TECHNISCHE UNIVERSITÄT DRESDEN

THE RETURN OF THE SMOOTH DIGRAPHS (MODULO PP-CONSTRUCTABILITY)

Florian Starke

Definitions

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

Definitions

Examples

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

h1: $f(x) \approx f(y), f(x, y, z) \approx g(x)$
not h1: $f(x, x) \approx x, f(f(x, y), z) \approx f(x, f(y, z))$

Definitions

Examples

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

h1: $f(x) \approx f(y), f(x, y, z) \approx g(x)$
not h1: $f(x, x) \approx x, f(f(x, y), z) \approx f(x, f(y, z))$

h1-condition: finite set of h1-identities

Definitions

Examples

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

h1: $f(x) \approx f(y)$, $f(x, y, z) \approx g(x)$
not h1: $f(x, x) \approx x$, $f(f(x, y), z) \approx f(x, f(y, z))$

h1-condition: finite set of h1-identities

$f(x, x, y) \approx f(y, x, x) \approx f(y, y, y)$

Definitions

Examples

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

h1: $f(x) \approx f(y)$, $f(x, y, z) \approx g(x)$
not h1: $f(x, x) \approx x$, $f(f(x, y), z) \approx f(x, f(y, z))$

h1-condition: finite set of h1-identities

$f(x, x, y) \approx f(y, x, x) \approx f(y, y, y)$

$\tilde{f}\colon A \to A$ **satisfies** $f(x) \approx f(y)$ if $\tilde{f}(a) = \tilde{f}(b)$ for all $a, b \in A$.

Definitions

Examples

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

h1: $f(x) \approx f(y)$, $f(x,y,z) \approx g(x)$
not h1: $f(x,x) \approx x$, $f(f(x,y),z) \approx f(x, f(y,z))$

h1-condition: finite set of h1-identities

$f(x,x,y) \approx f(y,x,x) \approx f(y,y,y)$

$\tilde{f} \colon A \to A$ **satisfies** $f(x) \approx f(y)$ if $\tilde{f}(a) = \tilde{f}(b)$ for all $a, b \in A$.

A set of functions **satisfies** a h1-condition Σ if it contains functions that satisfy Σ.

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

$$
\begin{array}{ccc}
\mathfrak{A} & & \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\
\Big| & \text{:iff} & \Uparrow \\
\mathfrak{B} & & \mathrm{Pol}(\mathfrak{B}) \models \Sigma
\end{array}
$$

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

$$\begin{array}{c}\mathfrak{A}\\ \Big| \\ \mathfrak{B}\end{array} \qquad :\text{iff} \qquad \begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma\end{array}$$

$$\text{iff} \qquad \begin{array}{c}\mathrm{Pol}(\mathfrak{A})\\ {\scriptstyle \text{minor}}\Big\uparrow \\ \mathrm{Pol}(\mathfrak{B})\end{array}$$

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

$$\begin{array}{c} \mathfrak{A} \\ \Big| \\ \mathfrak{B} \end{array} \quad :\text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array}$$

$$\text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \\ \text{minor}\uparrow \\ \mathrm{Pol}(\mathfrak{B}) \end{array} \quad \text{iff} \quad \mathfrak{B} \text{ pp-constructs } \mathfrak{A}$$

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

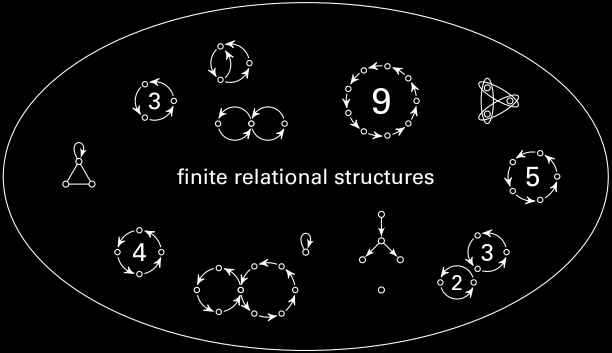
Central definition of this Talk

$$\boxed{\begin{array}{ccc} \mathfrak{A} & & \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Big| & :\text{iff} & \Uparrow \\ \mathfrak{B} & & \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array}}$$

$$\text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \\ \text{minor}\big\uparrow \\ \mathrm{Pol}(\mathfrak{B}) \end{array} \quad \text{iff} \quad \mathfrak{B} \text{ pp-constructs } \mathfrak{A}$$

