F- and H-Triangles for ν-Associahedra

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Root Systems

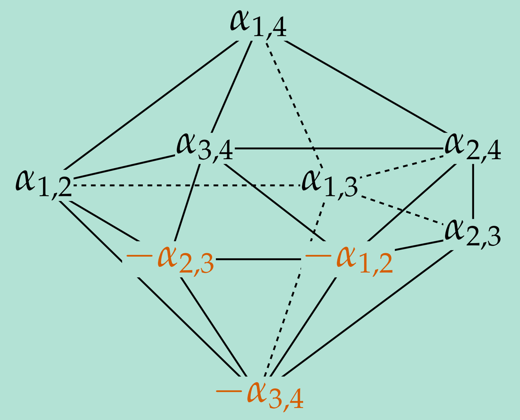
- ε_i .. i th unit vector in $\mathbb{R}^{d+1}$
- **positive root**: $\alpha_{i,j} \overset{\text{def}}{=} \varepsilon_i - \varepsilon_j$ for $i < j$
- **simple root**: $\alpha_{i,i+1}$

$$\Pi(d) \overset{\text{def}}{=} \{\alpha_{i,i+1} \mid 1 \leq i \leq d\}$$
$$\Phi_+(d) \overset{\text{def}}{=} \{\alpha_{i,j} \mid 1 \leq i < j \leq d+1\}$$
$$\Phi(d) \overset{\text{def}}{=} \Phi_+(d) \uplus (-\Phi_+(d))$$
$$\Phi_{\geq -1}(d) \overset{\text{def}}{=} \Phi_+(d) \uplus (-\Pi(d))$$

The Cluster Complex

- **cluster complex**: simplicial complex on $\Phi_{\geq -1}(d)$ determined by **compatibility**

Clus(3) :



$$F_3(x,y) = 5x^3+5x^2y+3xy^2+y^3 +10x^2+8xy+3y^2+6x+3y+1$$

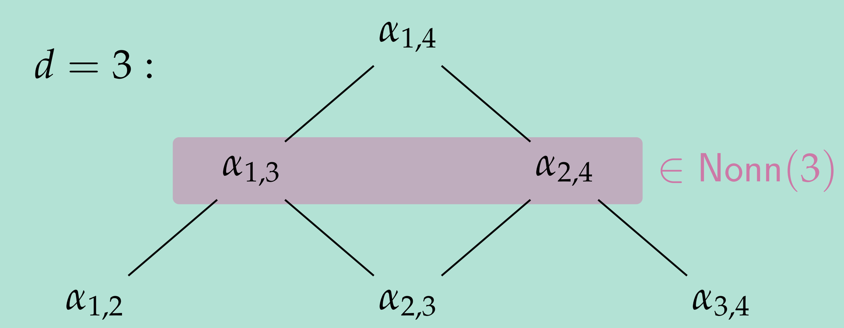
Chapoton's Triangles

- for $A \in \mathsf{Clus}(d)$:
  - $\mathsf{pos}(A) \overset{\text{def}}{=} |A \cap \Phi_+(d)|$
  - $\mathsf{neg}(A) \overset{\text{def}}{=} |A \cap (-\Pi(d))|$
- **F-triangle**: $F_d \overset{\text{def}}{=} \sum_{A \in \mathsf{Clus}(d)} x^{\mathsf{pos}(A)} y^{\mathsf{neg}(A)}$
- **H-triangle**: $H_d \overset{\text{def}}{=} \sum_{A \in \mathsf{Nonn}(d)} x^{|A|} y^{|A \cap \Pi(d)|}$

The Root Poset

- **root order**: $\alpha \preceq \beta$ if $\beta - \alpha \in \mathsf{Span}_{\mathbb{N}}\, \Pi(d)$
- **nonnesting partition**: antichain in **root poset**

