On the Lattice Property of Shard Orders

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Day's Doubling Construction

Let $\mathcal{P} = (P, \leq)$ be a poset and let $\mathbf{2}$ be the chain of length 2 whose elements are 0 and 1.
For $I \subseteq P$, define $P_{\leq I} = \{x \in P \mid x \leq y \text{ for some } y \in I\}$.
The **doubling** of $\mathcal{P}$ by I is the subposet $\mathcal{P}[I]$ of $\mathcal{P} \times \mathbf{2}$ given by the ground set $\bigl(P_{\leq I} \times \{0\}\bigr) \uplus \Bigl(\bigl((P \setminus P_{\leq I}) \cup I\bigr) \times \{1\}\Bigr)$.

Congruence-Uniform Lattices

A lattice is **congruence-uniform** if it can be obtained from the singleton lattice by a sequence of interval doublings.
Let us label the edges in the poset diagram according to the step in which they where created; and call this labeling λ.

The Alternate Order

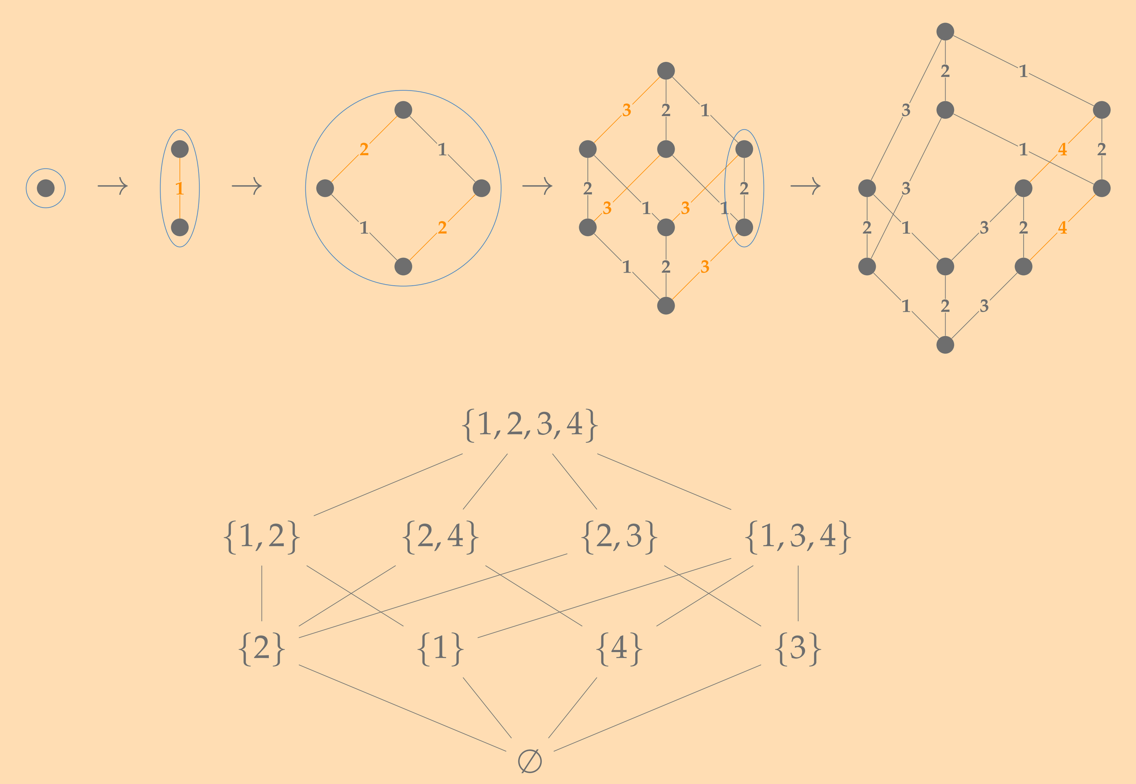
Let $\mathcal{P} = (P, \leq)$ be a congruence-uniform lattice. For $x \in P$, define $x_{\downarrow} = \bigwedge_{y \lessdot x} y$, and

$\Psi(x) = \{\lambda(u, v) \mid x_{\downarrow} \leq u \lessdot v \leq x\}$.

