The m-Cover Posets and the Strip-Decomposition of m-Dyck Paths

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Coxeter-Catalan Combinatorics

- Catalan object: a finite set associated with the symmetric group whose cardinality equals some Catalan number
- problem: given an integer $m \geq 1$, parametrize a Catalan object so that the resulting set has cardinality equal to the corresponding Fuß-Catalan number, and when $m = 1$ the set coincides with the corresponding classical Catalan object
- general problem: recall that the symmetric group is a Coxeter group of type A; generalize the above constructions to any finite Coxeter group

Motivation

- Tamari lattice: a classical Catalan object that can be realized as the set of Dyck paths equipped with the rotation order
- m-Tamari lattice: the set of m-Dyck paths equipped with the rotation order
- m-Tamari lattice associated with a Coxeter group W: ***open!*** we provide a new realization of the classical m-Tamari lattice in terms of m-tuples of Dyck paths; using the same construction (the m-cover posets), we provide an "m-Tamari like" lattice for the dihedral group $\mathfrak{D}_k$

The m-Cover Posets

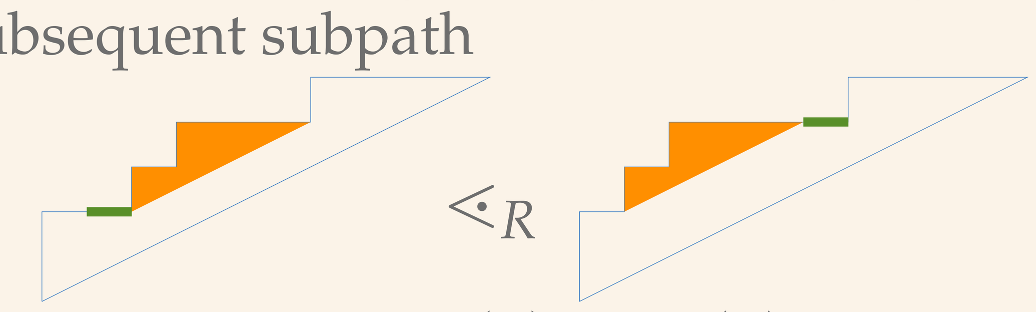
Let $\mathcal{P} = (P, \leq)$ be a bounded poset, let $\hat{0}$ denote the least element, and let $m > 0$.

- m-cover poset $\mathcal{P}^{\langle m\rangle}$: the subposet of the direct product $\mathcal{P}^m$ with ground set

$$P^{\langle m\rangle} = \Big\{ \big(\underbrace{\hat{0},\ldots,\hat{0}}_{l_1},\underbrace{p,\ldots,p}_{l_2},\underbrace{q,\ldots,q}_{l_3}\big) \mid p \lessdot q, l_1 + l_2 + l_3 = m \Big\}$$

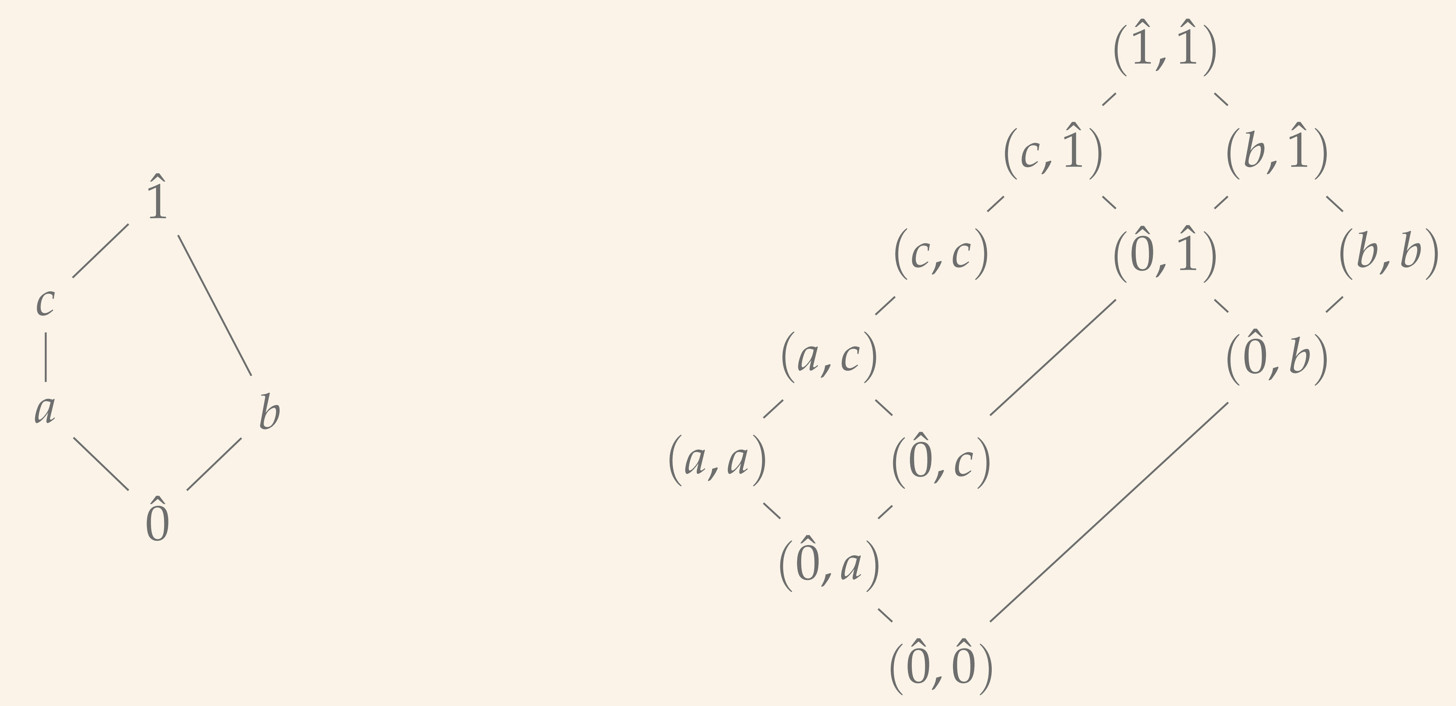
The m-Dyck Paths and the m-Tamari Lattices

