quad \text{iff} \quad \begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c}\end{array}$$

$$\begin{array}{c}\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15} \\ | \\ \mathfrak{B}\end{array} \quad \text{iff} \quad \begin{array}{c}\mathrm{Pol}(\mathfrak{B}) \not\models \Sigma_{6,20,15}, \\ \Sigma_{2,20,5}, \Sigma_{3,10,15}, \Sigma_{6,4,3}, \\ \Sigma_{3,5,15}, \Sigma_{3,2,3}\end{array}$$

Order

$$\begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma\end{array} \quad \text{iff} \quad \begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c}\end{array}$$

$\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}$ | $\mathfrak{B}$ iff $\mathrm{Pol}(\mathfrak{B}) \not\models$

$\Sigma_{3,5,15}$ $\Sigma_{3,2,3}$

$\Sigma_{2,20,5}$ $\Sigma_{3,10,15}$ $\Sigma_{6,4,3}$

$\Sigma_{6,20,15}$

Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array}$$

$$\begin{array}{c} \mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array} \quad \text{iff} \quad \mathrm{Pol}(\mathfrak{B}) \not\models$$

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$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{P} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{P} \end{array}$$

$$\begin{array}{c} \mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array} \quad \text{iff}$$

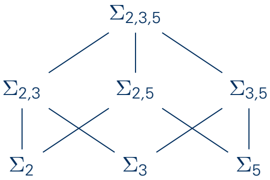
$\mathrm{Pol}(\mathfrak{B}) \not\models$ $\Sigma_{3,5}$ $\Sigma_{3,2}$ $\Sigma_{2,5}$ $\Sigma_{3,10,15}$ $\Sigma_{6,4,3}$ $\Sigma_{6,20,15}$

iff

$\mathrm{Pol}(\mathfrak{B}) \not\models$ $\Sigma_{2,5}$ $\Sigma_{3,5}$ $\Sigma_{2,3}$ Σ_5 Σ_2 Σ_3

Understanding Smooth Digraphs

$\mathfrak{C}_1$



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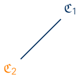

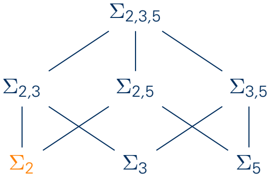

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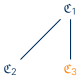

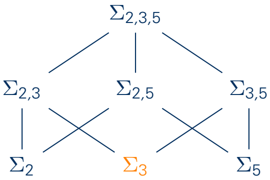

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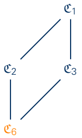

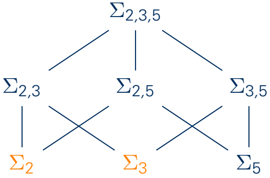

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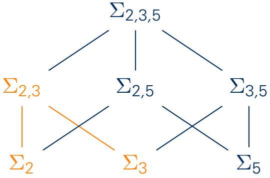

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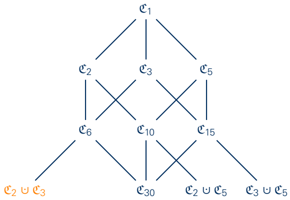

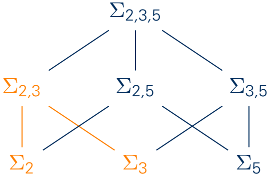

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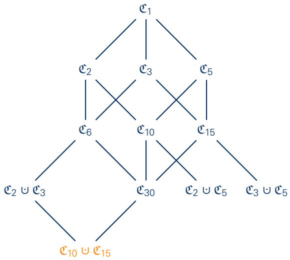

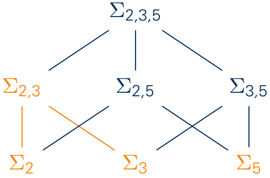

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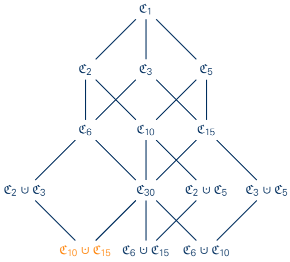

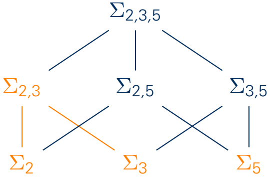

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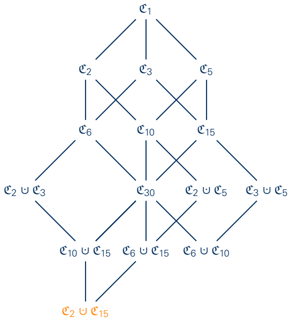