$d = 3$:



$$H_3(x,y) = x^3y^3+3x^2y^2+2x^2y+x^2+3xy+3x+1$$

The F=H-Correspondence

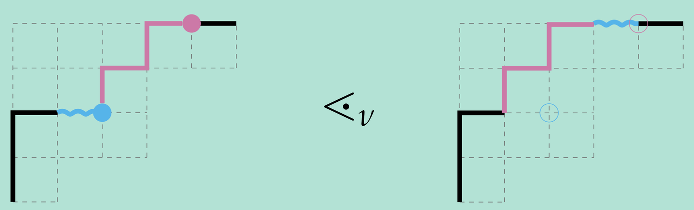
Theorem (Thiel, 2014).
For $d \geq 1$,

$$F_d(x,y) = x^d H_d\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

- conjectured by Chapoton in 2006
- proof uses differential equations and induction

ν-Paths and Rotation

- **northeast path**: lattice path using north and east steps of unit length
- **ν-path**: northeast path weakly above a fixed path ν
- **rotation**: exchanging parts of a ν-path with respect to a **valley**



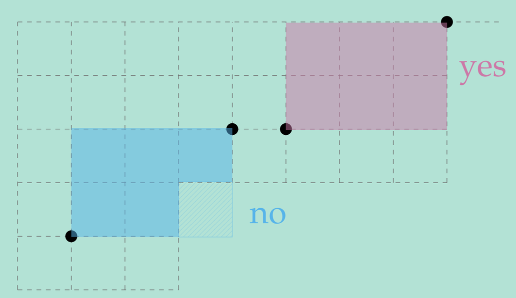
- **ν-Tamari lattice**: ν-paths ordered by rotation

Some Statistics

- **relevant position**: position in first column and row of a valley of ν
- **degree**: $\deg(\nu) \overset{\text{def}}{=} \max\{\mathsf{val}(\mu) \mid \mu \in \mathsf{Dyck}(\nu)\}$
- for $C \in \mathsf{Asso}(\nu)$ and ν using $m+n$ steps:
  - $\dim(C) \overset{\text{def}}{=} m+n+1-|C|$
  - $\mathsf{asc}(C)$ .. number of ascent nodes in C
  - $\mathsf{rel}(C)$ .. number of relevant positions filled by C
  - $\mathsf{corel}(C) \overset{\text{def}}{=} \deg(\nu) - \dim(C) - \mathsf{rel}(C)$

Compatibility and ν-Trees

- **ν-incompatible nodes**:



- **ν-face**: collection of ν-compatible nodes
- **ν-tree**: maximal ν-face
- **ν-Tamari complex**: simplicial complex of ν-faces