The **alternate order** of $\mathcal{P}$ is the poset $\mathrm{Alt}(\mathcal{P}) = (P, \sqsubseteq)$ determined by the order relation $x \sqsubseteq y$ if and only if $\Psi(x) \subseteq \Psi(y)$.

Problem 1 (N. Reading, 2016)
For which congruence-uniform lattices is their alternate order again a lattice?

Example



The Motivation

The Poset of Regions

Let $\mathcal{A}$ be a simplicial hyperplane arrangement, and fix a base region B. The **poset of regions** $\mathcal{P}(\mathcal{A}, B)$ is the reflexive and transitive closure of the adjacency graph of the regions of $\mathcal{A}$ oriented away from B.

A central hyperplane arrangement is simplicial if every region is a simplicial cone.

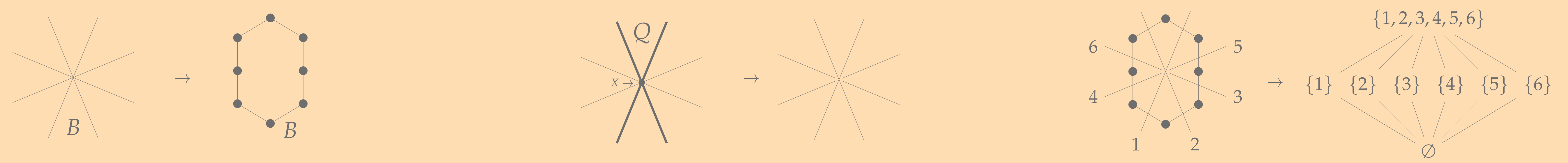
Shards of Hyperplanes

Let X be an intersection of hyperplanes of $\mathcal{A}$ of codimension 2. The regions containing X form a polygonal interval of $\mathcal{P}(\mathcal{A}, B)$ with a greatest element Q. The bounding hyperplanes of Q that contain X "cut" all the other hyperplanes containing X. All these cuts split the hyperplanes of $\mathcal{A}$ into **shards**.

Congruence-Uniform Lattices of Regions

If $\mathcal{A}$ and B are such that $\mathcal{P}(\mathcal{A}, B)$ is a congruence-uniform lattice, then we can identify the edge labels of $\mathcal{P}(\mathcal{A}, B)$ with the shards of $\mathcal{A}$. The sets $\Psi(\cdot)$ correspond to intersections of shards.

Theorem 2 (N. Reading, 2011)
If $\mathcal{P}(\mathcal{A}, B)$ is a congruence-uniform lattice, then $\mathrm{Alt}\bigl(\mathcal{P}(\mathcal{A}, B)\bigr)$ is a lattice, too.



Meet-Semidistributive Lattices

A lattice $\mathcal{P} = (P, \leq)$ is **meet-semidistributive** if for all $x, y, z \in P$ the following implication holds: if $x \wedge y = x \wedge z$, then $x \wedge y = x \wedge (y \vee z)$ for all $x, y \in P$.

Every congruence-uniform lattice is meet-semidistributive.

The Möbius Function

The **Möbius function** of a poset $\mathcal{P}$ is the function $\mu_{\mathcal{P}}$ defined recursively by:

$$\mu_{\mathcal{P}}(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x, z), & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases}$$

The Crosscut Theorem

Let $\mathcal{P} = (P, \leq)$ be a lattice with least element $\hat{0}$ and greatest element $\hat{1}$.
An antichain $C \subseteq P \setminus \{\hat{0}, \hat{1}\}$ is a **crosscut** if every maximal chain of $\mathcal{P}$ intersects C.
A crosscut C is **spanning** if $\bigvee C = \hat{1}$ and $\bigwedge C = \hat{0}$.

Theorem 3 (G.-C. Rota, 1964)
Let $\mathcal{P} = (P, \leq)$ be a lattice, and let $C \subseteq P$ be a crosscut. We have $\mu_{\mathcal{P}}(\hat{0}, \hat{1}) = \sum_{X \subseteq C \text{ spanning}} (-1)^{|X|}$.