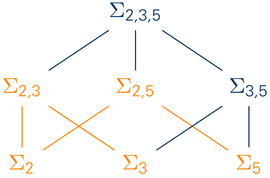

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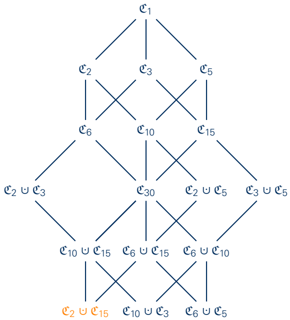

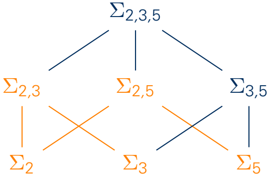

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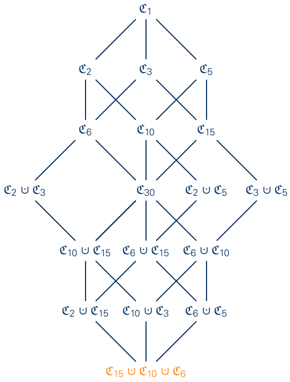

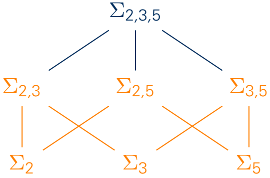

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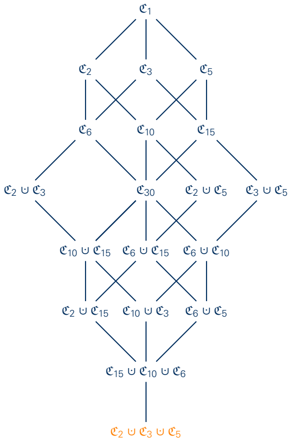

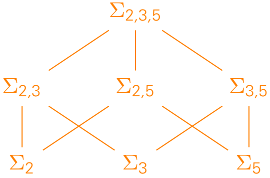

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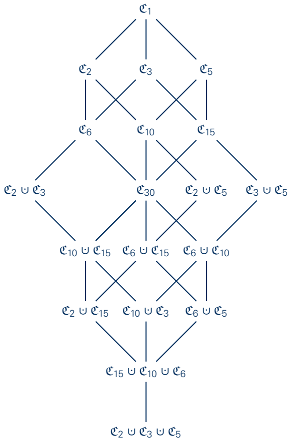

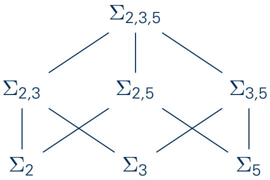

Theorem

The following map is an isomorphism of posets where the order on the image set is reverse inclusion:

$$\textit{Smooth Digraphs} \to {\downarrow}_{fin}\{\textit{prime cyclic loop conditions}\} \cup \{\textit{all pclc}\}$$
$$[\mathfrak{A}] \mapsto \{\Sigma_P \mid \mathrm{Pol}(\mathfrak{A}) \not\models \Sigma_P\}$$

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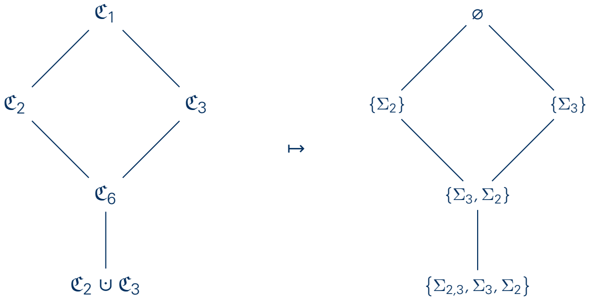
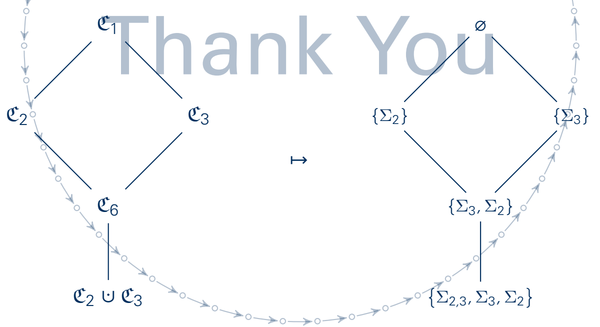


Theorem

The following map is an isomorphism of posets where the order on the image set is reverse inclusion:

$$\text{Smooth Digraphs} \to \downarrow_{fin}\{\text{prime cyclic loop conditions}\} \cup \{\text{all pclc}\}$$
$$[\mathfrak{A}] \mapsto \{\Sigma_P \mid \mathrm{Pol}(\mathfrak{A}) \not\models \Sigma_P\}$$



Paper: https://arxiv.org/abs/1906.05699