Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

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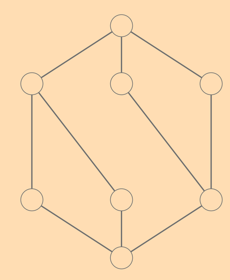
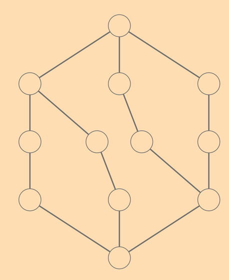
The Strong Sperner Property and Symmetric Chain Decompositions

Let $\mathcal{P} = (P, \leq)$ be a graded poset of rank n. For $k \geq 1$, a **k-family** is $X \subseteq P$ that does not contain a $(k+1)$-chain. $\mathcal{P}$ is **k-Sperner** if the size of a k-family does not exceed the sum of the k largest rank numbers, and it is **strongly Sperner** if it is k-Sperner for all $k \leq n$.
A **symmetric chain decomposition** of $\mathcal{P}$ is a partition of P with saturated chains that are symmetric about the middle rank(s). The existence of a symmetric chain decomposition implies the strong Sperner property.

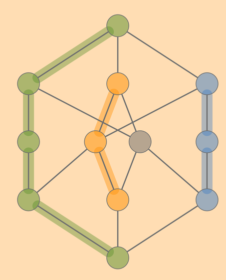
Some Examples



A non-Sperner poset.



A Sperner poset that is not 2-Sperner.



A strongly Sperner poset with a symmetric chain decomposition.

The Groups $G(d,d,n)$

For $d, n \geq 1$, the group $G(d,d,n)$ consists of monomial $(n \times n)$-matrices whose non-zero entries are d^{th} roots of unity, and whose product of non-zero entries is 1. We can view these groups as subgroups of the symmetric group $\mathfrak{S}_{dn}$, where the underlying set consists of n integers each appearing in d different colors.
Define $\left(\left(k_1^{(t_1)} \;\ldots\; k_r^{(t_r)}\right)\right) = \left(k_1^{(t_1)} \;\ldots\; k_r^{(t_r)}\right) \cdots \left(k_1^{(t_1+d-1)} \;\ldots\; k_r^{(t_r+d-1)}\right)$, and $\left[k_1^{(t_1)} \;\ldots\; k_r^{(t_r)}\right]_s = \left(k_1^{(t_1)} \;\ldots\; k_r^{(t_r)} \; k_1^{(t_1+s)} \; k_r^{(t_r+s)} \;\ldots\; k_1^{(t_1(d-1)s)} \;\ldots\; k_r^{(t_r+(d-1)s)}\right)$.

