Tamari Lattices for Parabolic Quotients of the Symmetric Group

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Parabolic 231-Avoiding Permutations

Let $\mathfrak{S}_n$ be the symmetric group on $[n] := \{1, 2, \ldots, n\}$, and let $S := \{s_i\}_{i=1}^{n-1}$ be the set of simple reflections $s_i := (i, i+1)$. The Cayley graph of $\mathfrak{S}_n$ generated by S may be oriented to form the *weak order*, which is a lattice. Fix $J := S \setminus \{s_{j_1}, s_{j_2}, \ldots, s_{j_r}\}$, and let $\mathrm{B}(J)$ be the set partition of $[n]$ (whose parts we call *J-regions*)

$$\{\{1, \ldots, j_1\}, \{j_1+1, \ldots, j_2\}, \ldots, \{j_{r-1}+1, \ldots, j_r\}, \{j_r+1, \ldots, n\}\}.$$

The *parabolic quotient* $\mathfrak{S}_n^J$ is the set of $w \in \mathfrak{S}_n$ whose one-line notation has the form

$$w = w_1 < \cdots < w_{j_1} \mid w_{j_1+1} < \cdots < w_{j_2} \mid \cdots \mid w_{j_r+1} < \cdots < w_n.$$

Definition 1 A permutation $w \in \mathfrak{S}_n^J$ is *J-231-avoiding* if there exist no three indices $i < j < k$, all of which lie in different J-regions, such that $w_k < w_i < w_j$ and $w_i = w_k + 1$. Let $\mathfrak{S}_n^J(231)$ denote the set of J-231-avoiding permutations of $\mathfrak{S}_n^J$.

Theorem 2 *For $J \subseteq S$, the restriction of the weak order to $\mathfrak{S}_n^J(231)$ forms a lattice. Furthermore, $\mathcal{T}_n^J$ is a lattice quotient of the weak order on $\mathfrak{S}_n^J$.*

When $J = \emptyset$, we recover the classical Tamari lattice on 231-avoiding permutations.

Parabolic Noncrossing Partitions

Let $\mathbf{P} = \{P_1, P_2, \ldots, P_s\}$ be a set partition of $[n]$. A pair (a, b) is a *bump* of $\mathbf{P}$ if $a, b \in P_i$ for some $i \in [s]$ and there is no $c \in P_i$ with $a < c < b$.

Definition 3 A set partition $\mathbf{P}$ of $[n]$ is *J-noncrossing* if it satisfies:

(NC1) If i and j lie in the same J-region, then they are not contained in the same part of $\mathbf{P}$.

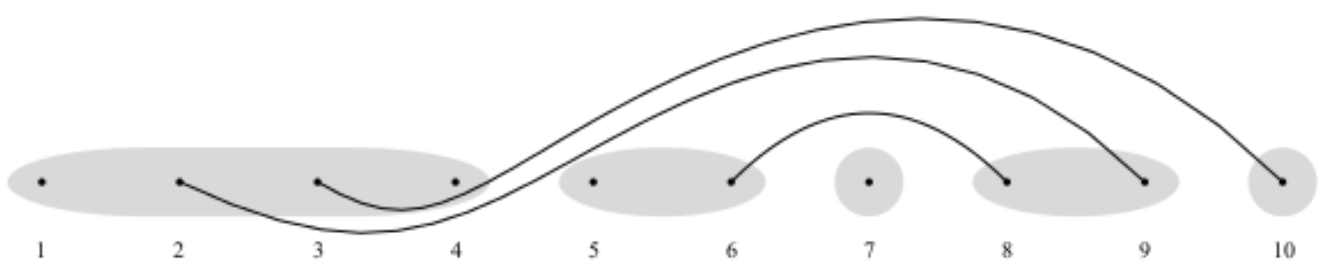
(NC2) If two distinct bumps (i_1, i_2) and (j_1, j_2) of $\mathbf{P}$ satisfy $i_1 < j_1 < i_2 < j_2$, then either i_1 and j_1 lie in the same J-region or i_2 and j_1 lie in the same J-region.

