Parabolic Tamari Lattices in Linear Type B (arXiv:2112.13400)

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Goal

Provide a **combinatorial model** of the construction of **parabolic Tamari lattices in type B**.
Starting point: Universal algebraic construction of parabolic Tamari posets for all types (Mühle and Williams 2019), universal construction of Tamari (Cambrian) lattices for all types, defined on c-aligned elements, combinatorial constructions using pattern-avoidance for classical Coxeter groups (Reading 2007).

Type-B permutations and Coxeter group

Type-B permutations: permutations π of $\pm[n] \stackrel{\text{def}}{=} \{-n, \dots, -1, 1, \dots, n\}$ that are sign-symmetric, *i.e.*, $\pi(-i) = -\pi(i)$. We also denote $-i$ by $\bar{i}$.
One-line notation: $\pi = \bar{9}\,\bar{7}\,\bar{8}\,5\,\bar{6}\,1\,\bar{3}\,\bar{4}\,2 \mid \bar{2}\,4\,3\,\bar{1}\,6\,\bar{5}\,8\,7\,9$.
They form the Coxeter group of type B, also called the hyperoctahedral group $\mathfrak{H}_n$.

Weak order on type-B Coxeter groups

Inversion of $\pi \in \mathfrak{H}_n$: indices $i, j \in \pm[n]$ with $i < j$ and $\pi(i) > \pi(j)$
By sign-symmetry, if i, j is an inversion of $\pi \in \mathfrak{H}_n$, then $-j, -i$ too.
Thus we denote it by $(\!(i\ j)\!)$ with $0 < i < j$ or $0 < j < -i$, and $[\![i]\!]$ when $j = -i$
Inversion set of π: set of inversions of π, denoted by $\mathsf{Inv}(\pi)$
Example:

$$\pi = \bar{4}\,\bar{3}\,\bar{5}\,1\,2 \mid \bar{2}\,\bar{1}\,5\,3\,4 \Rightarrow \mathsf{Inv}(\pi) = \{[\![1]\!], [\![2]\!], (\!(-2\ 1)\!), (\!(3\ 4)\!), (\!(3\ 5)\!)\}$$

Weak order (left), type B: $\pi \leq_{\mathsf{weak}} \sigma \Leftrightarrow \mathsf{Inv}(\pi) \subseteq \mathsf{Inv}(\sigma)$
Example:

$$\bar{4}\,5\,3\,\bar{1}\,2 \mid \bar{2}\,1\,3\,5\,4 \leq_{\mathsf{weak}} \bar{4}\,\bar{3}\,\bar{5}\,1\,2 \mid \bar{2}\,\bar{1}\,5\,3\,4$$

Generators and parabolic subgroups

- Generators: $S = \{s_0, s_1, \dots, s_{n-1}\}$
  - For $i \geq 1$, s_i exchanges i and $i+1$ (thus $-i$ and $-i-1$ as well);
  - s_0 exchanges 1 and -1.
- Type-B composition: $\alpha = (\alpha_1, \dots, \alpha_k)$, with possibly $\alpha_1 = 0$
- α is split when $\alpha_1 = 0$, join otherwise.
- Parabolic subgroup of $\mathfrak{H}_n$: generated by s_i except when $i = \alpha_1 + \cdots + \alpha_j$ for some j
- Parabolic quotient $\mathfrak{H}_\alpha$: formed by permutations that are increasing in each region.

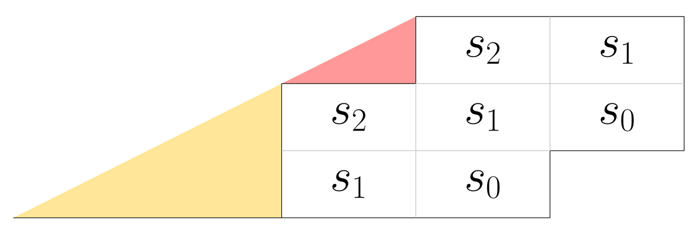
$\alpha = (0, 2, 1, 4, 2)$ (split) $\bar{9}\ \bar{8} \mid \bar{3}\ \bar{1}\ 5\ 6 \mid 7 \mid \bar{2}\ 4 \mid \bar{4}\ 2 \mid \bar{7} \mid \bar{6}\ \bar{5}\ 1\ 3 \mid 8\ 9$ (increasing)