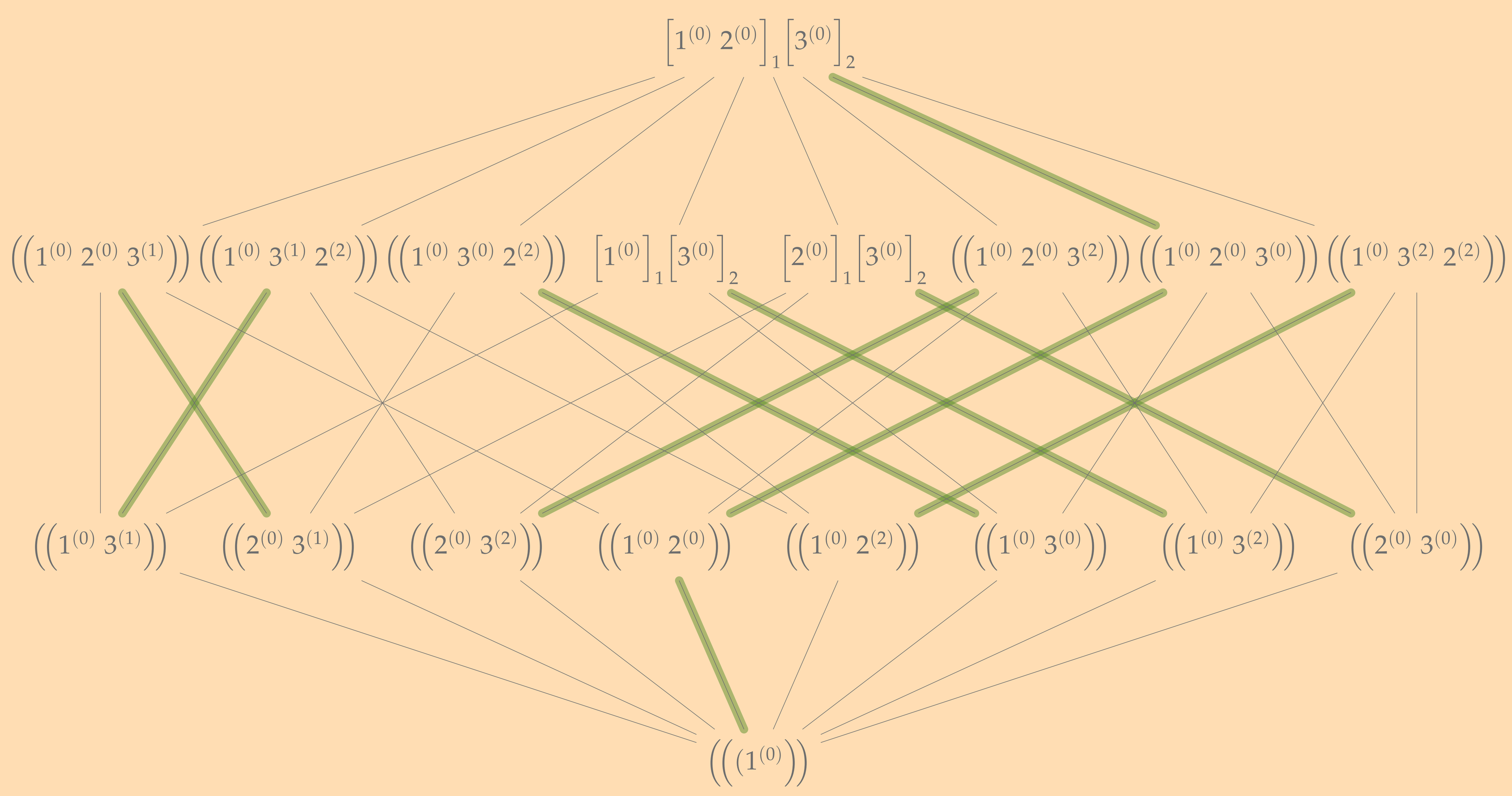
Facts
The group $G(1,1,n)$ is isomorphic to the symmetric group $\mathfrak{S}_n$.
The group $G(d,d,n)$ is isomorphic to an index-d subgroup of the wreath product $\boldsymbol{\mu}_d \wr \mathfrak{S}_n$, where $\boldsymbol{\mu}_d$ is the cyclic group of d^{th} roots of unity.

The Motivation

For $d = 1$, define $R_k = \{x \in NC_{G(1,1,n)} \mid x(1) = k\}$; let $\mathcal{R}_k = (R_k, \leq_T)$. Let $\mathbf{2}$ denote the 2-chain, and let $\uplus$ denote disjoint union.

Theorem 1 (R. Simion & D. Ullmann, 1991) *For $n \geq 1$, we have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, and $\mathcal{R}_k \cong \mathcal{NC}_{G(1,1,k-2)} \times \mathcal{NC}_{G(1,1,n-k+1)}$ whenever $3 \leq k \leq n$. Consequently, $\mathcal{NC}_{G(1,1,n)}$ admits a symmetric chain decomposition.*

Example: $d = n = 3$



Noncrossing Partition Lattices Associated with Well-Generated Complex Reflection Groups

The groups $\{G(d,d,n)\}_{d,n\geq 1}$ are irreducible **well-generated complex reflection groups**. There is one other infinite family of such groups, denoted by $\{G(d,1,n)\}_{d,n\geq 2}$, and 26 exceptional ones. We can define a **noncrossing partition lattice** $\mathcal{NC}_W$, for W being one of these groups, analogously as before.

Theorem 2 (V. Reiner, 1997) *For $d, n \geq 2$, the lattice $\mathcal{NC}_{G(d,1,n)}$ admits a symmetric chain decomposition.*

In principle, one could try to prove the existence of a symmetric chain decomposition for the noncrossing partition lattices associated with the irreducible exceptional well-generated complex reflection groups by computer. This is, however, quite a hard problem. Nevertheless, we managed to prove the strong Sperner property using a decomposition argument.

The Posets $\mathcal{NC}_{G(d,d,n)}$

$G(d,d,n)$ is generated by $T = \left\{\left(\left(a^{(0)}\; b^{(s)}\right)\right) \mid 1 \leq a < b \leq n, 0 \leq s < d\right\}$; let ℓ_T be the corresponding length function. For $x, y \in G(d,d,n)$ define $x \leq_T y$ if and only if $\ell_T(y) = \ell_T(x) + \ell_T(x^{-1}y)$.