(NC3) If two distinct bumps (i_1, i_2) and (j_1, j_2) of $\mathbf{P}$ satisfy $i_1 < j_1 < j_2 < i_2$, then i_1 and j_1 lie in different J-regions.

Let NC_n^J denote the set of all J-noncrossing set partitions of $[n]$.

When $J = \emptyset$, we recover the classical noncrossing set partitions.

Example 4 *For $J = \{s_1, s_2, s_3, s_5, s_8\}$,* $\{\{1\}, \{2, 9\}, \{3, 10\}, \{4\}, \{5\}, \{6, 8\}, \{7\}\} \in \mathrm{NC}_{10}^J$.



Parabolic Nonnesting Partitions

Definition 5 A set partition $\mathbf{P}$ of $[n]$ is *J-nonnesting* if it satisfies:

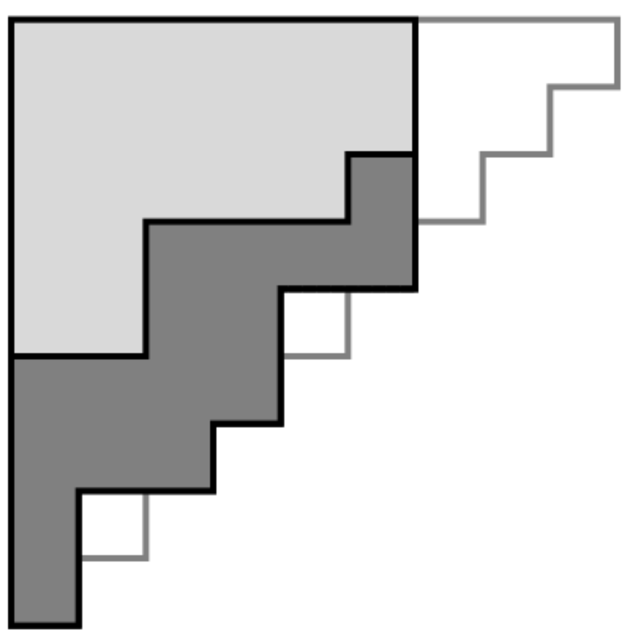
(NN1) If i and j lie in the same J-region, then they are not in the same part of $\mathbf{P}$.

(NN2) If (i_1, i_2) and (j_1, j_2) are two distinct bumps of $\mathbf{P}$, then it is not the case that $i_1 < j_1 < j_2 < i_2$.

Let NN_n^J denote the set of all J-nonnesting partitions of $[n]$.

When $J = \emptyset$, we recover the classical nonnesting set partitions.

Example 6 *For $J = \{s_1, s_2, s_3, s_5, s_8\}$,* $\{\{1\}, \{2\}, \{3, 5\}, \{4, 8\}, \{6, 10\}, \{7\}, \{8\}, \{9\}\} \in \mathrm{NN}_{10}^J$.



Parabolic Catalan Objects are Equinumerous

Although we no longer have a product formula for $\lvert\mathfrak{S}_n^J(231)\rvert$, our parabolic generalizations remain in bijection, generalizing the situation when $J = \emptyset$.

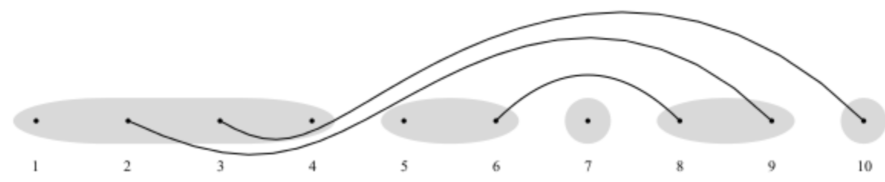
Theorem 7 *For $n > 0$ and $J \subseteq S$, we have* $\lvert\mathfrak{S}_n^J(231)\rvert = \lvert\mathrm{NC}_n^J\rvert = \lvert\mathrm{NN}_n^J\rvert$.

