

Two Posets of Noncrossing Partitions

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Eurocomb, Vienna

Outline

Two Posets of
Noncrossing
Partitions

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Noncrossing
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A Subposet of
Noncrossing
Partitions

1 Noncrossing Partitions

2 A Subposet of Noncrossing Partitions

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- **noncrossing partition**

$\rightsquigarrow NC_n$

Noncrossing Partitions

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- **noncrossing partition**

$\rightsquigarrow NC_n$

$$\left\{ \{1, 6, 7\}, \{2, 8, 14, 15\}, \{3, 4, 5\}, \{9, 10, 12, 13\}, \{11\}, \{16\} \right\}$$

Noncrossing Partitions

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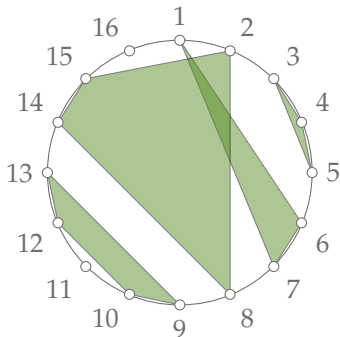
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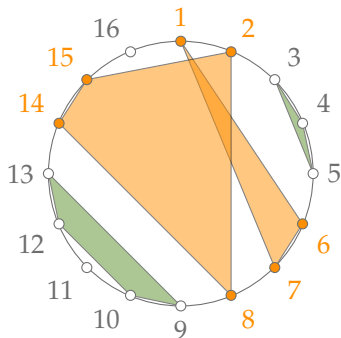
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Nope!

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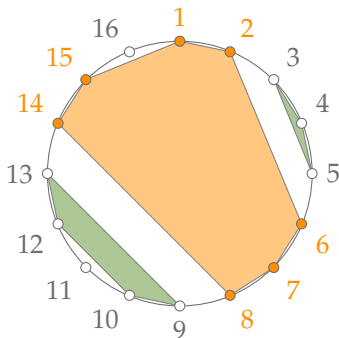
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First Properties

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Theorem (G. Kreweeras, 1972)

For $n \geq 0$, the cardinality of NC_n is

$$\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}.$$

Noncrossing Partitions

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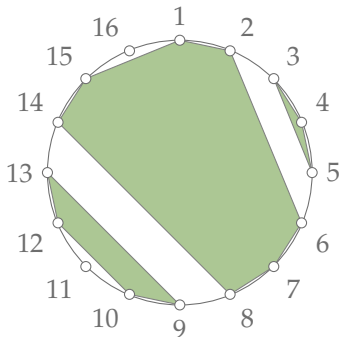
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● dual refinement order

$\rightsquigarrow \leq_{\text{dref}}$



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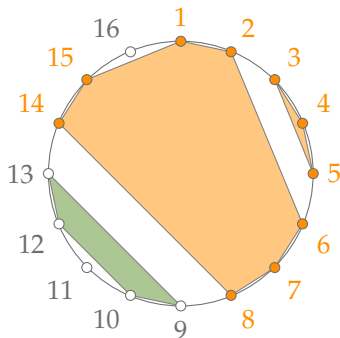
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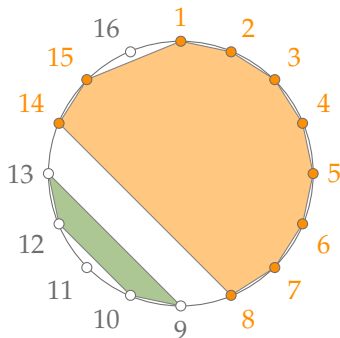
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- $\mathcal{NC}_n = (\mathcal{NC}_n, \leq_{\text{dref}})$
- let $\mathbf{0} = 1|2| \cdots |n$ and $\mathbf{1} = 12 \cdots n$
- let $[n] = \{1, 2, \dots, n\}$

Theorem (G. Kreweras, 1972)

For $n \geq 0$, the poset $\mathcal{NC}_n$ is a lattice.

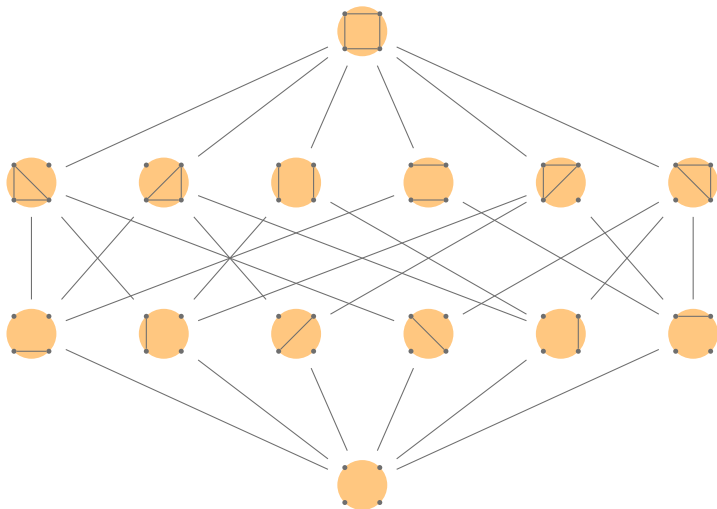
Example: $\mathcal{NC}_4$

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Supersolvability

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- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- **M -chain**: the chain D above