The Case $d = 1$
Let $c = (1\;2\;\ldots\;n)$, and define $NC_{G(1,1,n)} = \{x \in G(1,1,n) \mid x \leq_T c\}$; let $\mathcal{NC}_{G(1,1,n)} = \left(NC_{G(1,1,n)}, \leq_T\right)$.

The Case $d > 1$
Let $\gamma = \left[1^{(0)}\;2^{(0)}\;\ldots\;(n-1)^{(0)}\right]_1\left[n^{(0)}\right]_{-1}$, and define $NC_{G(d,d,n)} = \{x \in G(d,d,n) \mid x \leq_T \gamma\}$; let $\mathcal{NC}_{G(d,d,n)} = \left(NC_{G(d,d,n)}, \leq_T\right)$.

The Main Result

For $d > 1$, define $R_k^{(s)} = \left\{x \in G(d,d,n) \mid x(1^{(0)}) = k^{(s)}\right\}$; let $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$. Let $\emptyset$ denote the empty poset.

Lemma 3 (, 2015) *For $d, n \geq 2$, we have the following isomorphisms:*
- $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)} \cong \mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$;
- $\mathcal{R}_1^{(1)} \uplus \mathcal{R}_2^{(d-1)} \cong \mathcal{NC}_{G(1,1,n-2)} \uplus \mathcal{NC}_{G(1,1,n-2)}$;
- $\mathcal{R}_n^{(s)} \cong \mathcal{NC}_{G(1,1,n-1)}$, *for* $0 \leq s < d$;
- $\mathcal{R}_k^{(0)} \cong \mathcal{NC}_{G(d,d,n-k+1)} \times \mathcal{NC}_{G(1,1,k-2)}$, *for* $3 \leq k < n$;
- $\mathcal{R}_k^{(d-1)} \cong \mathcal{NC}_{G(d,d,n-k)} \times \mathcal{NC}_{G(1,1,k-1)}$, *for* $3 \leq k < n$;
- $\mathcal{R}_k^{(s)} = \emptyset$ *otherwise.*

Disconnected, asymmetric!

We therefore need to fix the parts $\mathcal{R}_1^{(1)}$ and $\mathcal{R}_2^{(d-1)}$. For this, observe that left-multiplication by $\left(\left(1^{(0)}\; n^{(d-2)}\right)\right)$ respectively $\left(\left(2^{(0)}\; n^{(0)}\right)\right)$ embeds $\mathcal{R}_1^{(1)}$ and $\mathcal{R}_2^{(d-1)}$ into $\mathcal{R}_n^{d-1}$. Denote these maps by f_1 and f_2, respectively, and define $\mathcal{D}_1 = \mathcal{R}_1^{(1)} \uplus f(\mathcal{R}_1^{(1)})$, $\mathcal{D}_2 = \mathcal{R}_2^{(d-1)} \uplus f_2(\mathcal{R}_2^{(d-1)})$, and $\mathcal{D} = \mathcal{R}_n^{(d-1)} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.

Lemma 4 (, 2015) *For $d, n \geq 2$, we have $\mathcal{D}_1 \cong \mathcal{D}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$, and $\mathcal{D} \cong \biguplus_{k=3}^{n-1} \mathcal{NC}_{G(1,1,k-2)} \times \mathcal{NC}_{G(1,1,n-k)}$.*

Theorem 5 (, 2015) *For $d, n \geq 2$, the poset $\mathcal{NC}_{G(d,d,n)}$ admits a symmetric chain decomposition.*

The Decomposition Argument

Let $\mathcal{P} = (P, \leq)$ be a graded poset of rank n with rank numbers $r_0, r_1, \ldots, r_n$. Say that s is the index of the largest rank number, and let R be the set of poset elements of rank s. Let $P[1] = P \setminus R$, and more generally, define $P[k] = \underbrace{(\cdots((P[1])[1])\cdots)[1]}_{k \text{ times}}$. Set $\mathcal{P}[k] = (P[k], \leq)$.

Proposition 6 (, 2015) *A graded poset $\mathcal{P}$ of rank n is strongly Sperner if and only if $\mathcal{P}[k]$ is 1-Sperner for each $k \in \{0, 1, \ldots, n\}$.*

In order to check whether a poset is 1-Sperner one basically needs to compute the size of the largest antichain, and there are fast algorithms for that.

Theorem 7 (, 2015) *The lattice $\mathcal{NC}_W$ is strongly Sperner for each well-generated complex reflection group W.*