Connectivity Properties of Factorization Posets in Generated Groups

Henri Mühle[1] and Vivien Ripoll[2]

[1]Technische Universität Dresden, Institut für Algebra, Zellescher Weg 12–14, 01069 Dresden, Germany.
[2]Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria.

The lattice of (generalized) noncrossing partitions enjoys many structural, combinatorial and topological properties. The purpose of this poster is to explore the interplay between some of these properties, that may perhaps be described as “connectivity properties”, on the abstract level of generated groups.

Generated Groups

Generated group: a pair (G, A) with G a group and $A \subseteq G$ a set that generates G as a monoid.

On this poster, all generated groups (G, A) have the property that A is closed under G-conjugation.

Reduced Factorizations

A-length: the function $\ell_A : G \to \mathbb{N}$ defined by $g \mapsto \min\{k \mid g = a_1a_2\cdots a_k \text{ with } a_i \in A\}$.
A-reduced factorization: for $g \in G$, a tuple $(a_1, a_2, \ldots, a_k)$ with entries in A such that $g = a_1a_2\cdots a_k$ and $\ell_A(g) = k$.
$\mathrm{Red}_A(g)$: set of A-reduced factorizations of g.

On this poster, we consider only such $g \in G$ for which $\mathrm{Red}_A(g)$ is finite.

Factorization Posets

Prefix order: for $g, h \in G$ define $g \leq_A h$ if and only if $\ell_A(g) + \ell_A(g^{-1}h) = \ell_A(h)$.
Factorization poset: the principal order ideal of $(G, \leq_A)$ generated by some $g \in G$; denoted by $\mathcal{P}_g(G, A)$.

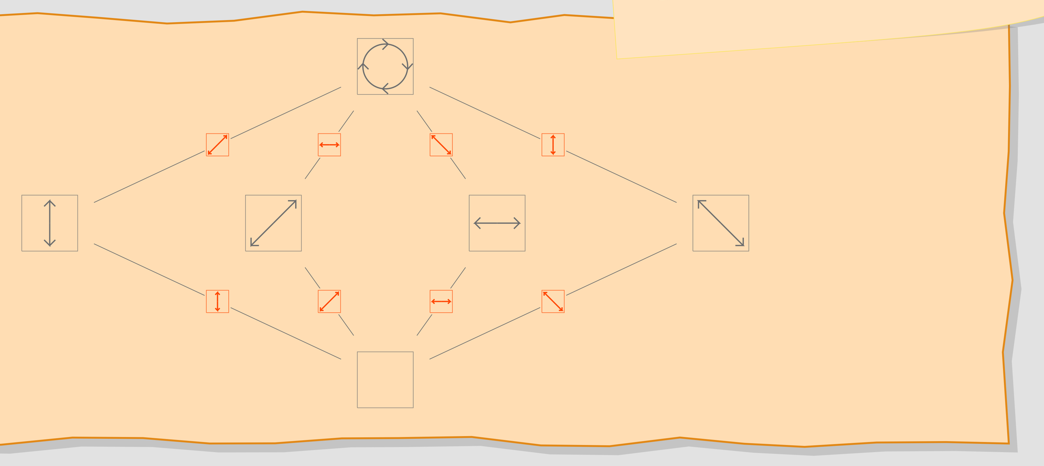
Maximal chains of $\mathcal{P}_g(G, A)$ are in bijection with A-reduced factorizations for g:
$\{x_0 \lessdot_A x_1 \lessdot_A \cdots \lessdot_A x_k\} \mapsto (x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{k-1}^{-1}x_k)$

Induces an edge-labeling.

An Example

Let G be the (dihedral) group of symmetries of a square, and let $A = \{↕, ⤢, ↔, ⤡\}$ be the set of reflections.

The four A-reduced factorizations of the (clockwise) quarter turn $g = ⟳$ are:
(↕, ⤢), (⤢, ↔), (↔, ⤡), and (⤡, ↕).



The Chain Graph

Let $\mathcal{P}$ be a graded poset with least and greatest element.

Maximal chain: a maximal unrefinable chain of $\mathcal{P}$.
Chain graph: the graph on the set of maximal chains of $\mathcal{P}$, where two maximal chains are adjacent if they differ in exactly one element.
Chain-connected: a poset whose chain graph is connected.

The Hurwitz Graph

Extends to a braid group action.

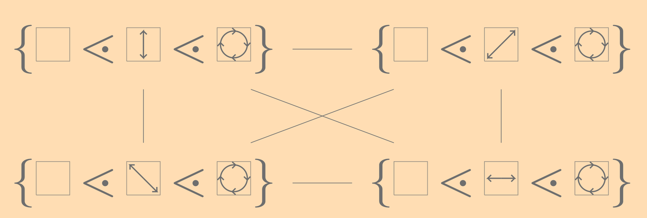
Let $(a_1, a_2, \ldots, a_k) \in \mathrm{Red}_A(g)$.