Supersolvability

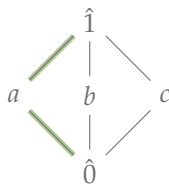
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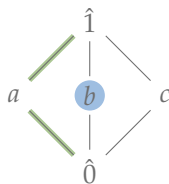
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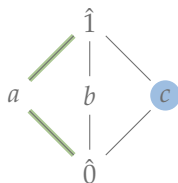
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Theorem (P. Hersh, 1999)

For $n \geq 1$, the lattice $\mathcal{NC}_n$ is supersolvable.

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- x_i .. noncrossing partition with only non-singleton block $[i - 1] \cup \{n\}$

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Proposition (✂, 2017)

For $n \geq 1$, the chain $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is an M-chain of $\mathcal{NC}_n$.

Möbius Function

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- $\mathcal{P} = (P, \leq)$.. a (finite) poset
- **Möbius function:** the map $\mu_{\mathcal{P}} : P \times P \rightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x, y) = \begin{cases} 1, & \text{if } x = y, \\ - \sum_{x \leq z < y} \mu_{\mathcal{P}}(x, z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

Möbius Function

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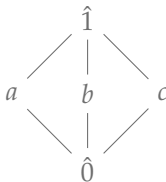
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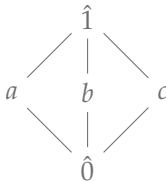
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$$- \mu_{\mathcal{P}}(\hat{0}, \hat{1}) = \mu_{\mathcal{P}}(\hat{0}, \hat{0}) + \mu_{\mathcal{P}}(\hat{0}, a) + \mu_{\mathcal{P}}(\hat{0}, b) + \mu_{\mathcal{P}}(\hat{0}, c)$$

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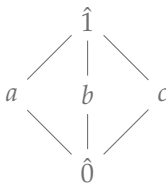
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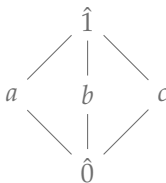
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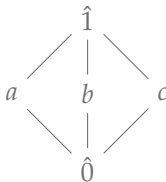
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$$- \mu_{\mathcal{P}}(\hat{0}, \hat{1}) = -2\mu_{\mathcal{P}}(\hat{0}, \hat{0})$$

Möbius Function

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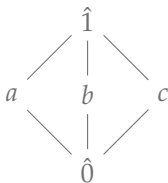
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$$\mu_{\mathcal{P}}(\hat{0}, \hat{1}) = 2\mu_{\mathcal{P}}(\hat{0}, \hat{0})$$

Möbius Function

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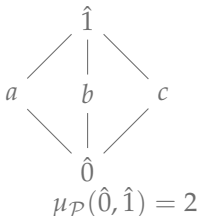
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Theorem (G. Kreweras, 1972)

For $n \geq 1$ we have

$$\mu_{\mathcal{NC}_n}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} \text{Cat}(n-1).$$

NBB-Bases of $\mathcal{NC}_n$

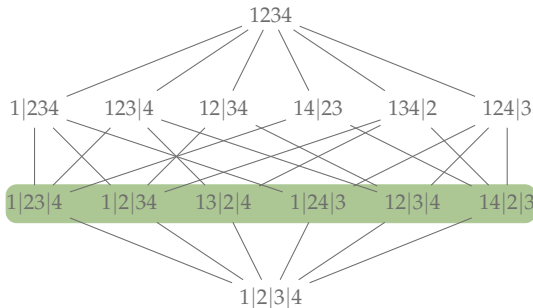
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- $\mathbf{a}_{i,j}$.. noncrossing partition with only non-singleton block $\{i,j\}$
- $\mathcal{A}_n = \{\mathbf{a}_{i,j} \mid 1 \leq i < j \leq n\}$
- let $\leq$ be any partial order on $\mathcal{A}_n$; $X \subseteq \mathcal{A}_n$