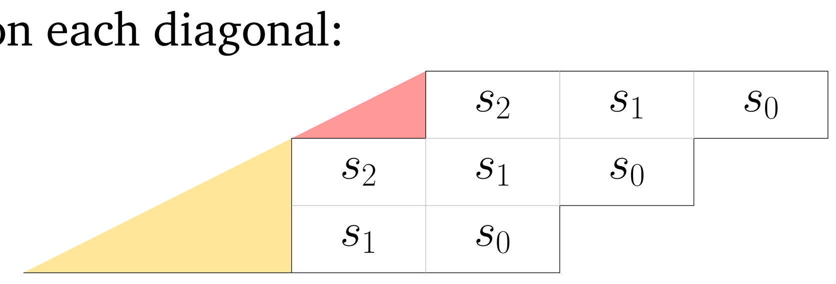
$\alpha = (2, 1, 4, 2)$ (join) $\bar{9}\ \bar{8} \mid \bar{7}\ \bar{4}\ 1\ 2 \mid 6 \mid \bar{5}\ \bar{3} \mid 3\ 5 \mid \bar{6} \mid \bar{2}\ \bar{1}\ 4\ 7 \mid 8\ 9$ (increasing)

- Weak order on $\mathfrak{H}_\alpha$: restriction of $\leq_{\mathsf{weak}}$ on $\mathfrak{H}_\alpha$

Algebraic construction of type-B Tamari lattice

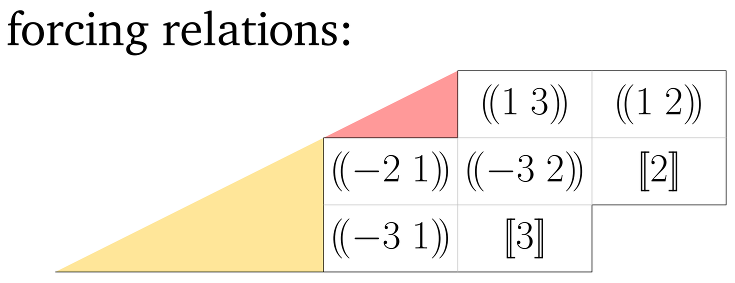
Type B: take the Coxeter element $c = s_{n-1}s_{n-2}\cdots s_1 s_0$
π is c-aligned $\Leftrightarrow$ π satisfies the forcing relations: every $t \in \mathsf{Cov}(\pi)$ implies several other $t' \in \mathsf{Inv}(\pi)$, determined by a linear order of inversions given by the c-sorting word of the longest element in $\mathfrak{H}_n$, and also by positive linear combinations of the root related to t.
Type B, **parabolic**: replace the longest element in $\mathfrak{H}_n$ by that in $\mathfrak{H}_\alpha$, denoted by $\omega_{\mathsf{o};\alpha}$.
The c-sorting word $u = u_1 \dots u_k$ can be read from left to right, bottom to top, in a skew tableau filled with the same s_i on each diagonal:

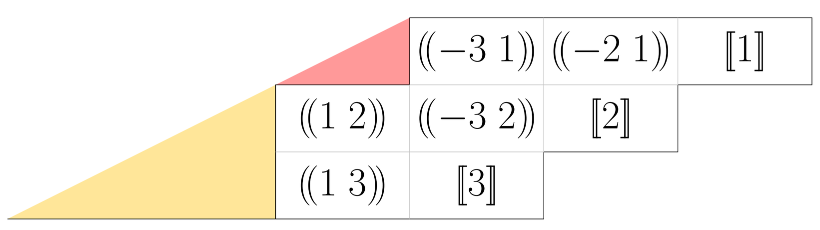




$\alpha = (1, 2) \Rightarrow u = s_1 s_0 \mid s_2 s_1 s_0 \mid s_2 s_1$ $\quad$ $\alpha = (0, 1, 2) \Rightarrow u = s_1 s_0 \mid s_2 s_1 s_0 \mid s_2 s_1 s_0$

All inversions of $\pi \in \mathfrak{H}_\alpha$ take the form $t_i = u_k \dots u_{k-i+1} \dots u_k$, and the order is given by $t_1 \prec \cdots \prec t_k$. Putting each inversion t_i in the cell of u_{k-i+1} helps computing the forcing relations:





Equivalent combinatorial construction

Type-B $(\alpha, 231)$-pattern in $\pi \in \mathfrak{H}_\alpha$: indices $i < j < k$ in $\pm[n]$ with $j > 0$ and i, j, k in different regions such that

- $\pi(i) = \pi(k) + 1$ or $\pi(i) = -\pi(k) = 1$; (cover inversion)
- $\pi(j) > \pi(i)$ when α is split or $j > \alpha_1$; (231)
- $\pi(j) < \pi(k)$ when α is join and $j \leq \alpha_1$. (312)

Split case:

Pattern $4 \mid \bar{7}\ \bar{3} \mid \mathbf{1} \mid \bar{6}\ \bar{2}\ 5 \mid \bar{5}\ 2\ \mathbf{6} \mid \mathbf{\bar{1}} \mid 3\ 7 \mid \bar{4}$

Pattern $4 \mid \bar{5}\ \bar{1} \mid 6 \mid \bar{2}\ 3\ 7 \mid \bar{7}\ \mathbf{\bar{3}}\ 2 \mid \bar{6} \mid 1\ \mathbf{5} \mid \mathbf{\bar{4}}$

Join case:

Not pattern $\mathbf{\bar{6}}\ \bar{4} \mid \bar{8}\ 7 \mid \bar{5}\ \bar{3}\ \bar{2}\ \bar{1}\ \mathbf{1}\ 2\ 3\ 5 \mid \mathbf{\bar{7}}\ 8 \mid 4\ 6$

Pattern $\bar{7}\ \mathbf{5} \mid \bar{4}\ 3 \mid \bar{8}\ \bar{6}\ \bar{2}\ \bar{1}\ 1\ \mathbf{2}\ 6\ 8 \mid \bar{3}\ \mathbf{4} \mid \bar{5}\ 7$

$\mathfrak{H}_\alpha(231)$: the set of type-B $(\alpha, 231)$-avoiding permutations
Type-B parabolic Tamari lattice $\mathsf{Tam}_B(\alpha)$: restriction of the weak order to $\mathfrak{H}_\alpha(231)$

Main result 1: $\mathsf{Tam}_B(\alpha)$ is a lattice

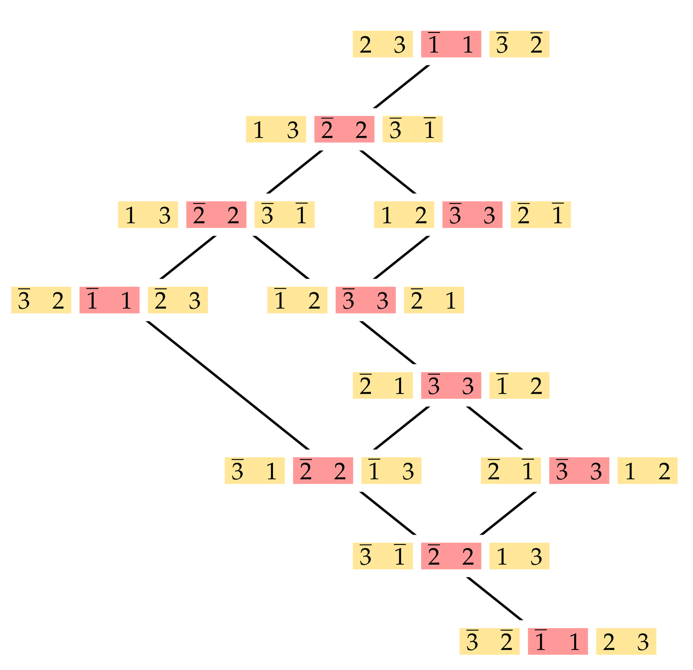
Theorem. *For every type-B composition α, $\mathsf{Tam}_B(\alpha)$ is a lattice. Moreover, it is a quotient lattice of the weak order on the parabolic quotient $\mathfrak{H}_\alpha$.*

Congruence classes defined by downward projection $\Pi_\downarrow$:

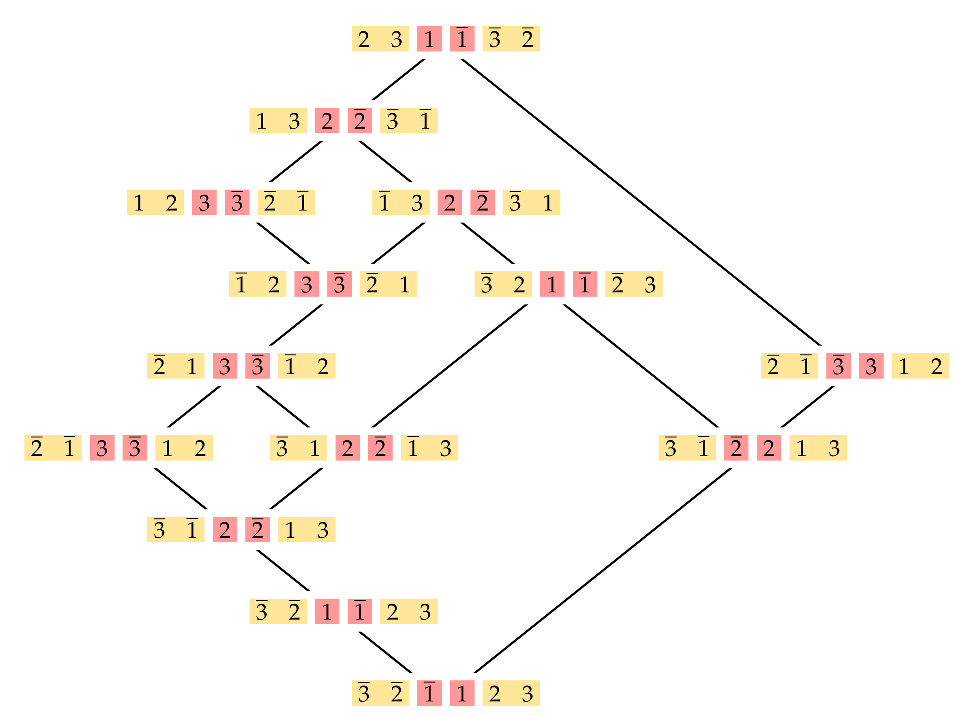
1. For each $(\alpha, 231)$-pattern i, j, k, exchange $\pi(i)$ and $\pi(k)$.
2. Repeat Step 1 until no such pattern exists.

$\Pi_\downarrow$ gives the smallest element in the class. We also define an upward projection $\Pi_\uparrow$ using $(\alpha, 312)$-patterns, giving the largest element.
These projections define the same poset on respectively the $(\alpha, 231)$- and the $(\alpha, 312)$-avoiding permutations.

Examples of type-B parabolic Tamari lattices



$\alpha = (1, 2)$



$\alpha = (0, 1, 2)$

Main result 2: lattice properties of $\mathsf{Tam}_B(\alpha)$

Theorem. *For every type-B composition α, $\mathsf{Tam}_B(\alpha)$ is congruence uniform and trim.*

- Congruence uniform: quotient lattice of $\mathfrak{H}_n$
- Semi-distributive: from congruence uniformity
- Extremal: explicit counting of join-irreducibles and the length of the lattice

$\omega_{\mathsf{o};(0,2,1,4,2)} = 8\ 9 \mid 4\ 5\ 6\ 7 \mid 3 \mid 1\ 2 \mid \bar{2}\ \bar{1} \mid \bar{3} \mid \bar{7}\ \bar{6}\ \bar{5}\ \bar{4} \mid \bar{9}\ \bar{8}$

$\omega_{\mathsf{o};(2,1,4,2)} = 8\ 9 \mid 4\ 5\ 6\ 7 \mid 3 \mid \bar{2}\ \bar{1}\ 1\ 2 \mid \bar{3} \mid \bar{7}\ \bar{6}\ \bar{5}\ \bar{4} \mid \bar{9}\ \bar{8}$

$$|\mathsf{Inv}(\omega_{\mathsf{o};\alpha})| = n^2 - \sum_i \binom{\alpha_i}{2} - \binom{\alpha_1 + 1}{2}.$$

- Trim: from extremality and semi-distributivity

References

H. Mühle and N. Williams, *Tamari Lattices for Parabolic Quotients of the Symmetric Group*, Electron. J. Combin. 26 (2019).

N. Reading, *Clusters, Coxeter-Sortable Elements and Noncrossing Partitions*, Trans. Amer. Math. Soc. 359 (2007).