TECHNISCHE UNIVERSITÄT DRESDEN

THE RETURN OF THE SMOOTH DIGRAPHS (MODULO PP-CONSTRUCTABILITY)

Florian Starke

pp-constructability

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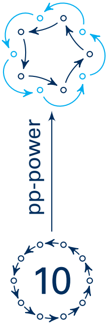
$\Phi_E(x, y) = x \xrightarrow{2} y$



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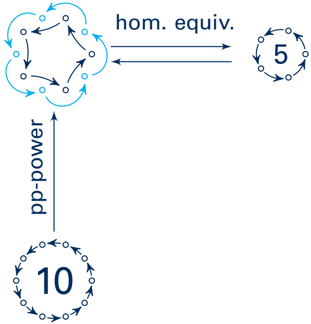

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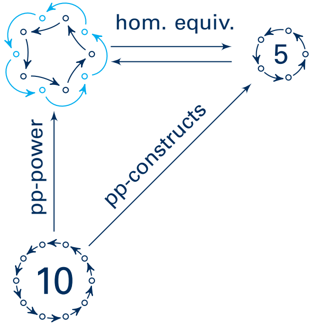

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.

$\mathfrak{A}$

|

$\mathfrak{B}$

:iff $\mathfrak{B}$ pp-constructs $\mathfrak{A}$

Define the Order

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite relational structures.
The pp-constructability order is a quasi order.
If $\mathfrak{B} \leq \mathfrak{A}$, then $\mathrm{CSP}(\mathfrak{A}) \leq_{\text{log-space}} \mathrm{CSP}(\mathfrak{B})$.

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$\mathrm{Pol}(\mathfrak{A})$
↑ minor
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iff

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Central definition of this Talk

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Definitions

h1-identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$
(x_i, y_j not necessarily distinct)

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h1: $f(x) \approx f(y)$, $f(x, y, z) \approx g(x)$
not h1: $f(x, x) \approx x$, $f(f(x, y), z) \approx f(x, f(y, z))$

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$\tilde{f}\colon A \to A$ **satisfies** $f(x) \approx f(y)$ if $\tilde{f}(a) = \tilde{f}(b)$ for all $a, b \in A$.

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A set of functions **satisfies** a h1-condition Σ if it contains functions that satisfy Σ.

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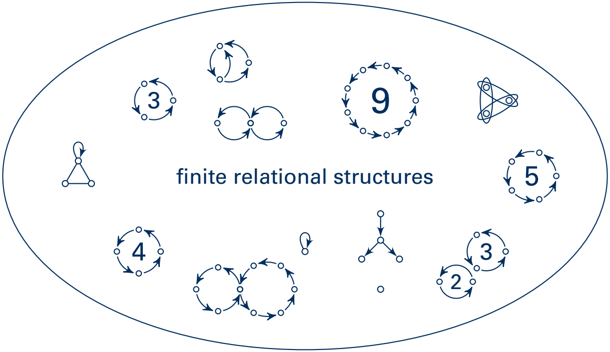
"When you go higher, then more h1-conditions are satisfied and the CSPs get easier."

Poset-Introduction

Goal: understand finite structures ordered by pp-constructability



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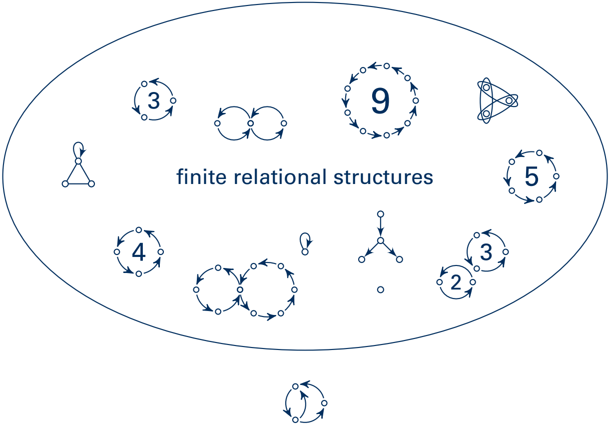