$\mathcal{A}_4$

NBB-Bases of $\mathcal{NC}_n$

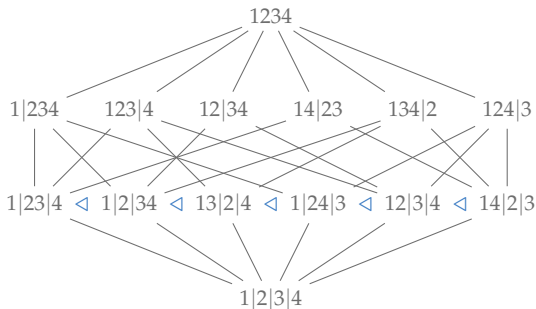
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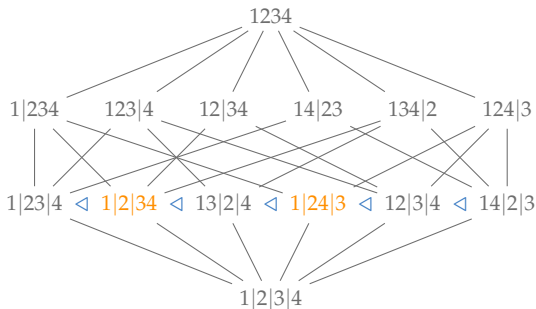
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- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \triangleleft \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$



NBB-Bases of $\mathcal{NC}_n$

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BB

NBB-Bases of $\mathcal{NC}_n$

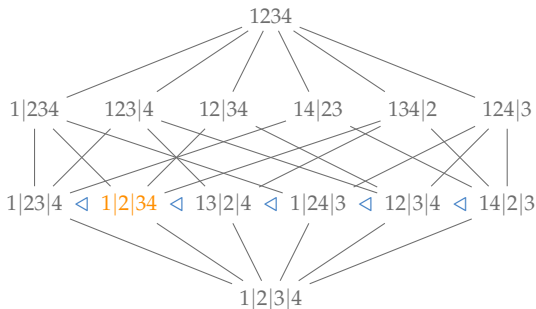
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not BB

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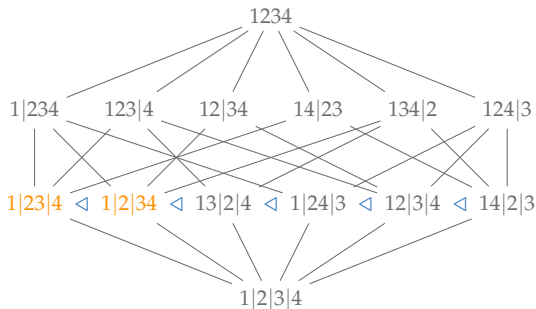
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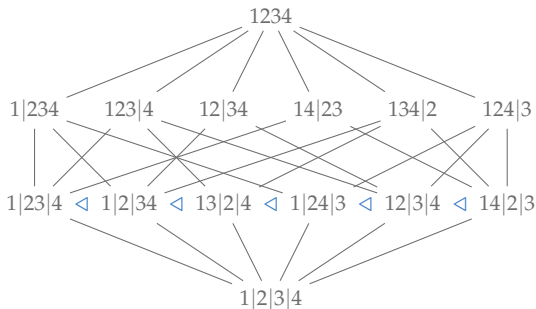
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- **NBB**: no nonempty subset of X is BB



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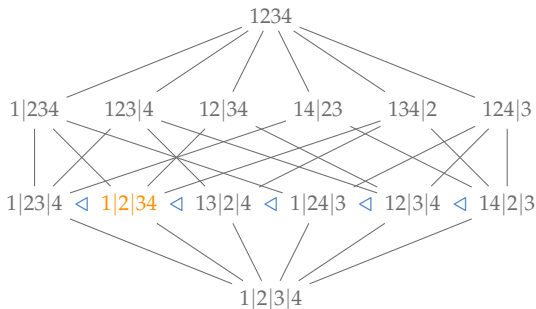
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NBB

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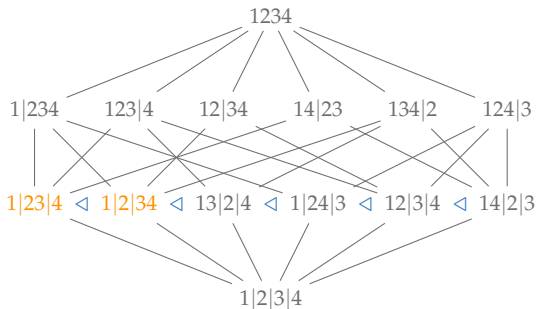
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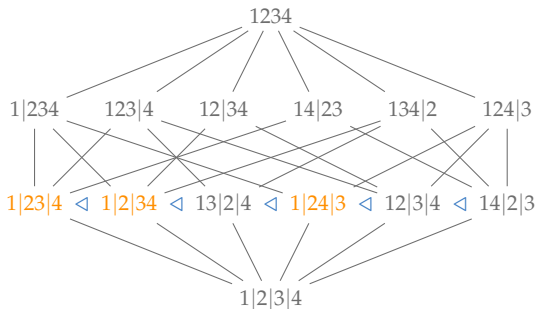
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NBB-Bases of $\mathcal{NC}_n$