From $\mathfrak{S}_n^J(231)$ to NC_n^J

A permutation $w \in \mathfrak{S}_n^J(231)$ corresponds to the J-noncrossing partition $\mathbf{P} \in \mathrm{NC}_n^J$ whose bumps are determined by the descents of w.

Example 8 *For $n = 10$ and $J = \{s_1, s_2, s_3, s_5, s_8\}$,*

$1\,7\,9\,10 \mid 2\,5 \mid 3 \mid 4\,6 \mid 8 \quad \in \mathfrak{S}_{10}^J$ *gives*

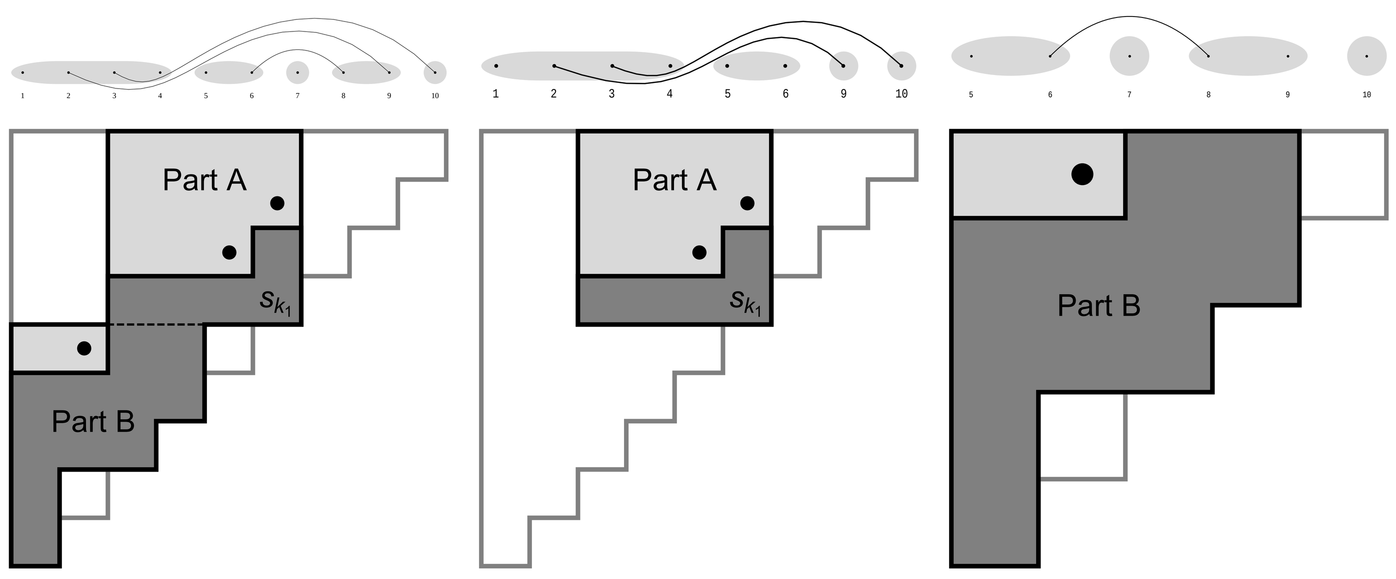


$\in \mathrm{NC}_{10}^J$.