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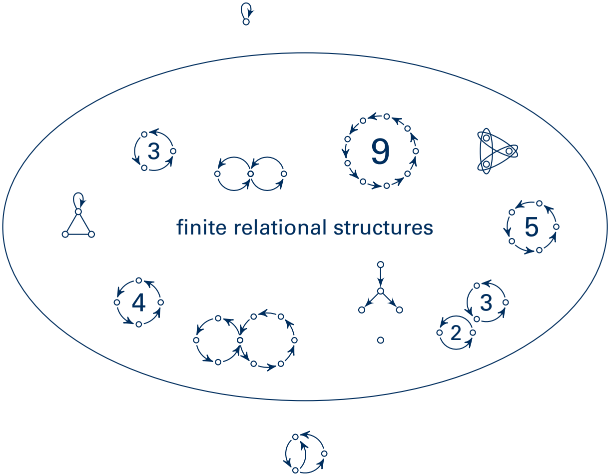

Poset-Introduction

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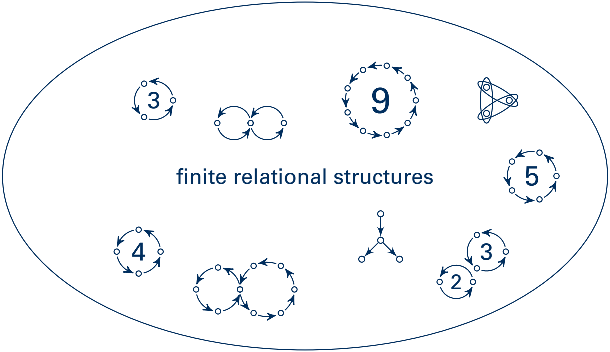
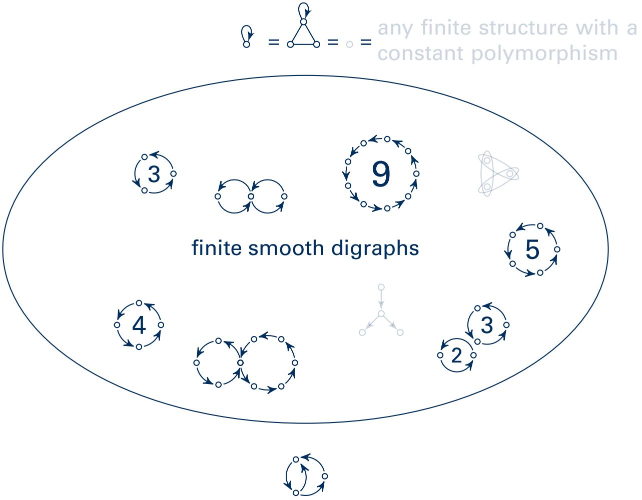
any finite structure with a constant polymorphism



Poset-Introduction

goal: understand finite smooth digraphs ordered by pp-constructability



Poset ... duction

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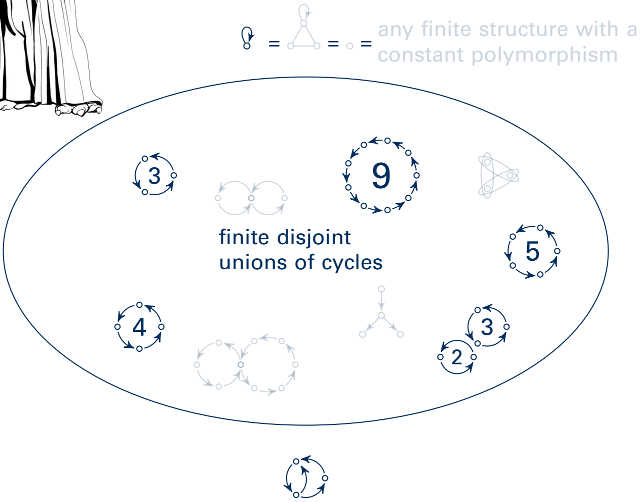

Cyclic loop conditions

Pol() $\not\models$ $\begin{matrix} f(x_0, x_1, x_2) \\ \approx \\ f(x_1, x_2, x_0) \end{matrix}$

Cyclic loop conditions

$$\mathrm{Pol}\left(\vec{C}_3\right) \not\models \begin{array}{c} f(\,0\,,\,1\,,\,2\,) \\ \approx \quad \downarrow \quad \downarrow \quad \downarrow \\ f(\,1\,,\,2\,,\,0\,) \end{array}$$