- m-Dyck path of length $(m+1)n$: north-east path from $(0,0)$ to (mn, n) that stays weakly above the line $x = my$
- $\mathcal{D}_n^{(m)}$ denotes the set of all m-Dyck paths of length $(m+1)n$
- $\big|\mathcal{D}_n^{(m)}\big| = \mathrm{Cat}^{(m)}(\mathfrak{S}_n) = \frac{1}{mn+1}\binom{(m+1)n}{n}$
- rotation order $\leq_R$: covers given by exchanging an east step with the subsequent subpath



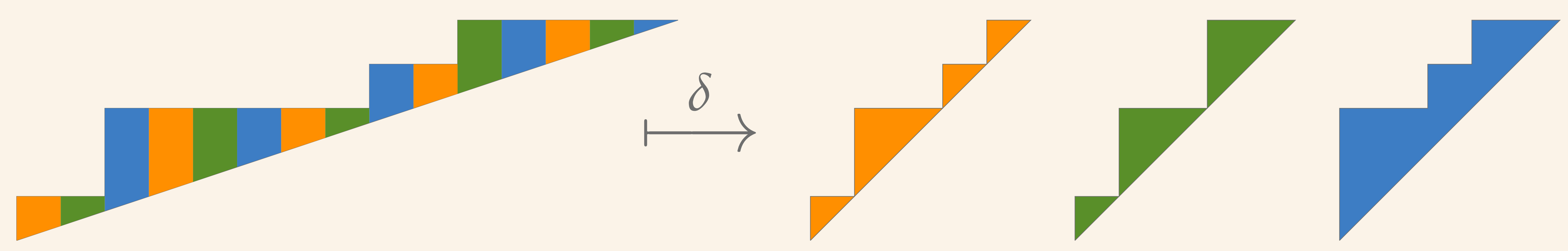
- m-Tamari lattice: the poset $\mathcal{T}_n^{(m)} = \big(\mathcal{D}_n^{(m)}, \leq_R\big)$

Example: The pentagon lattice and its 2-cover poset



The Strip-Decomposition

- strip-decomposition: the map $\delta : \mathcal{D}_n^{(m)} \to \big(\mathcal{D}_n\big)^m$ given by "cutting the path into strips and regrouping"



Theorem: A new Realization of the m-Tamari lattice

For $m, n > 0$ we have $\mathcal{T}_n^{(m)} \cong \mathbf{DM}\big(\mathcal{T}_n^{\langle m\rangle}\big)$, where **DM** denotes the Dedekind-MacNeille completion.

Sketch of the proof. Combine the following.

- $\mathcal{P} \cong \mathbf{DM}\big(\mathcal{M}(\mathcal{P}) \cup \mathcal{J}(\mathcal{P})\big)$, where $\mathcal{P}$ is a finite lattice [known]
- $\Big(\mathcal{M}\big(\mathcal{T}_n^{(m)}\big), \leq_R\Big) \cong \Big(\mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_R\Big)$
- $\Big(\mathcal{J}\big(\mathcal{T}_n^{(m)}\big), \leq_R\Big) \cong \Big(\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_R\Big)$
- every $\mathfrak{p} \in \mathcal{D}_n^{\langle m\rangle}$ can be written as a join of elements in $\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big)$

A Conjecture

For $m, n > 0$ we conjecture $\mathcal{T}_n^{(m)} \cong \beta \circ \delta\big(\mathcal{T}_n^{(m)}\big)$, where