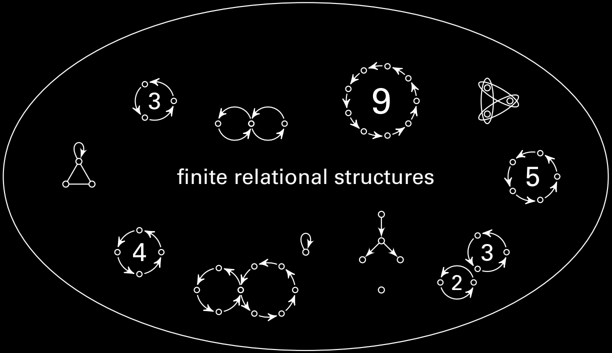
Poset-Introduction

Goal: understand finite structures ordered by pp-constructability



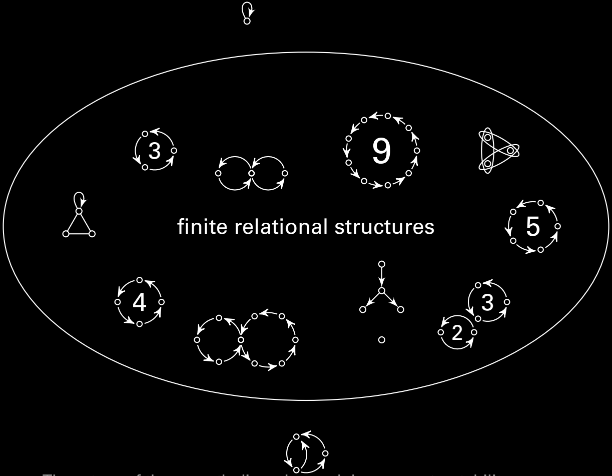
Poset-Introduction

Goal: understand finite structures ordered by pp-constructability



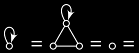
Poset-Introduction

Goal: understand finite structures ordered by pp-constructability

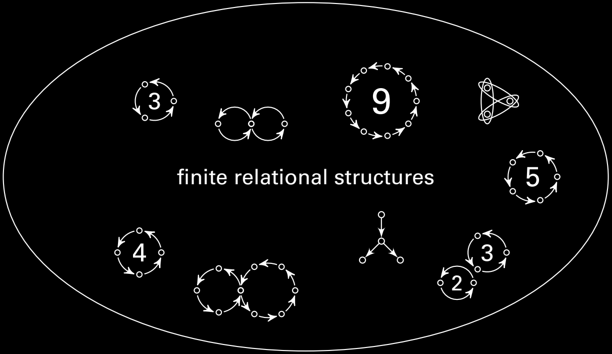


Poset-Introduction

Goal: understand finite structures ordered by pp-constructability

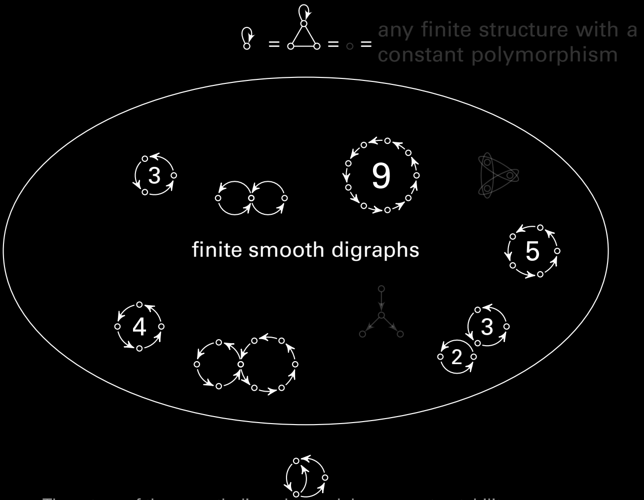


any finite structure with a constant polymorphism



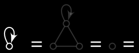
Poset-Introduction

goal: understand finite smooth digraphs ordered by pp-constructability

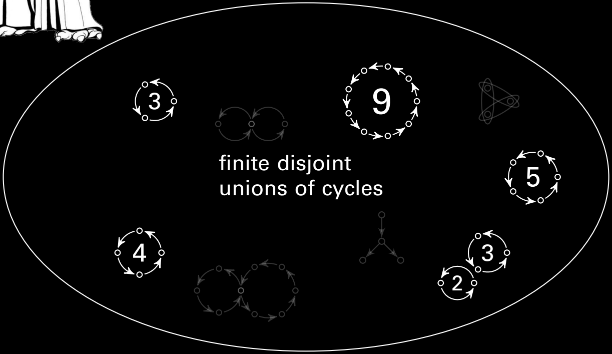


Pos troduction

goal: te smooth digraphs ordered by pp-constructability



= = = any finite structure with a constant polymorphism



Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models$ $\begin{array}{c} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array}$