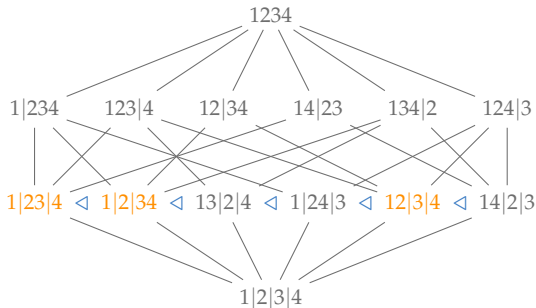
Two Posets of
Noncrossing
Partitions

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Noncrossing
Partitions

A Subposet of
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- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \triangleleft \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB
- **NBB-base** for $\mathbf{x}$: X is NBB and $\bigvee X = \mathbf{x}$



NBB-base for 1

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Theorem (A. Blass, B. Sagan, 1997)

Let $\mathcal{P} = (P, \leq)$ be a finite lattice and $\trianglelefteq$ any partial order on the atoms of $\mathcal{P}$. For $x \in P$ we have

$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_{\mathbf{X}} (-1)^{|\mathbf{X}|},$$

where the sum runs over the NBB-bases for x .

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- **NBB**: no nonempty subset of X is BB
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Corollary (The Crosscut Theorem; G.-C. Rota, 1964)

Let $\mathcal{P} = (P, \leq)$ be a finite lattice. For $x \in P$ we have

$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_X (-1)^{|X|},$$

where the sum runs over all subsets of atoms of $\mathcal{P}$ whose join is x .

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

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- subsets of $\mathcal{A}_n$ correspond to certain graphs on $[n]$



NBB-Bases for 1 in $\mathcal{NC}_n$

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- let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be the M -chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\text{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\text{dref}} \mathbf{x}_{i+1}\}$

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- let $\mathbf{a} \leq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$

NBB-Bases for 1 in $\mathcal{NC}_n$

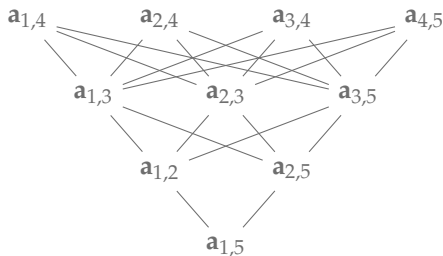
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NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

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Proposition (✂, 2017)

For $n \geq 1$ the NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$ are precisely those maximal chains of $(\mathcal{A}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \dots, n\}$ for some $k \in [n-1]$.

NBB-Bases for 1 in $\mathcal{NC}_n$

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- let $\mathbf{a} \leq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$

Corollary

For $n \geq 1$ we have

$$\mu_{\mathcal{NC}_n}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} \text{Cat}(n-1).$$

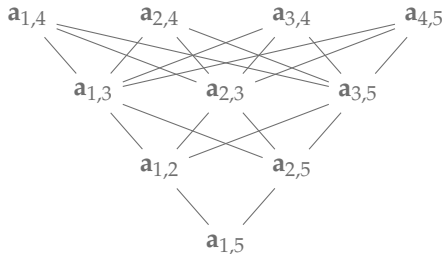
Example: NBB-Bases for 1 in $\mathcal{NC}_5$

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$$\{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{1,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{3,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{4,5}\}$$

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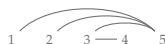
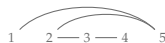
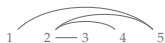
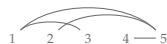
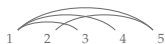
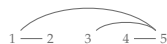
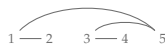
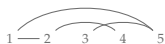
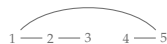
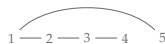
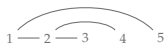
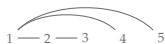
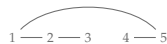
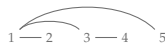
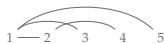
Example: NBB-Bases for 1 in $\mathcal{NC}_5$

