RESEARCH STATEMENT

HENRI MÜHLE

BACKGROUND

Combinatorial families hardly ever arise as plain sets. Usually, they are naturally equipped with some sort of partial order that allows for comparison of its members. To name a few, words can be ordered lexicographically, integer partitions can be ordered by comparing components, and permutations can be ordered by inclusion of their inversion sets. More often than not, these partial orders help to understand the combinatorial objects, their intrinsic structure, and their relations to other combinatorial families.

Quite frequently it happens that these partial orders have a lattice structure, that is, for every two elements there exists a unique maximal element that is smaller than the two, and a unique minimal element that is greater than the two. This extra structure is helpful, because it enables us to approach these combinatorial families from an algebraic point of view so that we may form quotients or consider homomorphisms and substructures.

I like to describe my research area as **combinatorial lattice theory**, since most of the lattices arising in my work can be defined combinatorially, and many of their structural, topological and combinatorial properties can be obtained through certain suitable edge labelings. Most of the posets that I have worked with in the past years, arise from one of the following two constructions.

POSETS FROM GENERATED GROUPS

The first source has a very group-theoretic flavor. If G is a group that is generated by a (finite) set $A \subseteq G$, then we may use the *word length* (with respect to A) ℓ_A to define two partial orders on G:

- the *A-postfix order*: $g \leq_{\mathsf{post}} h$ if and only if $\ell_A(hg^{-1}) + \ell_A(g) = \ell_A(h)$;
- the *A-prefix order*: $g \leq_{\mathsf{pre}} h$ if and only if $\ell_A(g) + \ell_A(g^{-1}h) = \ell_A(h)$.

These partial orders recover orientations of the Cayley graph of G with respect to A.

From now on, we focus on the case, where A is closed under taking inverses and under conjugation. In this case, the A-postfix order and the A-prefix order actually agree and can be understood as a subword order on G. The identity of G is the unique minimal element in this order, and the maximal chains from the identity to some $g \in G$ are in bijection with the set of A-reduced words for g. We call the restriction of this order to the interval from the identity to g the *factorization poset* of g, denoted by $\mathbf{P}(g)$.

If $\ell_A(g) = n$, then the i^{th} generator σ_i of the n-strand braid group $\mathfrak{B}_n$ acts on a word $g = a_1a_2\cdots a_n$ by a *Hurwitz move*:

$$\begin{aligned} \sigma_i \cdot &(a_1, a_2, \ldots, a_{i-1}, \quad a_i, \quad a_{i+1}, \quad a_{i+2}, a_{i+3}, \ldots, a_n) \\ \stackrel{\mathsf{def}}{=} &(a_1, a_2, \ldots, a_{i-1}, \quad a_{i+1}, \quad a_{i+1}^{-1}a_ia_{i+1}, \quad a_{i+2}, a_{i+3}, \ldots, a_n). \end{aligned}$$

It is straightforward to verify that this action extends to an action of $\mathfrak{B}_n$ on the set $\mathsf{Red}_A(g)$ of A-reduced words for g. An important question, that goes back to Hurwitz' study of branched coverings of a Riemann surface [23] is whether this action is *transitive*, i.e. whether all A-reduced words for g lie in the same orbit of this action. Viewing this from the poset perspective, the number of *Hurwitz orbits* gives a "connectivity coefficient" for the element g. Indeed, if the Hurwitz action is transitive for g, then poset diagram of $\mathbf{P}(g)$ is connected (as a graph).

The prototypical example of a factorization poset comes from the setting where G is the symmetric group, A is the set of transpositions and g is a long cycle [8]. A natural extension of this setting considers a well-generated complex reflection group G, with A being the set of all reflections of G and g a Coxeter element. In this situation, it turns out that the factorization poset is always a lattice [6,9] and that the Hurwitz action is indeed transitive [6,19]. Moreover, these lattices have other beautiful topological and structural properties [1,30,31,47,48,52].

In [40], V. Ripoll and I have further studied the connectivity aspect of factorization posets by relating it to the topological concept of shellability. We have investigated the interaction of these two notions of connectivity, and we have given necessary graph-theoretical and combinatorial conditions.