Cyclic loop conditions

Pol() $\not\models \begin{matrix} & f(\,0, & 1, & 2\,) \\ \approx & \downarrow & \downarrow & \downarrow \\ & f(\,1, & 2, & 0\,) \end{matrix}$

Cyclic loop conditions

$$\mathrm{Pol}\left(3\right) \not\models \begin{array}{l} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array} \qquad \mathrm{Pol}\left(2\right) \not\models \begin{array}{l} f(x_0, x_1, x_2, x_3) \\ \approx \\ f(x_1, x_2, x_3, x_0) \end{array}$$

Cyclic loop conditions

Pol 

$$\not\models \begin{array}{l} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array}$$

Pol 

$$\not\models \begin{array}{lcccc} f(& 0, & 1, & 0, & 1\,) \\ \approx & \downarrow & \downarrow & \downarrow & \downarrow \\ f(& 1, & 0, & 1, & 0\,) \end{array}$$

Cyclic loop conditions

$$\mathrm{Pol}\left(3 \right) \not\models \begin{array}{c} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array} \qquad \mathrm{Pol}\left(2 \right) \not\models \begin{array}{c} f(x_0, x_1, x_2, x_3) \\ \approx \\ f(x_1, x_2, x_3, x_0) \end{array}$$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \begin{array}{c} f(x_0, x_1, y_0, y_1, y_2) \\ \approx \\ f(x_1, x_0, y_1, y_2, y_0) \end{array}$$

Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models \begin{array}{l} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{array}$ $\qquad$ $\mathrm{Pol}\Big($  $\Big) \not\models \begin{array}{l} f(x_0, x_1, x_2, x_3) \\ \approx \\ f(x_1, x_2, x_3, x_0) \end{array}$

$$\mathrm{Pol}(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \begin{array}{lccccc} f(& 0', & 10', & 0, & 4, & 2\,) \\ \approx & {\scriptstyle 10}\downarrow & {\scriptstyle 10}\downarrow & {\scriptstyle 10}\downarrow & {\scriptstyle 10}\downarrow & {\scriptstyle 10}\downarrow \\ f(& 10', & 0', & 4, & 2, & 0\,) \end{array}$$

Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models f(x_0, x_1, x_2) \approx f(x_1, x_2, x_0)$

$\mathrm{Pol}\Big($  $\Big) \not\models f(x_0, x_1, x_2, x_3) \approx f(x_1, x_2, x_3, x_0)$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models f(x_0, x_1, y_0, y_1, y_2) \approx f(x_1, x_0, y_1, y_2, y_0)$$

Cyclic loop conditions

$$\mathrm{Pol}\left(3 \right) \not\models \Sigma_3 \qquad\qquad \mathrm{Pol}\left(2 \right) \not\models \begin{array}{l} f(x_0, x_1, x_2, x_3) \\ \approx \\ f(x_1, x_2, x_3, x_0) \end{array}$$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \mathbin{\dot\cup} \mathfrak{C}_{20} \mathbin{\dot\cup} \mathfrak{C}_{15}) \not\models \begin{array}{l} f(x_0, x_1, y_0, y_1, y_2) \\ \approx \\ f(x_1, x_0, y_1, y_2, y_0) \end{array}$$

Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_3$ $\qquad$ $\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

Cyclic loop conditions

$\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_3$

$\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

$\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{b_1,\dots,b_n}$ iff for all $\mathfrak{C}_{a_1},\dots,\mathfrak{C}_{a_n} \hookrightarrow \mathfrak{A}$ exists $\mathfrak{C}_a \hookrightarrow \mathfrak{A}$ such that a divides $\mathrm{lcm}(a_1 \dot{-} b_1,\dots,a_n \dot{-} b_n)$

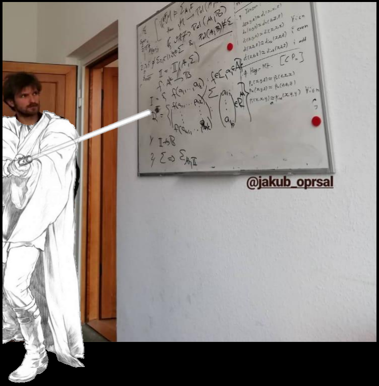
Order

$$\begin{gathered}\mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma\end{gathered}$$

$$\begin{gathered}\mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15} \\ | \\ \mathfrak{B}\end{gathered}$$