Two Posets of
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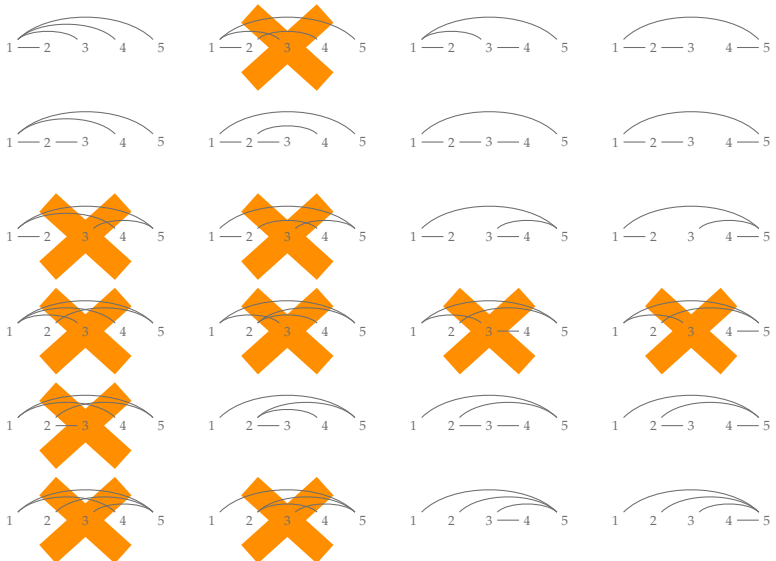
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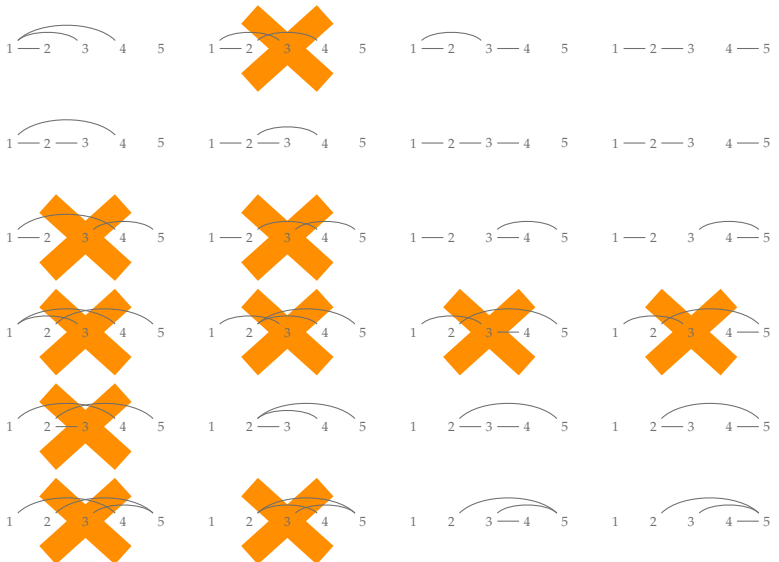
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Outline

Two Posets of
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A Subposet of
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1 Noncrossing Partitions

2 A Subposet of Noncrossing Partitions

Patterns in Noncrossing Partitions

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- for $\mathbf{x} \in NC_n$ write $i \sim_{\mathbf{x}} j$ if $\{i, j\} \subseteq B \in \mathbf{x}$

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- for $\mathbf{x} \in \text{NC}_n$ write $i \sim_{\mathbf{x}} j$ if $\{i, j\} \subseteq B \in \mathbf{x}$
- **incidence pattern**: $(i, j) \in X \subseteq [n] \times [n]$
- **block pattern**: $Y \in \mathcal{Y} \subseteq \wp([n])$
- for $Z_n \subseteq \text{NC}_n$ define
$$Z_n[X; \mathcal{Y}] = \{ \mathbf{x} \in Z_n \mid i \sim_{\mathbf{x}} j \text{ for } (i, j) \in X, \\ \text{and } B \in \mathbf{x} \text{ for } B \in \mathcal{Y} \}$$

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Idea

For $X_1, X_2, \dots, X_s \subseteq [n] \times [n]$ and $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_s \subseteq \wp([n])$ let

$$\mathbb{P}_n = \bigcup_{i=1}^s \text{NC}_n[X_s; \mathcal{Y}_s].$$

Study the poset $(\text{NC}_n \setminus \mathbb{P}_n, \leq_{\text{dref}})$ of noncrossing partitions avoiding the “patterns” $\mathbb{P}_n$.

Patterns in Noncrossing Partitions

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- easy examples
 - $\text{NC}_n[\emptyset; \{B\}] \cong \prod_{i=1}^s \text{NC}_{n_i}$, where the n_i depend on B
 - $\text{NC}_n[\{(i, j)\}; \emptyset] = \{\mathbf{x} \in \text{NC}_n \mid \mathbf{a}_{i,j} \leq_{\text{dref}} \mathbf{x}\}$

Another Example

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- consider the following patterns:
 - let $X_1 = \emptyset, \mathcal{Y}_1 = \{\{n-1, n\}\}$
 - let $X_2 = \{(1, n-1)\}, \mathcal{Y}_2 = \{\{n\}\}$
- straightforward:
 - $|\mathrm{NC}_n[X_1; \mathcal{Y}_1]| = \mathrm{Cat}(n-2) = |\mathrm{NC}_n[X_2; \mathcal{Y}_2]|$
- define $PE_n = \mathrm{NC}_n \setminus \left(\mathrm{NC}_n[X_1; \mathcal{Y}_1] \cup \mathrm{NC}_n[X_2; \mathcal{Y}_2] \right)$
- let $\mathcal{PE}_n = (PE_n, \leq_{\mathrm{dref}})$