The Hurwitz-transitivity of factorization posets arising from reflection groups for g not necessarily a Coxeter element was studied for instance in [4,5,29,54]. In my articles [38,39], I have extended this construction to subgroups of the symmetric group generated by k-cycles. With my collaborators, we have explicitly counted Hurwitz orbits for $k = 3$. We have combinatorially realized the factorization posets coming from a long cycle in terms of certain *noncrossing set partitions*.

Classically, a set partition of $[n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}$ is *noncrossing* if there do not exist indices $1 \leq a < b < c < d \leq n$ such that a and c belong to one part and b and d belong to another. Ordering noncrossing set partitions by refinement yields a lattice; the *noncrossing partition lattice* $\textsf{Nonc}(n)$ [27]. It turns out that $\textsf{Nonc}(n)$ is isomorphic to the factorization poset $\mathbf{P}(g)$ arising from the long cycle $g = (1\ 2\ \ldots\ n)$ in the symmetric group generated by all transpositions [7,8]. This is illustrated in the right poset in Figure 1.

Remarkably, if we drop the condition that the generating set is closed under conjugation, then we obtain another fascinating partial order when G is a reflection group; the *weak (Bruhat) order*. This brings us directly to the next source of examples.

Posets from Polytopes

We may associate a partial order with a polytope by assigning a cost function to the vertices that is not constant along edges. This yields an acyclic graph on the vertices of the polytope. My interest lies in the case where this graph *is* the poset diagram of some poset [22], and even more so if this poset is a lattice. From a very general point of view I am interested in lattice properties shared by all lattices arising in this manner, or at least in geometric properties of polytopes that ensure certain lattice properties.

For instance, all the lattices I am aware of that arise in this manner are *semidistributive*, i.e. for every three elements p, q, r it holds that

$$p \vee q = p \vee r \quad \text{implies} \quad (p \vee q) \wedge (p \vee r) = p \vee (q \wedge r),$$
$$p \wedge q = p \wedge r \quad \text{implies} \quad (p \wedge q) \vee (p \wedge r) = p \wedge (q \vee r).$$

A prototypical example of this construction is the *weak (Bruhat) order* on the symmetric group arising from a certain orientation of the permutohedron. Somewhat surprisingly, the weak order arises as a factorization poset, too, where G is the symmetric group, A is the set of *adjacent* transpositions and g is the reverse permutation.

The weak order on the symmetric group has an important quotient lattice: the *Tamari lattice* $\mathsf{Tam}(n)$. The Tamari lattice arises from a certain orientation of the associahedron, and can be defined via a rotation operation on binary trees. This construction generalizes nicely to real reflection groups. The resulting polytopes are generalized associahedra and are related (by duality) to the cluster complexes introduced in [21], and the resulting lattices are the *Cambrian lattices* of [43]. I have studied topological and structural properties of Cambrian lattices in [25, 32], and other publications investigating these lattices are for instance [24, 44, 46].

Other families of polytopes, exhibiting a behavior similar to the associahedra, are the freehedra [17, 49, 50], the ν-associahedra [12, 13], the grid-associahedra [28], Stokes polytopes [3, 16], Grassmann-associahedra [51] or graph associahedra [2]. The lattices arising from freehedra are called *Hochschild lattices*, and I have studied enumerative and structural aspects of them in [36]. In the following months, I plan to investigate lattices of shuffles and lattices of synchronized Tamari intervals which arise in an analogous fashion.

Another family of lattices arising as quotients of the weak order appear in the context of parabolic quotients of Coxeter groups. I am very active in the exploration of this area [10, 20, 26, 34, 37, 41], and pursuing these constructions is one of my main research objectives in the near future.