Order

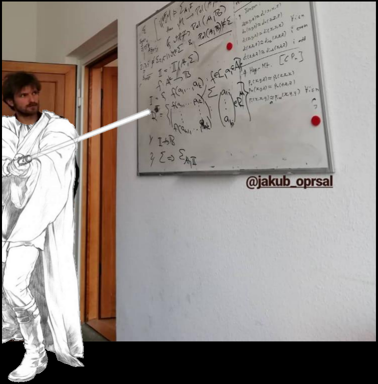
$$\mathrm{Pol}(\mathfrak{A}) \models \Sigma$$
$$\Uparrow$$
$$\mathrm{Pol}(\mathfrak{B}) \models \Sigma$$



$$\mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15}$$
$$|$$
$$\mathfrak{B}$$

Order

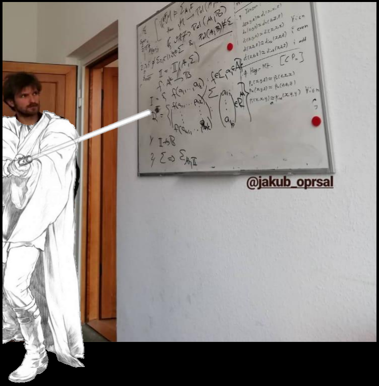
$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array}$$



$$\begin{array}{c} \mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{B}) \not\models \Sigma_{6,20,15}, \\ \Sigma_{2,20,5}, \Sigma_{3,10,15}, \Sigma_{6,4,3}, \\ \Sigma_{3,5,15}, \Sigma_{3,2,3} \end{array}$$

Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array}$$



$$\begin{tikzcd} \mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15} \arrow[d, no head] \\ \mathfrak{B} \end{tikzcd} \quad \text{iff} \quad \mathrm{Pol}(\mathfrak{B}) \not\models \begin{tikzcd}[column sep=small] & \Sigma_{3,5,15} \arrow[d, no head] & \Sigma_{3,2,3} \arrow[dl, no head] \arrow[d, no head] \\ \Sigma_{2,20,5} \arrow[dr, no head] & \Sigma_{3,10,15} \arrow[d, no head] & \Sigma_{6,4,3} \arrow[dl, no head] \\ & \Sigma_{6,20,15} & \end{tikzcd}$$

Order

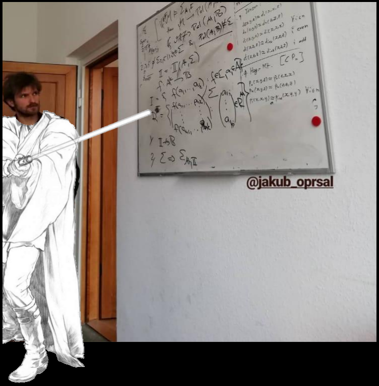
$$\begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma\end{array} \quad \text{iff} \quad \begin{array}{c}\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c}\end{array}$$

$$\begin{array}{c}\mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15} \\ | \\ \mathfrak{B}\end{array} \quad \text{iff} \quad \mathrm{Pol}(\mathfrak{B}) \not\models$$

$$\begin{array}{ccc} & \Sigma_{3,5,15} & \Sigma_{3,2,3} \\ \Sigma_{2,20,5} & \Sigma_{3,10,15} & \Sigma_{6,4,3} \\ & \Sigma_{6,20,15} & \end{array}$$

Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \qquad \text{iff} \qquad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array}$$



$$\begin{array}{c} \mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array} \qquad \text{iff} \qquad \mathrm{Pol}(\mathfrak{B}) \not\models \begin{array}{ccc} & \Sigma_{3,5} & \Sigma_{3,2} \\ \Sigma_{2,5} & \Sigma_{3,10,15} & \Sigma_{6,4,3} \\ & \Sigma_{6,20,15} & \end{array}$$

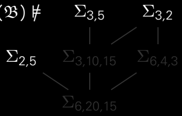
Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_{A \dot{-} c} \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_{A \dot{-} c} \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_P \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_P \end{array}$$

$$\begin{array}{c} \mathfrak{C}_6 \cup \mathfrak{C}_{20} \cup \mathfrak{C}_{15} \\ | \\ \mathfrak{B} \end{array}$$