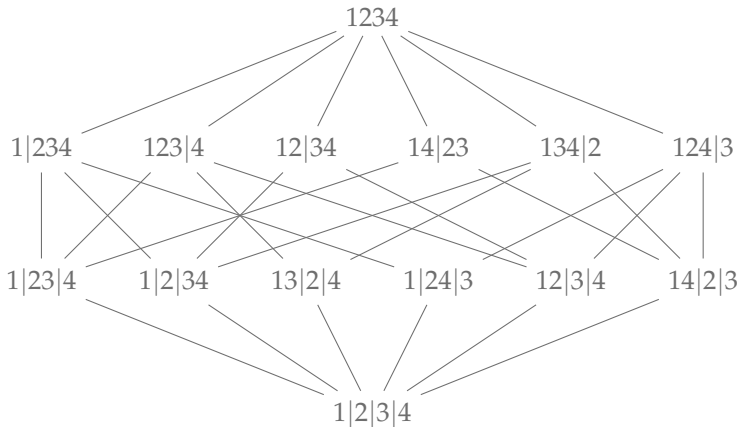
Example: $\mathcal{NC}_4$

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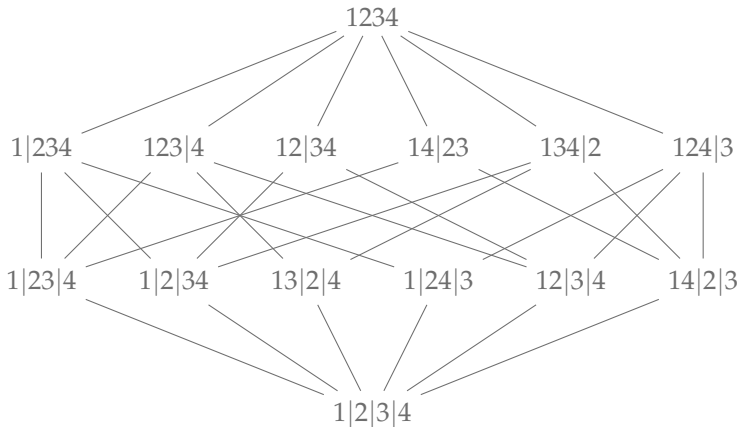
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$$\mathcal{NC}_4[\emptyset; \{\{3,4\}\}]$$

$$\mathcal{NC}_4[\{(1,3)\}; \{4\}]$$

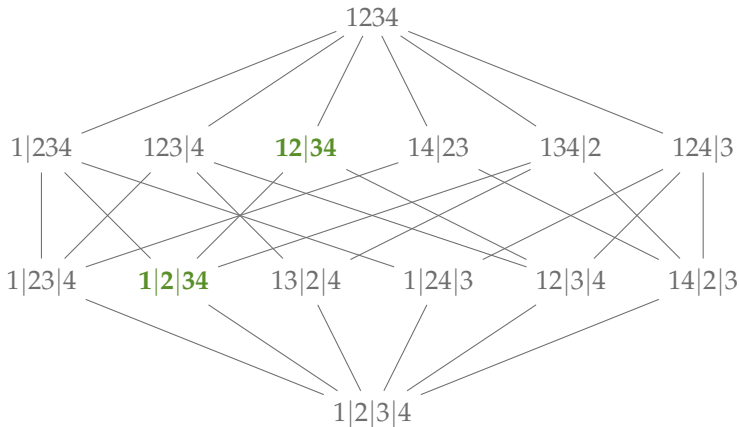
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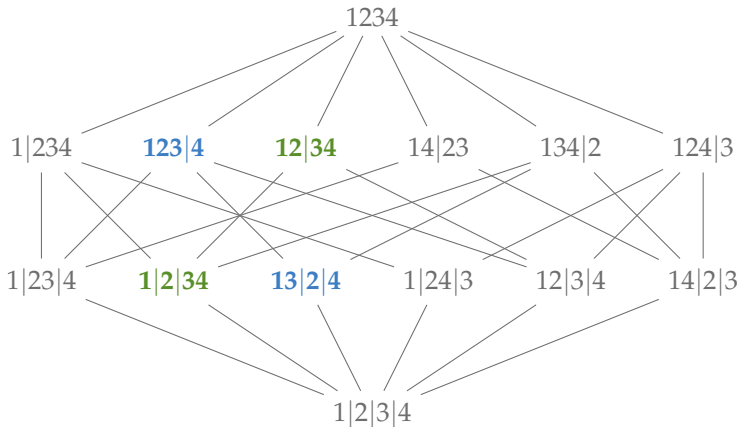
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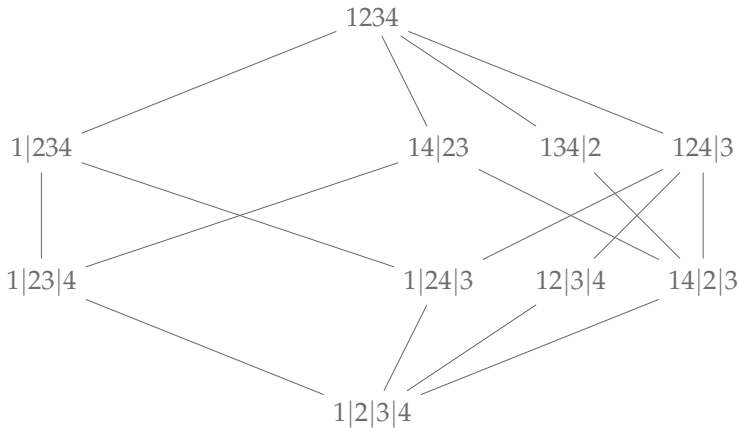
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$$\mathbf{NC}_4[\emptyset; \{\{3, 4\}\}]$$