The ν-Associahedron

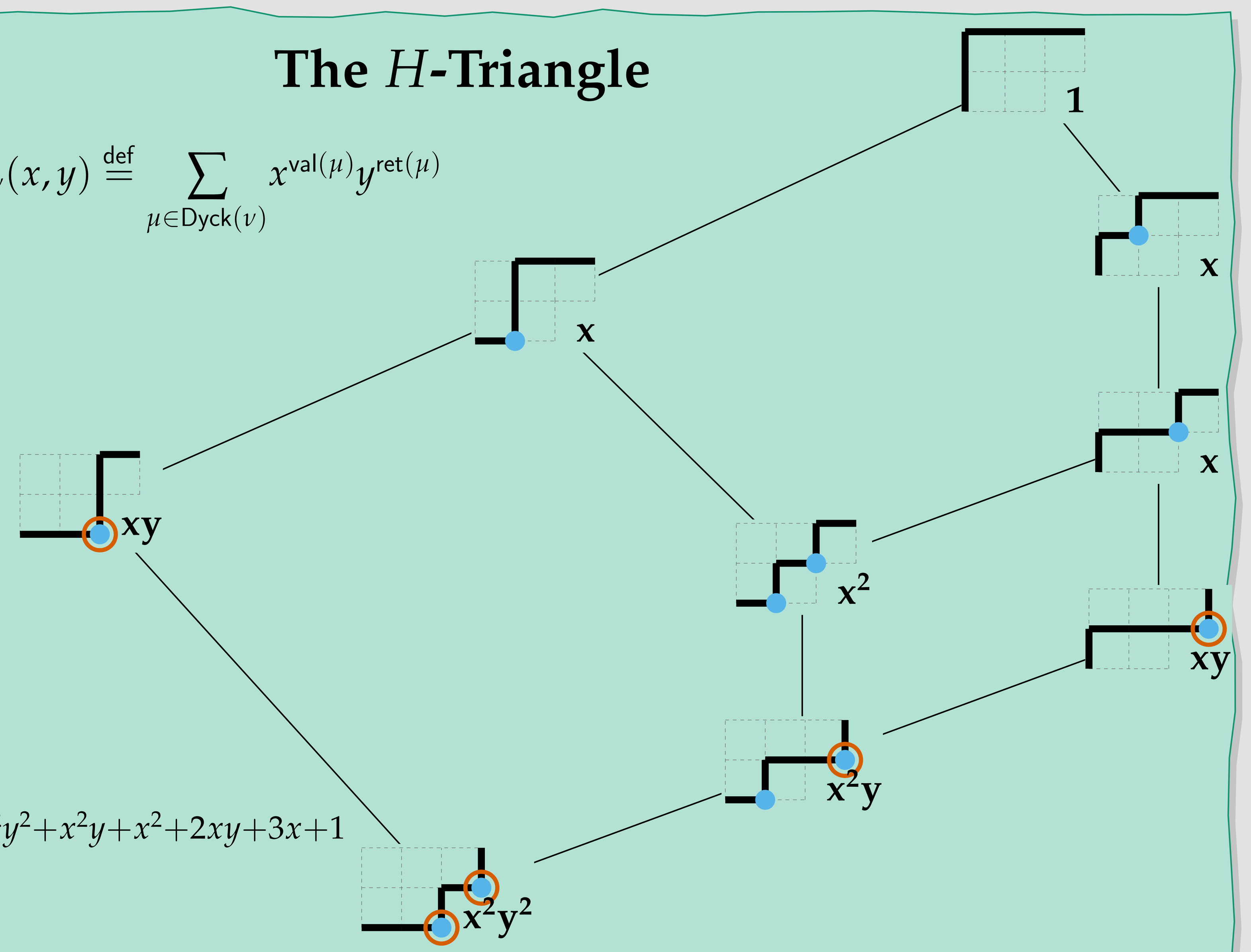
- **covering ν-face**: ν-face containing top-left corner and at least one node per row and column
- **ν-associahedron**: complex of covering ν-faces

Theorem (CC+Padrol+Sarmiento, 2019).

- *The ν-associahedron is a polytopal complex.*
- *It is the dual of the subcomplex of interior faces of the ν-Tamari complex.*
- *Its line graph realizes the ν-Tamari lattice.*

The H-Triangle

- **H-triangle**: $H_\nu(x,y) \overset{\text{def}}{=} \sum_{\mu \in \mathsf{Dyck}(\nu)} x^{\mathsf{val}(\mu)} y^{\mathsf{ret}(\mu)}$



$$H_{EEENEN}(x,y) = x^2y^2+x^2y+x^2+2xy+3x+1$$

The F=H-Correspondence for ν-Associahedra

Proposition (CC+HM, 2021).
For every northeast path ν,

$$H_\nu(x,y) = \sum_{T \in \mathsf{Tree}(\nu)} x^{\mathsf{asc}(T)} y^{\mathsf{rel}(T)}$$