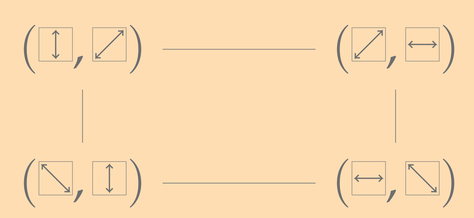
Hurwitz operator: σ_i, for $1 \leq i < k$, that acts by
$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_k) \mapsto$
$(a_1, \ldots, a_{i-1}, a_{i+1}, a_{i+1}^{-1}a_ia_{i+1}, a_{i+2}, \ldots, a_k)$.
(This is valid since A is closed under G-conjugation.)
Hurwitz graph: the graph on $\mathrm{Red}_A(g)$, where two A-reduced factorizations are adjacent if applying a Hurwitz operator to one of them yields the other.
Hurwitz-connected: an element $g \in G$ whose Hurwitz graph is connected.

Shellability

Shelling: a linear order $\prec$ on the maximal chains of $\mathcal{P}$ such that: whenever $M \prec M'$, there is $N \prec M'$ such that N is adjacent to M' in the chain graph and $N \cap M' \supseteq M \cap M'$.
Shellable: a poset that admits a shelling.
Edge-labeling: an assignment of (ordered) labels to edges in the poset diagram of $\mathcal{P}$.
EL-labeling: for every interval of $\mathcal{P}$ there is a unique maximal chain with increasing label sequence, and this label sequence is lexicographically smaller than any other label sequence of a maximal chain in this interval.

An Example





A shelling is for instance $\{□ \lessdot ↕ \lessdot ⟳\} \prec \{□ \lessdot ⤢ \lessdot ⟳\} \prec \{□ \lessdot ↔ \lessdot ⟳\} \prec \{□ \lessdot ⤡ \lessdot ⟳\}$.

The order ↕ $\prec$ ⤡ $\prec$ ↔ $\prec$ ⤢ yields an EL-labeling.

Let **prop** be a poset property. A poset $\mathcal{P}$ is **totally prop** if every interval of $\mathcal{P}$ satisfies **prop**.

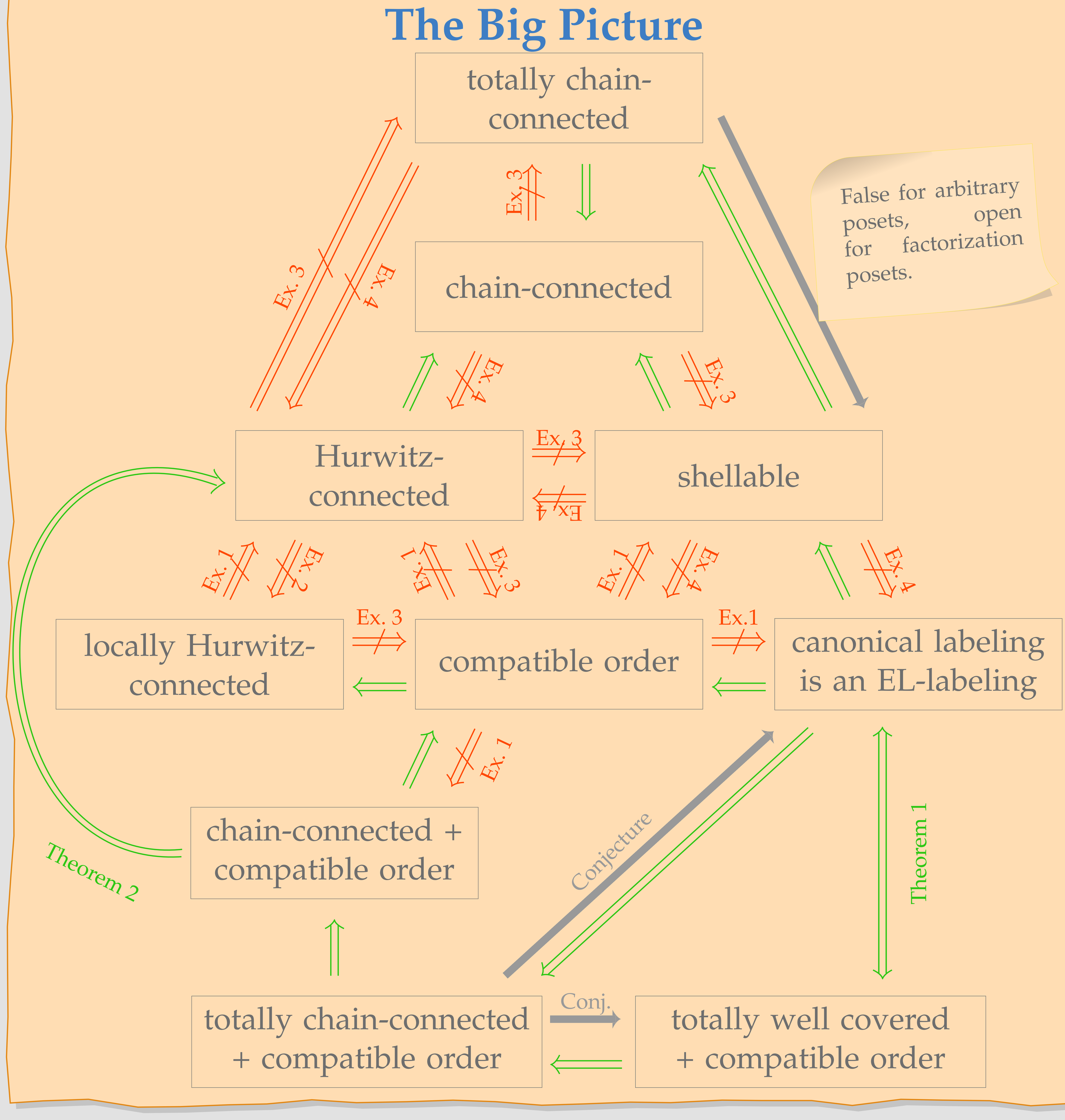
A poset $\mathcal{P}$ is **well covered** with respect to a total order $\prec$ of its atoms if for every atom a (except the minimal one for $\prec$), there exists an upper cover g' of a in $\mathcal{P}$ and another atom $a' \prec a$ such that $a' \leq g'$.