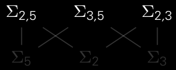
iff

$\mathrm{Pol}(\mathfrak{B}) \not\models$



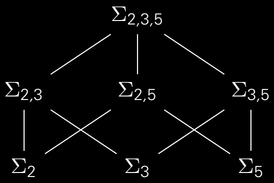
iff

$\mathrm{Pol}(\mathfrak{B}) \not\models$

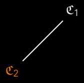


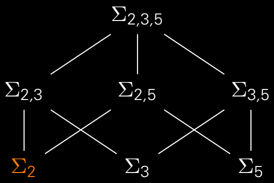
Understanding Smooth Digraphs

$\mathfrak{C}_1$

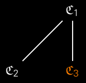


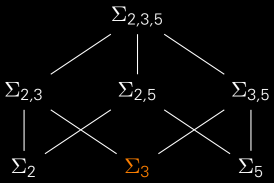
Understanding Smooth Digraphs



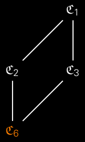


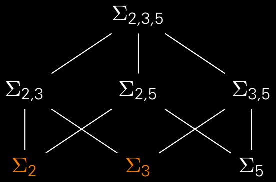
Understanding Smooth Digraphs



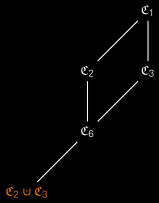


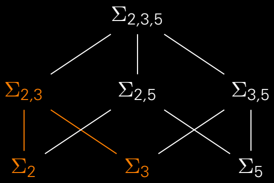
Understanding Smooth Digraphs



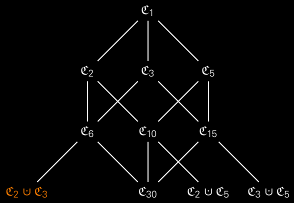


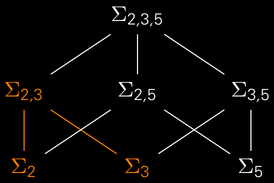
Understanding Smooth Digraphs



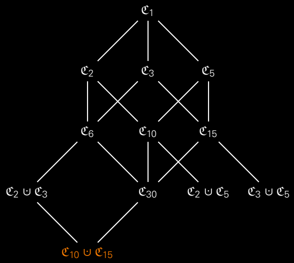


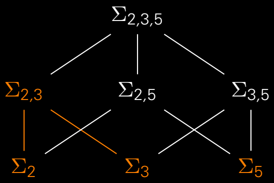
Understanding Smooth Digraphs



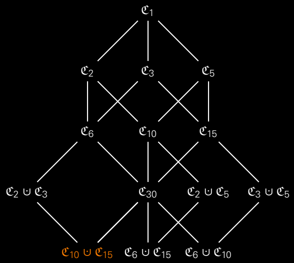


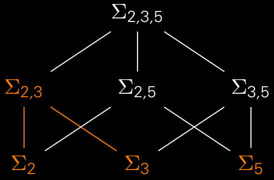
Understanding Smooth Digraphs



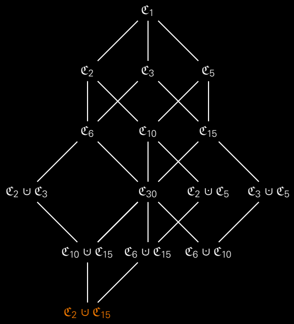


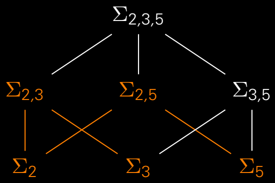
Understanding Smooth Digraphs



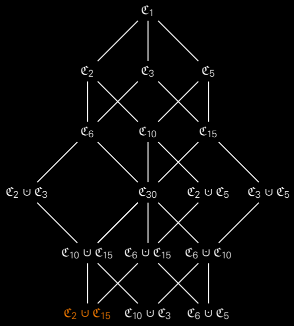


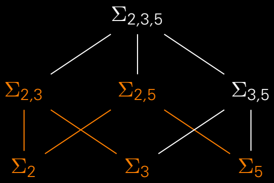
Understanding Smooth Digraphs



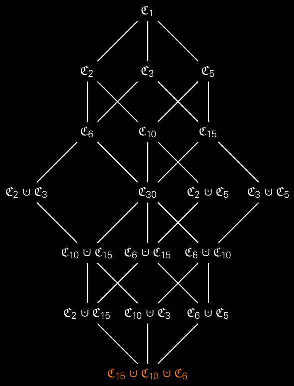


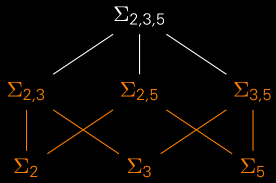
Understanding Smooth Digraphs



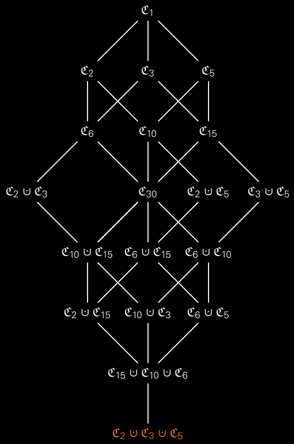


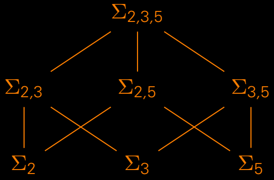
Understanding Smooth Digraphs



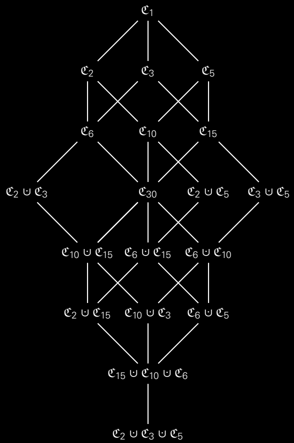


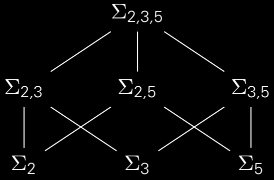
Understanding Smooth Digraphs





Understanding Smooth Digraphs





Theorem

The following map is an isomorphism of posets where the order on the image set is reverse inclusion:

$$\begin{aligned} \textit{Smooth Digraphs} &\to {\downarrow_{fin}}\{\textit{prime cyclic loop conditions}\} \cup \{\textit{all pclc}\} \\ [\mathfrak{A}] &\mapsto \{\Sigma_P \mid \mathrm{Pol}(\mathfrak{A}) \not\models \Sigma_P\} \end{aligned}$$