Cyclic loop conditions

Pol (

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Pol  $\not\models \begin{array}{lcccc} f(& 0, & 1, & 0, & 1\,) \\ \approx & \downarrow & \downarrow & \downarrow & \downarrow \\ f(& 1, & 0, & 1, & 0\,) \end{array}$

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$$\mathrm{Pol}\,(\mathfrak{C}_6 \mathbin{\dot\cup} \mathfrak{C}_{20} \mathbin{\dot\cup} \mathfrak{C}_{15}) \not\models f(x_0, x_1, y_0, y_1, y_2) \approx f(x_1, x_0, y_1, y_2, y_0)$$

Cyclic loop conditions

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$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \begin{array}{l} f(\;0',\; 10',\; 0,\; 4,\; 2\;) \\ \approx \;\; {\scriptstyle 10}{\downarrow} \;\; {\scriptstyle 10}{\downarrow} \;\; {\scriptstyle 10}{\downarrow} \;\; {\scriptstyle 10}{\downarrow} \;\; {\scriptstyle 10}{\downarrow} \\ f(\;10',\; 0',\; 4,\; 2,\; 0\;) \end{array}$$

Cyclic loop conditions

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$\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_3$

$\mathrm{Pol}\Big($  $\Big) \not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \,\dot\cup\, \mathfrak{C}_{20} \,\dot\cup\, \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

Cyclic loop conditions

Pol () $\not\models \Sigma_3$

Pol () $\not\models \Sigma_4$

$$\mathrm{Pol}\,(\mathfrak{C}_6 \uplus \mathfrak{C}_{20} \uplus \mathfrak{C}_{15}) \not\models \Sigma_{2,3}$$

$\mathrm{Pol}(\mathfrak{A}) \models \Sigma_{b_1,\dots,b_n}$ **iff** **for all** $\mathfrak{C}_{a_1}, \dots, \mathfrak{C}_{a_n} \hookrightarrow \mathfrak{A}$ **exists** $\mathfrak{C}_a \hookrightarrow \mathfrak{A}$ **such that** a **divides** $\mathrm{lcm}(a_1 \dot{-} b_1, \dots, a_n \dot{-} b_n)$

Order

$$\mathrm{Pol}(\mathfrak{A}) \models \Sigma$$
$$\Uparrow$$
$$\mathrm{Pol}(\mathfrak{B}) \models \Sigma$$

$$\mathfrak{C}_6$$
$$|$$
$$\mathfrak{B}$$

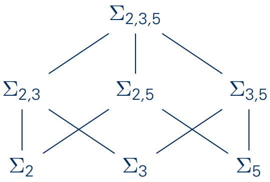
Order

$$\begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathfrak{A}) \models \Sigma_P \\ \Uparrow \\ \mathrm{Pol}(\mathfrak{B}) \models \Sigma_P \end{array}$$

$$\begin{array}{c} \mathfrak{C}_6 \\ | \\ \mathfrak{B} \end{array} \quad \text{iff} \quad \mathrm{Pol}(\mathfrak{B}) \not\models \Sigma_2, \Sigma_3$$

Understanding Smooth Digraphs

$\mathfrak{C}_1$



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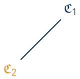

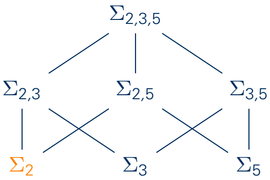

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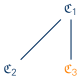

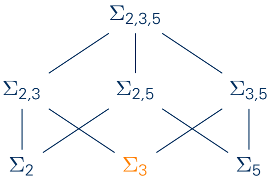

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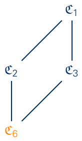

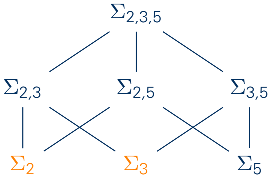

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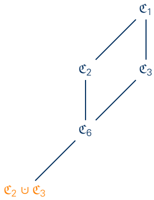

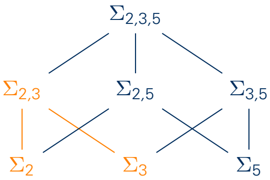

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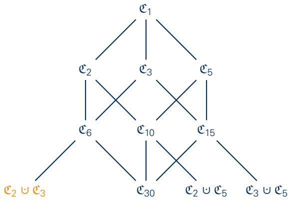