Connections among these Posets

N. Reading explained in [45] that the noncrossing partition lattice arises via a certain geometrically defined reordering of the Tamari lattice. This construction can be abstracted in purely lattice-theoretic terms. We consider a finite lattice $\mathbf{P} = (P, \leq)$ together with an edge labeling λ, and for any lattice element $p \in P$ we define its *nucleus* by

$$p_{\downarrow} \overset{\mathsf{def}}{=} p \wedge \bigwedge_{p' \lessdot p} p'.$$

The *core* of p is the interval $[p_{\downarrow}, p]$ and the *core label set* associated with p is the set $\Psi(p)$ of labels appearing in the core of p. If λ has the property that the assignment $p \mapsto \Psi(p)$ is injective, then we may define the core label order on $\mathbf{P}$ by $\mathsf{CLO}(\mathbf{P}) \overset{\mathsf{def}}{=} (P, \leq_{\mathsf{clo}})$, where $p \leq_{\mathsf{clo}} q$ if and only if $\Psi(p) \subseteq \Psi(q)$. In particular, when $\mathbf{P}$ is a semidistributive lattice, there exists a natural edge labeling which allows for a definition of a core label order. In many cases, when $\mathbf{P}$ is a combinatorially defined lattice, the core label order has remarkable combinatorial properties, see for instance [36, 37, 45].

The definition of the nucleus was independently discovered in [18] in the context of dynamical systems, and has its origins in Reading's geometric construction of shard intersections [45]. I have studied the core label order for congruence-uniform and meet-distributive lattices [33, 35]. Figure 1 shows the core label order of $\mathsf{Tam}(3)$ and how it realizes $\mathsf{Nonc}(3)$.

In recent work with C. Ceballos [11], we have given an elementary explanation of a relationship between two bivariate polynomials, the *F*- and the *H-triangle*, arising on ν-associahedra. Originally, such polynomials were first considered by F. Chapoton in the context of cluster complexes, root posets and noncrossing partition lattices [14, 15]. The corresponding relation, which is some sort of *combinatorial reciprocity*, was proven by M. Thiel by means of differential equations and essentially generalizes the well-known relation between the f- and the h-polynomial of a polytopal complex [53].

Our proof is completely combinatorial and has the advantage that it *explains* this relation. In essence, the polynomials in question depend on a marking of the edges of the ν-associahedron and

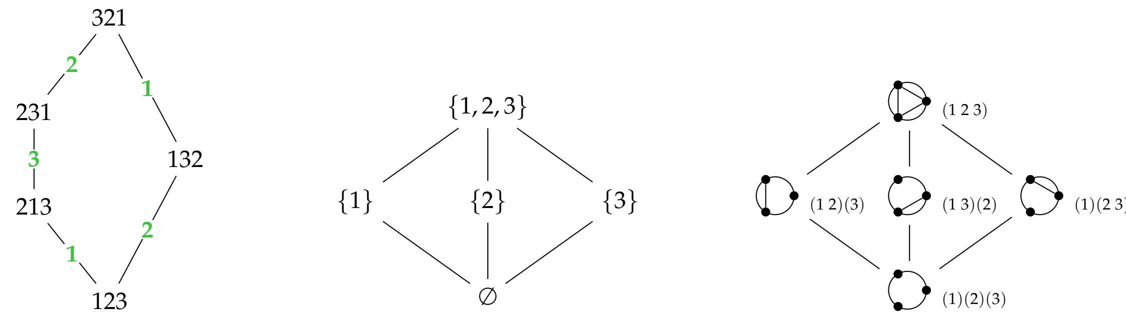

Figure 1. A labeled Tamari lattice, the corresponding core label order, and a noncrossing partition lattice.

constitute a refined face enumeration in two different ways. This construction extends straightforwardly to arbitrary edge labelings of finite posets, and was for instance applied to the Hochschild lattices [36]. It is a challenging task to exhibit natural edge labelings for other posets arising from polytopes which have a similar combinatorial impact. The underlying combinatorial reciprocity generalizes the famous Dehn–Sommerville relations.

For a northeast path ν, the *ν-Tamari lattice* $\mathsf{Tam}(\nu)$ is a semidistributive lattice arising from an orientation of the ν-associahedron. Its core label order is a generalization of the noncrossing partition lattice, which arises when ν is a staircase path. Once again, motivated by a construction of Chapoton's, we may consider a bivariate variant of the characteristic polynomial of $\mathsf{CLO}\bigl(\mathsf{Tam}(\nu)\bigr)$. For some paths ν, this polynomial can be obtained from the corresponding F- and H-triangles. I am currently investigating this connection which is conjecturally closely related to the *pureness* of the ν-associahedra. This adds a geometric flavor to the core label construction, and may impact applications of ν-associahedra, for instance in the context of diagonal harmonics [42].

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