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Properties of $\mathcal{PE}_n$

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- $|\mathcal{PE}_n| = \text{Cat}(n) - 2 \text{Cat}(n-2)$

Lemma (M. Bruce, M. Dougherty, M. Hlavacek,
R. Kudo, I. Nicolas, 2016)

We have $|\mathcal{PE}_3| = 3$ and for $n \geq 4$ we have

$$|\mathcal{PE}_n| = \left(\frac{5}{n+1} + \frac{9}{n-3} \right) \binom{2n-4}{n-4}.$$

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- recall: $\mathbf{x}_i$ has non-singleton block $[i-1] \cup \{n\}$

Theorem (✂, 2017)

For $n \geq 3$, the poset $\mathcal{PE}_n$ is a supersolvable lattice with M -chain $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$.

Properties of $\mathcal{PE}_n$

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- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$
- let $\bar{\mathcal{A}}_n = \mathcal{A}_n \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$

Proposition (, 2017)

For $n \geq 3$ the NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are precisely those maximal chains of $(\bar{\mathcal{A}}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \dots, n\}$ for some $k \in [n-2]$.

Properties of $\mathcal{PE}_n$

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Proposition (✂, 2017)

For $n \geq 3$ the NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are precisely those maximal chains of $(\bar{\mathcal{A}}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \dots, n\}$ for some $k \in [n-2]$.

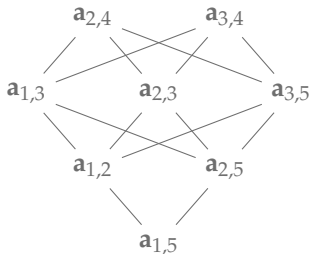
Example: NBB-Bases for 1 in $\mathcal{PE}_5$

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Example: NBB-Bases for 1 in $\mathcal{PE}_5$

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$$\{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{3,4}\}$$

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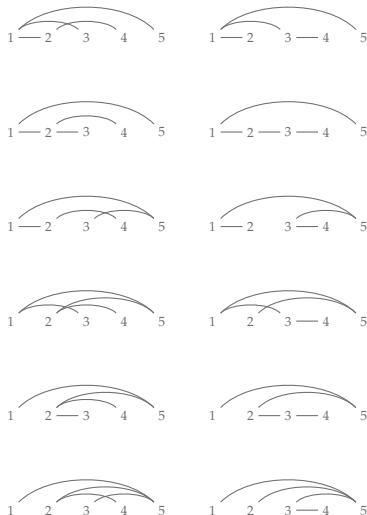
Example: NBB-Bases for 1 in $\mathcal{PE}_5$