Theorem

The following map is an isomorphism of posets where the order on the image set is reverse inclusion:

$$\textit{Smooth Digraphs} \to {\downarrow_{fin}}\{\textit{prime cyclic loop conditions}\} \,\dot\cup\, \{\textit{all pclc}\}$$
$$[\mathfrak{A}] \mapsto \{\Sigma_P \mid \mathrm{Pol}(\mathfrak{A}) \not\models \Sigma_P\}$$

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{C}_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{C}_2 \arrow[dr, dash] & & \mathfrak{C}_3 \arrow[dl, dash] \\
 & \mathfrak{C}_6 \arrow[d, dash] & \\
 & \mathfrak{C}_2 \,\dot\cup\, \mathfrak{C}_3 &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=small]
 & \varnothing \arrow[dl, dash] \arrow[dr, dash] & \\
\{\Sigma_2\} \arrow[dr, dash] & & \{\Sigma_3\} \arrow[dl, dash] \\
 & \{\Sigma_3, \Sigma_2\} \arrow[d, dash] & \\
 & \{\Sigma_{2,3}, \Sigma_3, \Sigma_2\} &
\end{tikzcd}$$

Theorem

The following map is an isomorphism of posets where the order on the image set is reverse inclusion:

$$\textit{Smooth Digraphs} \to \downarrow_{fin}\{\textit{prime cyclic loop conditions}\} \cup \{\textit{all pclc}\}$$
$$[\mathfrak{A}] \mapsto \{\Sigma_P \mid \mathrm{Pol}(\mathfrak{A}) \not\models \Sigma_P\}$$

Thank You

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{C}_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{C}_2 \arrow[dr, dash] & & \mathfrak{C}_3 \arrow[dl, dash] \\
 & \mathfrak{C}_6 \arrow[d, dash] & \\
 & \mathfrak{C}_2 \cup \mathfrak{C}_3 &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=small]
 & \emptyset \arrow[dl, dash] \arrow[dr, dash] & \\
\{\Sigma_2\} \arrow[dr, dash] & & \{\Sigma_3\} \arrow[dl, dash] \\
 & \{\Sigma_3, \Sigma_2\} \arrow[d, dash] & \\
 & \{\Sigma_{2,3}, \Sigma_3, \Sigma_2\} &
\end{tikzcd}$$

Paper:
https://arxiv.org/abs/1906.05699