An element $g \in G$ is called **locally Hurwitz-connected** if every subword of length 2 is Hurwitz-connected.

Let $A_g = \{a \in A \mid a \leq_A g\}$.

A total order $\prec$ of A_g is **g-compatible** if every $g' \leq_A g$ with $\ell_A(g') = 2$ has a unique A_g-reduced factorization that is increasing with respect to $\prec$.

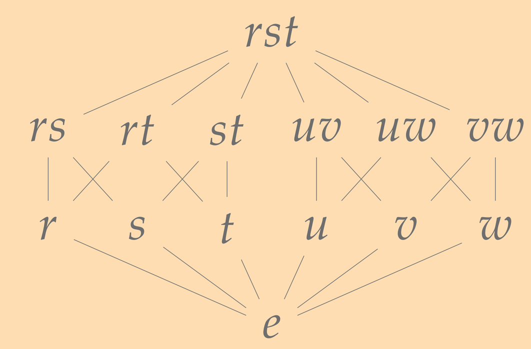
The Big Picture



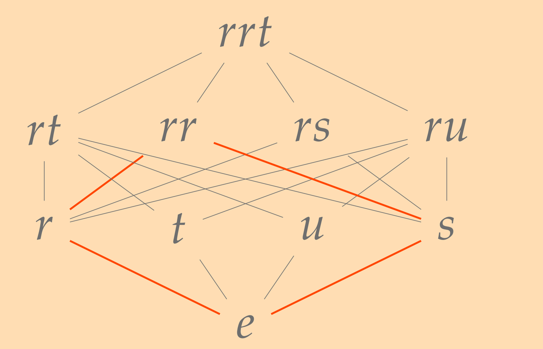
Theorem 1 (M.-R., 2017)
The canonical labeling of $\mathcal{P}_g(G, A)$ is an EL-labeling if and only if $\mathcal{P}_g(G, A)$ is totally well covered with respect to a g-compatible total order of A_g.

Theorem 2 (M.-R., 2017)
If $\mathcal{P}_g(G, A)$ is chain-connected and admits a g-compatible order of A_g, then g is Hurwitz-connected.

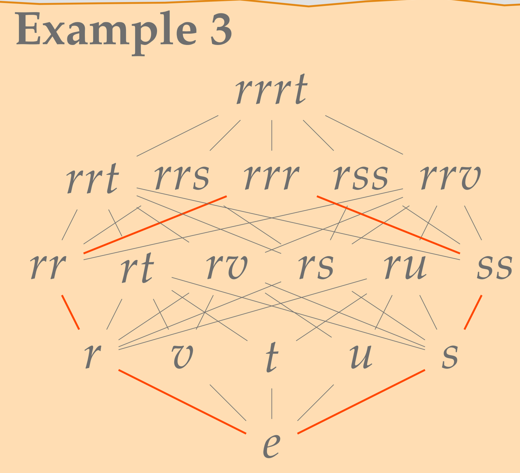
Example 1



Example 2



Example 3



Example 4