On the Topology of the Cambrian Semilattices

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Cambrian Semilattices

For an arbitrary Coxeter group W and an arbitrary Coxeter element $\gamma \in W$, Reading and Speyer defined the γ-Cambrian semilattice $\mathcal{C}_\gamma$ as a sublattice of the weak order semilattice. Cambrian semilattices constitute generalizations of the Tamari lattice $\mathcal{T}_n$, to which they reduce when W is the symmetric group $\mathfrak{S}_n$ and γ is the long cycle $\gamma = (1\ 2\ \cdots\ n)$.

What is known – The Finite Case

If W is a finite Coxeter group, and $\gamma \in W$ is a Coxeter element, then $\mathcal{C}_\gamma$ is a lattice. Considering its topological properties it is known that:

- $\mathcal{C}_\gamma$ is Cohen-Macaulay (in fact it is EL-shellable), and
- every open interval of $\mathcal{C}_\gamma$ is either contractible or spherical.

Question

What can be said about the topology of $\mathcal{C}_\gamma$ in general? More precisely, can the previous results be generalized to infinite Coxeter groups W?

Results

We give an affirmative answer to the above question!
Let W be a (possibly infinite) Coxeter group and let $\gamma \in W$ be a Coxeter element.

Theorem 1

Every closed interval in $\mathcal{C}_\gamma$ is EL-shellable.

Theorem 2

Every finite open interval in $\mathcal{C}_\gamma$ is either contractible or spherical.

Definitions

Let W be a Coxeter group of rank n with simple generators $s_1, s_2, \ldots, s_n$, and let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element of W. Let $\gamma^\infty = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | \cdots$.
Every $w \in W$ can be written as a subword of γ^∞, in the form

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{k,1}} s_2^{\delta_{k,2}} \cdots s_n^{\delta_{k,n}}, \qquad (1)$$

where $\delta_{i,j} \in \{0,1\}$ and $k \geq 0$.

- *i-th block of w*: the set $b_i(w) = \{s_j \mid \delta_{i,j} = 1\}$
- *γ-sorting word of w*: the lexicographically first subword of γ^∞ among all reduced words for w
- *γ-sortable element*: some $w \in W$ such that the γ-sorting word of w satisfies $b_1(w) \supseteq b_2(w) \supseteq \cdots \supseteq b_k(w)$
- *γ-Cambrian semilattice $\mathcal{C}_\gamma$*: the sub-semilattice of the weak-order semilattice consisting of all γ-sortable elements

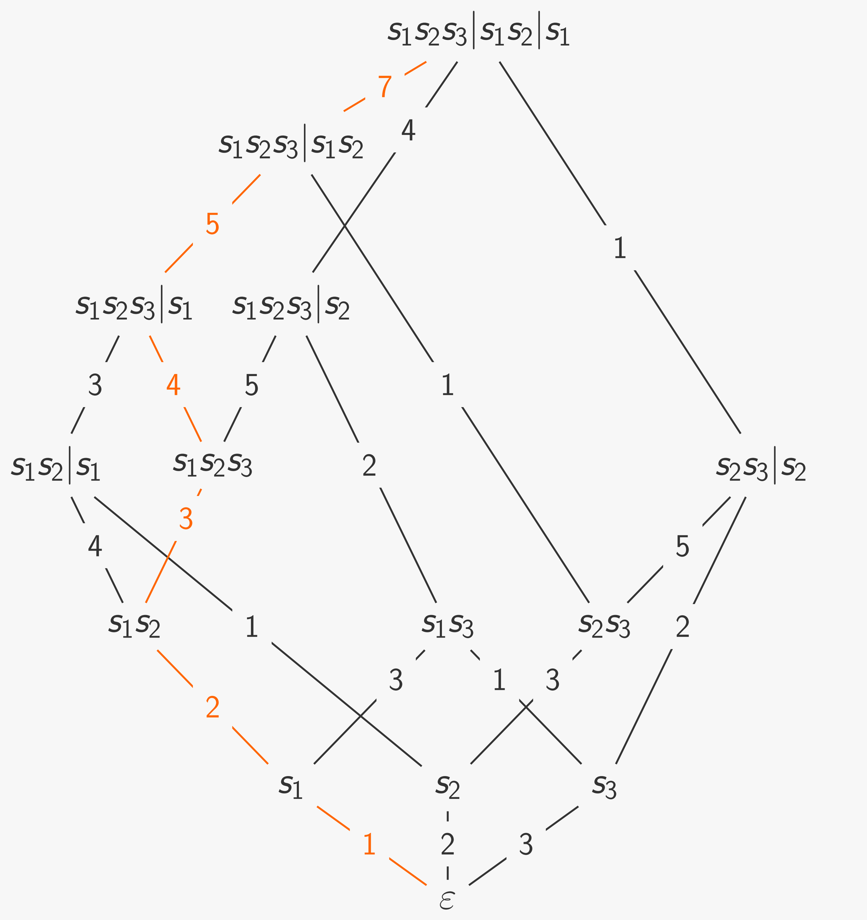
Example – γ-Sorting Words

Let $W = \mathfrak{S}_4$, generated by $s_i = (i\ i{+}1)$ for $i \in \{1,2,3\}$, and let $\gamma = s_1 s_2 s_3$. The following are reduced words of the same element $w \in W$:

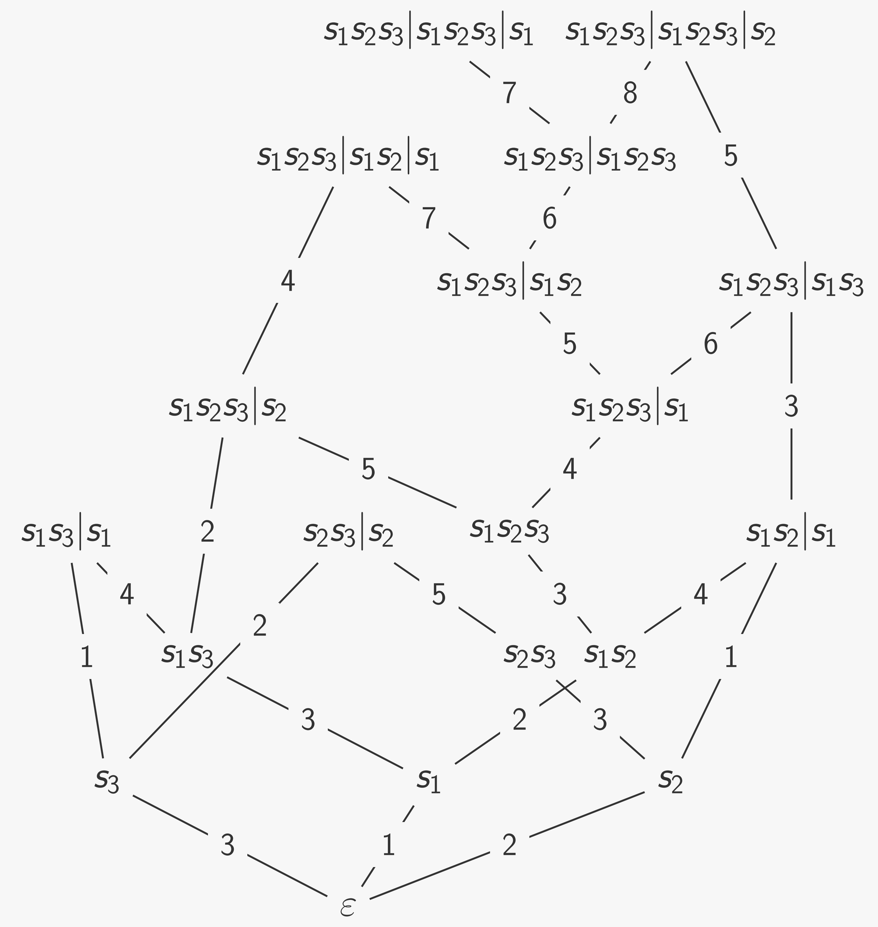
$\mathbf{w_1 = s_1 s_2 s_3 | s_1 s_2}$, $w_2 = s_1 s_2 | s_1 s_3 | s_2$, $w_3 = s_2 | s_1 s_2 s_3 | s_2$,
$w_4 = s_2 | s_1 s_3 | s_2 s_3$, $w_5 = s_2 s_3 | s_1 s_2 s_3$.

The γ-sorting word of w is w_1, and we have $b_1(w_1) = \{s_1, s_2, s_3\}$ and $b_2(w_1) = \{s_1, s_2\}$, with $b_1(w_1) \supseteq b_2(w_1)$.

Example – γ-Cambrian Semilattices



a. The Tamari lattice $\mathcal{T}_4$ is a Cambrian lattice of the symmetric group $\mathfrak{S}_4$.



b. A Cambrian semilattice of the affine Coxeter group $\tilde{A}_2$.

The Key Lemma

Lemma

Let $u, v \in \mathcal{C}_\gamma$ with $u \leq_\gamma v$. If $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$, then the join $s_1 \vee_\gamma u$ covers u in $\mathcal{C}_\gamma$.

Sketch of Proof – Theorem 1

Define the set of positions of the γ-sorting word of w as

$$\alpha_\gamma(w) = \{(i-1)\cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N},$$

where the $\delta_{i,j}$'s are the exponents from (1). Note that $\alpha_\gamma(w)$ depends on a reduced word for γ!

Denote by $\mathcal{E}(\mathcal{C}_\gamma)$ the set of covering relations of $\mathcal{C}_\gamma$, and define an edge-labeling of $\mathcal{C}_\gamma$ by

$$\lambda_\gamma : \mathcal{E}(\mathcal{C}_\gamma) \to \mathbb{N}, \quad (u,v) \mapsto \min\{i \mid i \in \alpha_\gamma(v) \smallsetminus \alpha_\gamma(u)\}.$$

Using induction on rank and length and the key lemma, we show that for every closed interval of $\mathcal{C}_\gamma$ there exists a unique rising maximal chain with respect to λ_γ which is lexicographically first among all maximal chains in this interval. Thus, λ_γ is an EL-labeling of $\mathcal{C}_\gamma$.

Example – The Labeling

Let $W = \mathfrak{S}_4$, and $\gamma = s_1 s_2 s_3$. Consider the following:

$$u = s_3 = s_1^0 s_2^0 s_3^1 \rightsquigarrow \alpha_\gamma(u) = \{3\}, \quad \text{and}$$
$$v = s_2 s_3 | s_2 = s_1^0 s_2^1 s_3^1 | s_1^0 s_2^1 s_3^0 \rightsquigarrow \alpha_\gamma(v) = \{2,3,5\}.$$

Thus, we have $\lambda_\gamma(u,v) = 2$.

Sketch of Proof – Theorem 2

A classical result on EL-shellable posets states that the dimension of the k-th homology group of the corresponding truncated order complex is given by the number of falling maximal chains of length $k+2$ (with respect to the EL-labeling).

Using induction on rank and length and the key lemma, we show that there exists at most one falling maximal chain in every closed interval of $\mathcal{C}_\gamma$.

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