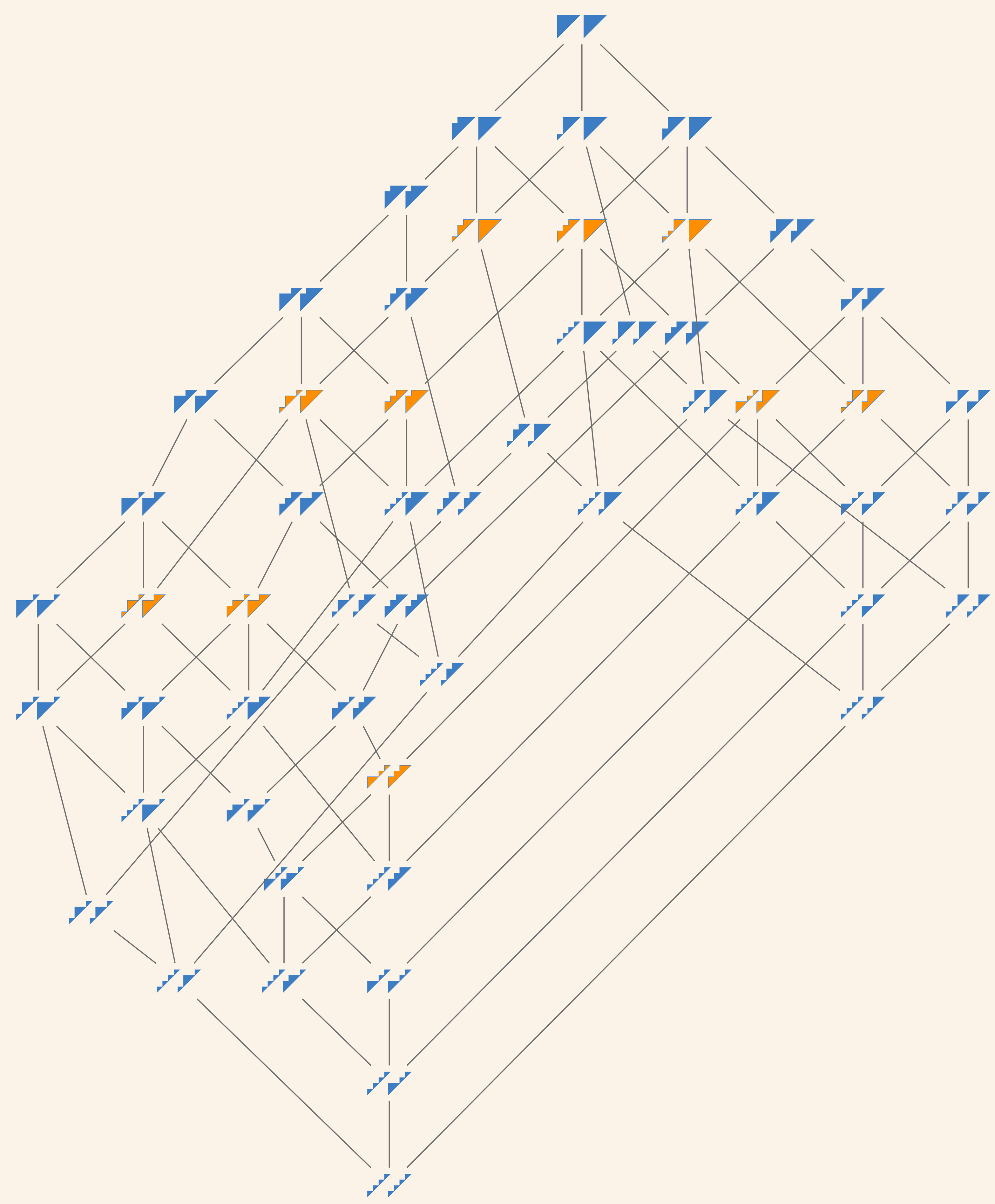
- $\beta = \beta_{m-1,m} \circ \cdots \circ \beta_{2,3} \circ \beta_{1,m} \circ \cdots \circ \beta_{1,2}$ and
- $\beta_{i,j} : \big(\mathcal{D}_n\big)^m \to \big(\mathcal{D}_n\big)^m$ is defined by replacing in $(\mathfrak{q}_1, \ldots, \mathfrak{q}_m)$ the i-th and j-th component by $\mathfrak{q}_i \wedge_R \mathfrak{q}_j$ and $\mathfrak{q}_i \vee_R \mathfrak{q}_j$, respectively

Theorem: Tamari-like Lattices for Dihedral Groups

For $k > 1$ let $\mathcal{C}_k$ denote the bounded poset whose proper part is the cardinal sum of a $(k-1)$-chain and an atom.

Then for every $m > 0$, the poset $\mathcal{C}_k^{\langle m\rangle}$ is a trim lattice, and its cardinality is $\mathrm{Cat}^{(m)}(\mathfrak{D}_k) = \binom{m+1}{2}k + m + 1$.

Example: The Lattice $\mathbf{DM}\big(\mathcal{T}_4^{\langle 2\rangle}\big) \cong \mathcal{T}_4^{(2)}$



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