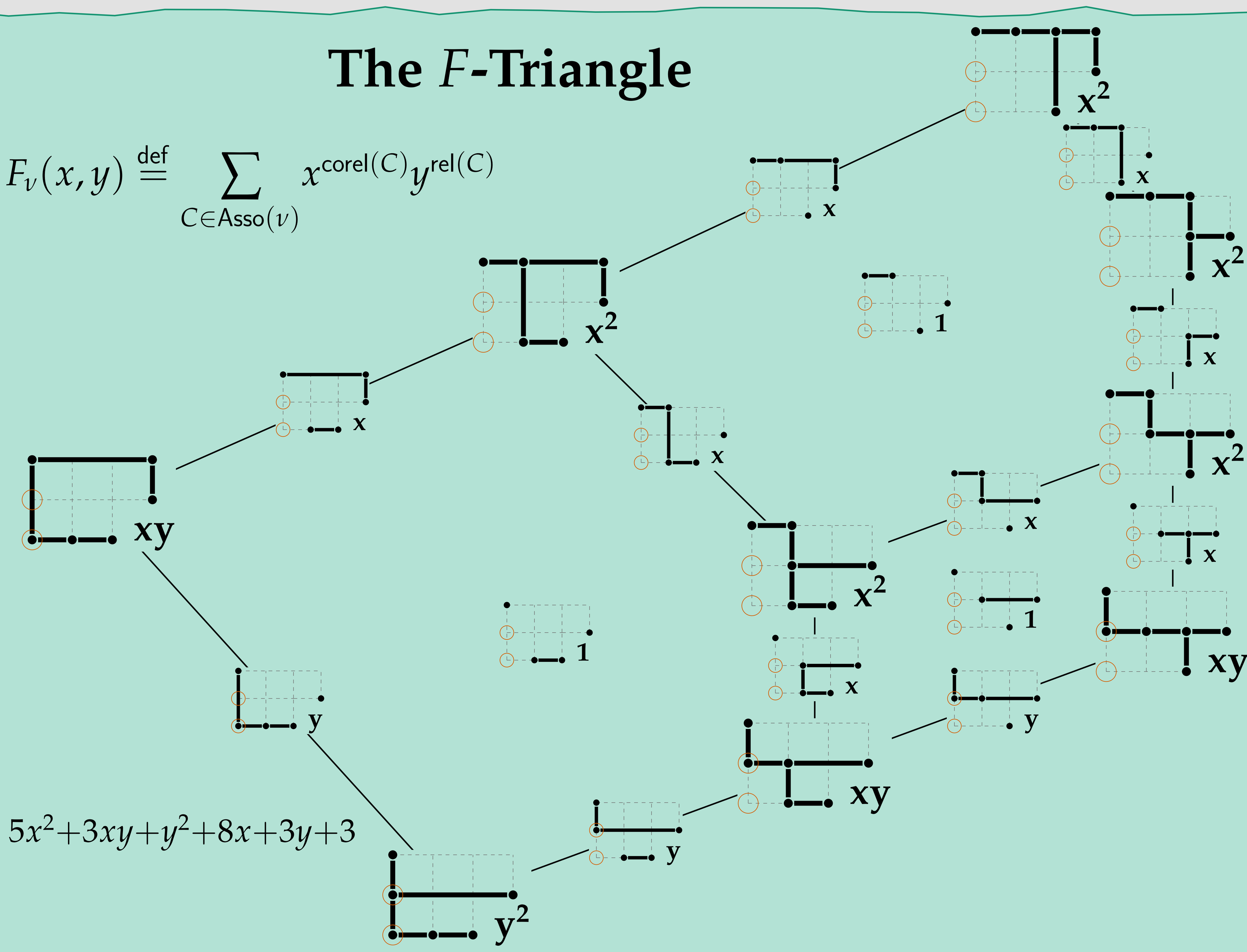
Theorem (CC+HM, 2021).
For every northeast path ν,

$$F_\nu(x,y) = x^{\deg(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

- extends to a multivariate version for finite posets

The F-Triangle

- **F-Triangle**: $F_\nu(x,y) \overset{\text{def}}{=} \sum_{C \in \mathsf{Asso}(\nu)} x^{\mathsf{corel}(C)} y^{\mathsf{rel}(C)}$



$$F_{EEENEN}(x,y) = 5x^2+3xy+y^2+8x+3y+3$$