Spherical Meet-Semidistributive Lattices

Proposition 4
Every meet-semidistributive lattice $\mathcal{P}$ satisfies $\mu_{\mathcal{P}}(\hat{0}, \hat{1}) \in \{-1, 0, 1\}$.

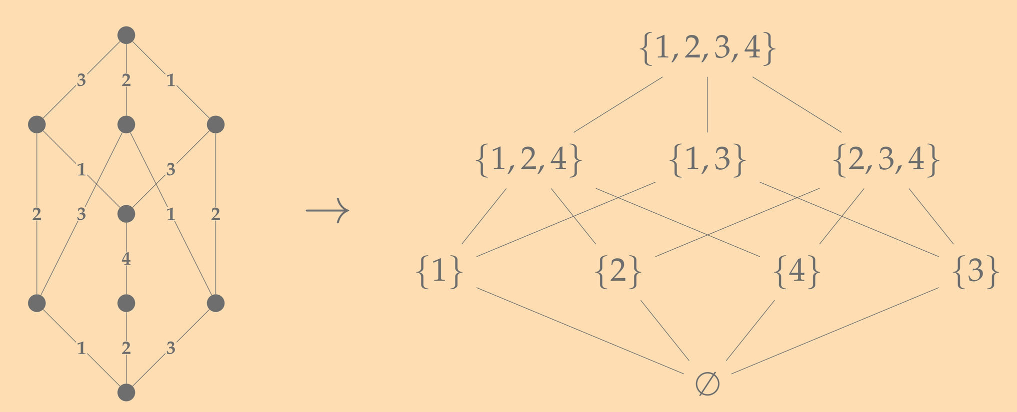
A meet-semidistributive lattice $\mathcal{P}$ is **spherical** if $\mu_{\mathcal{P}}(\hat{0}, \hat{1}) \neq 0$.

A Necessary Condition

Theorem 5 (, 2017)
Let $\mathcal{P}$ be a congruence-uniform lattice. If $\mathrm{Alt}(\mathcal{P})$ is a lattice, then $\mathcal{P}$ is spherical.

Sketch of proof: use meet-semidistributivity of $\mathcal{P}$ and Theorem 3 to show that $\mathrm{Alt}(\mathcal{P})$ has a greatest element if and only if $\mathcal{P}$ is spherical.

Another Example



A Particular Doubling

Let $\mathcal{P} = (P, \leq)$ be a lattice. An element $j \in P \setminus \{\hat{0}\}$ is **join-irreducible** if $j = x \vee y$ implies $j \in \{x, y\}$.

Proposition 6
*Let $\mathcal{P} = (P, \leq)$ be a congruence-uniform lattice, and let $x, y \in P$ such that there exists a join-irreducible element $j \in P$ with $j \in [x_{\downarrow}, x] \cap [y_{\downarrow}, y]$.
If $\Psi(j) \subseteq \Psi(x) \cap \Psi(y)$, then $\mathrm{Alt}\bigl(\mathcal{P}\bigl[\{j\}\bigr]\bigr)$ is not a lattice.*

Theorem 7 (, 2017)
Let $\mathcal{P}$ be a spherical congruence-uniform lattice with at least three atoms. There exists a spherical congruence-uniform lattice $\mathcal{P}'$ with $|\mathcal{P}'| = |\mathcal{P}| + 1$ such that $\mathrm{Alt}(\mathcal{P}')$ is not a lattice.

The Intersection Property

Congruence-uniform lattices of regions have the intersection property.

A congruence-uniform lattice $\mathcal{P} = (P, \leq)$ has the **intersection property** if for every $x, y \in P$ there exists some $z \in P$ such that $\Psi(x) \cap \Psi(y) = \Psi(z)$.

Proposition 8
Let $\mathcal{P}$ be a congruence-uniform lattice. If $\mathcal{P}$ has the intersection property, then $\mathrm{Alt}(\mathcal{P})$ is a meet-semilattice.

Problem 9
Which congruence-uniform lattices have the intersection property?

Problem 10
Find a spherical congruence-uniform lattice $\mathcal{P}$ without the intersection property for which $\mathrm{Alt}(\mathcal{P})$ is a lattice.