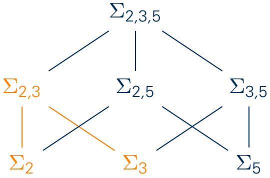

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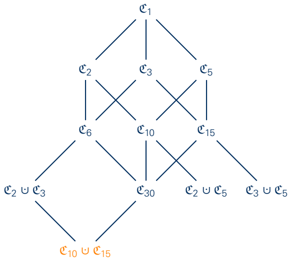

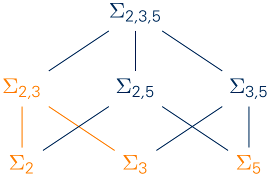

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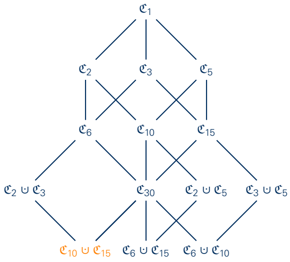

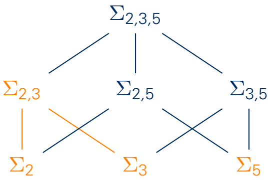

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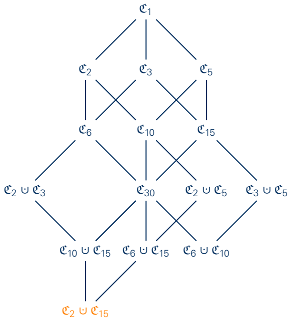

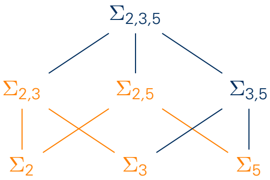

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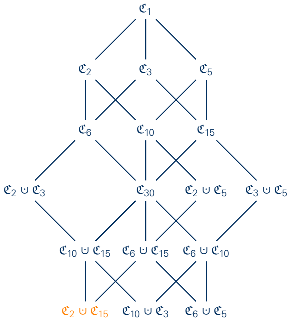

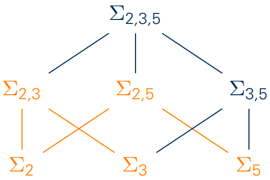

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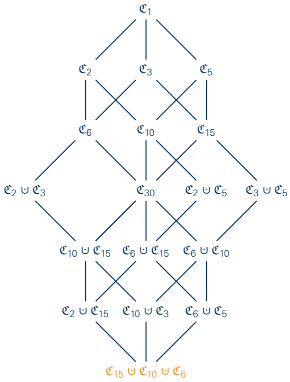

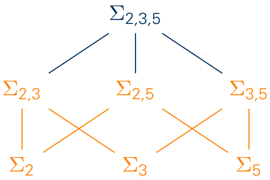

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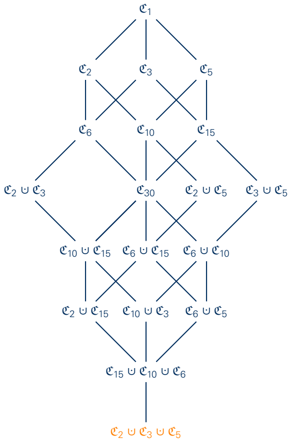

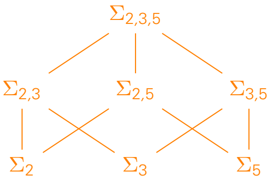

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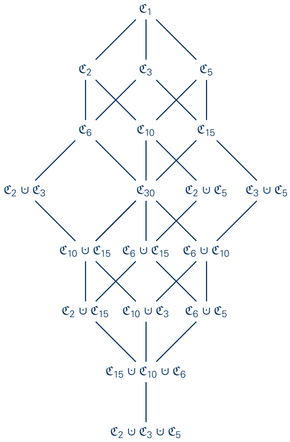

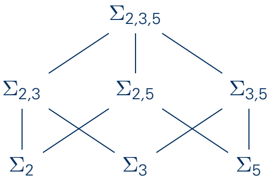

Understanding Smooth Digraphs