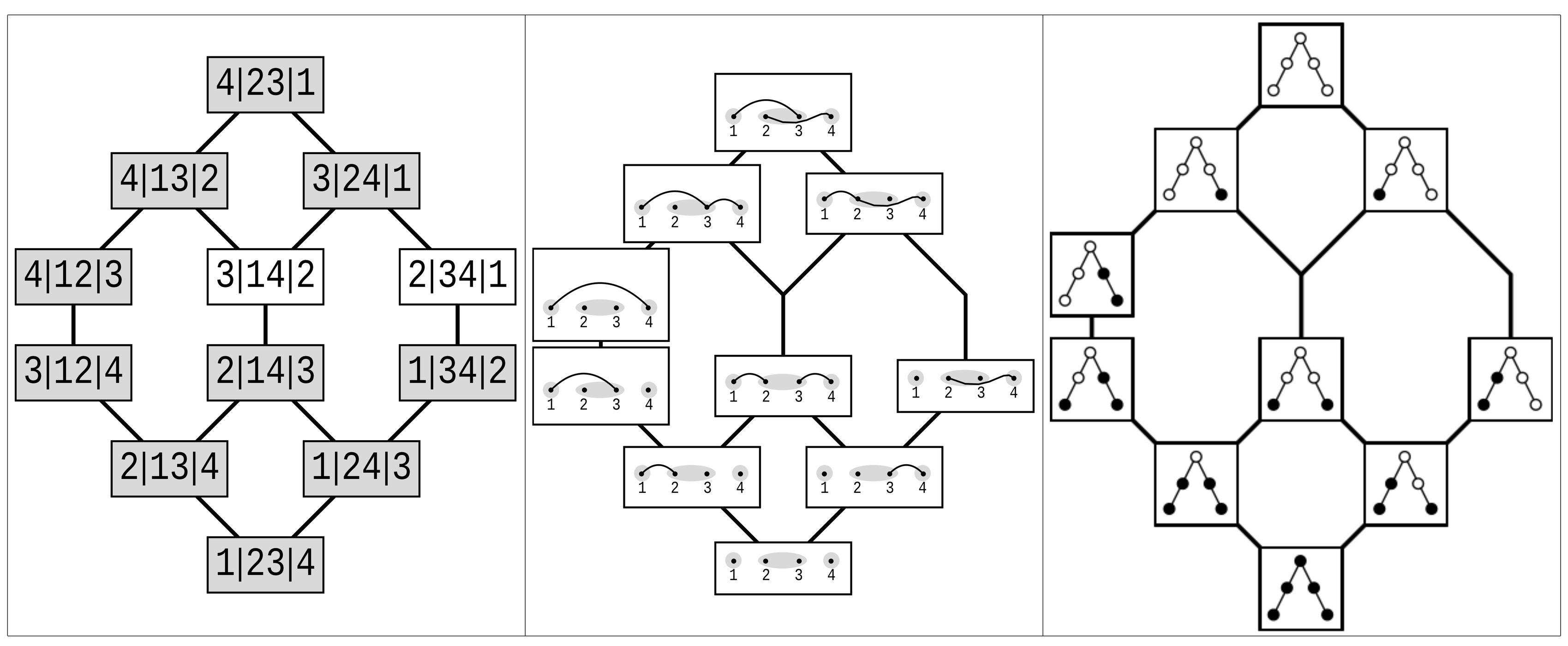
From NC_n^J to NN_n^J

A J-noncrossing partition $\mathbf{P}$ corresponds to the J-nonnesting partition $\mathbf{P}'$, in which the minimal elements outside $\mathbf{P}'$ are determined by the bumps of $\mathbf{P}$.

Example 9 *For $n = 10$ and $J = \{s_1, s_2, s_3, s_5, s_8\}$, we compute*



Example: $\mathfrak{S}_4, J = \{s_2\}$



Outlook

We can generalize the definition of J-231-sortable elements of $\mathfrak{S}_n^J$ to parabolic quotients of any finite Coxeter group and to any Coxeter element.
The definition of parabolic noncrossing and nonnesting partitions—as well as the c-cluster complex—can also be generalized, but—in contrast to the classical case when $J = \emptyset$—the four sets are not always equinumerous.