Two Posets of
Noncrossing
Partitions

Henri Mühle

Noncrossing
Partitions

A Subposet of
Noncrossing
Partitions



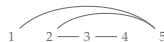
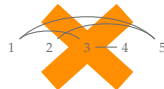
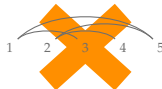
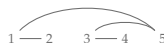
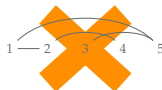
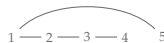
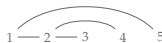
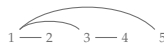
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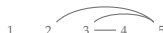
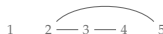
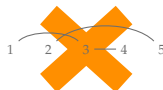
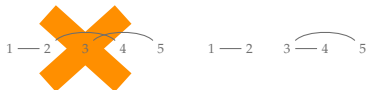
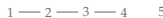
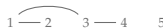
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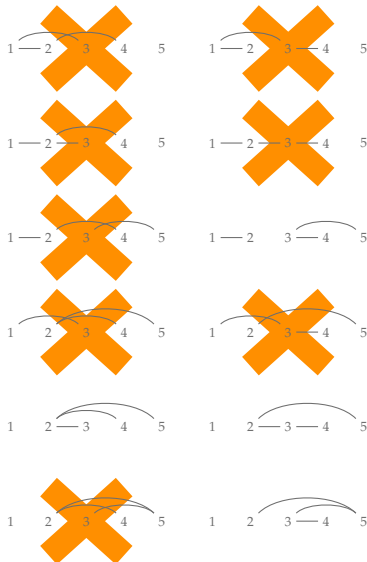
Example: NBB-Bases for 1 in $\mathcal{PE}_5$

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Properties of $\mathcal{PE}_n$

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- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$

Theorem (✂, 2017)

For $n \geq 3$, we have

$$\mu_{\mathcal{PE}_n}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} \left(\text{Cat}(n-1) - 2 \text{Cat}(n-2) \right).$$

Properties of $\mathcal{PE}_n$

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For $n \geq 3$, we have

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Two Posets of
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A Subposet of
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Partitions

Thank You.

Parking Functions

Two Posets of
Noncrossing
Partitions

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Parking
Functions

- $[n] = \{1, 2, \dots, n\}$
- **parking function**: a map $f : [n] \rightarrow [n]$ such that for all $k \in [n]$ the set $f^{-1}([k])$ has at least k elements
- PF_n .. set of all parking functions of length n

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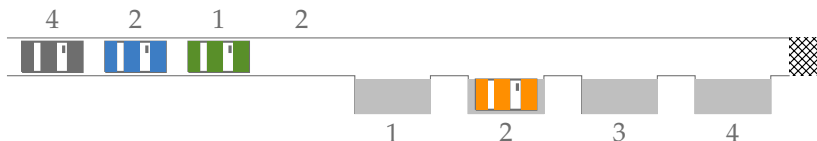
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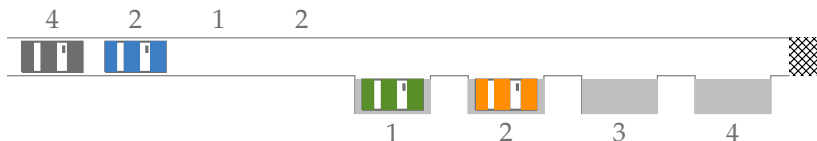
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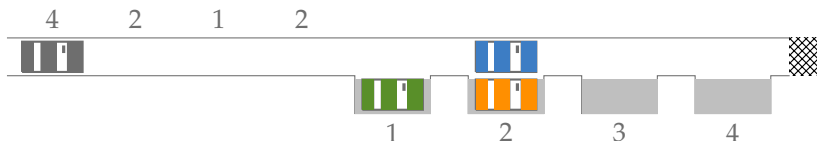
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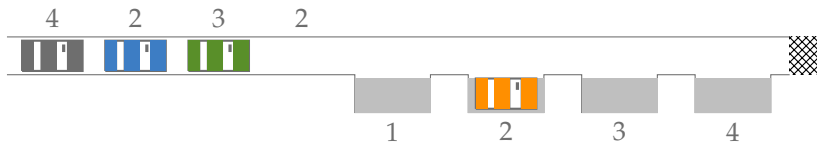
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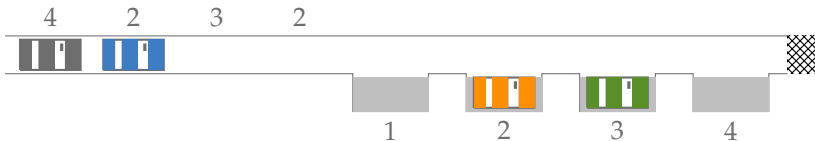
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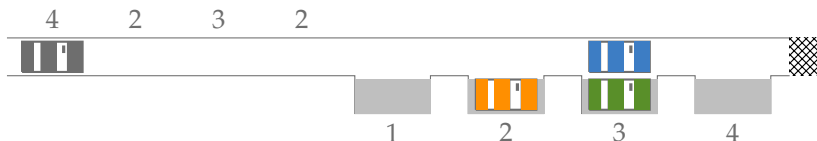
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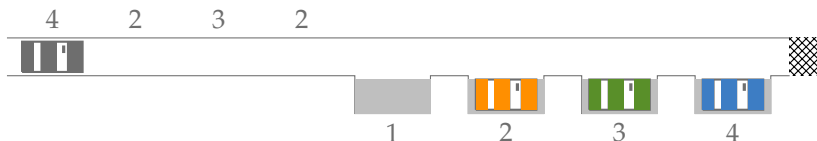
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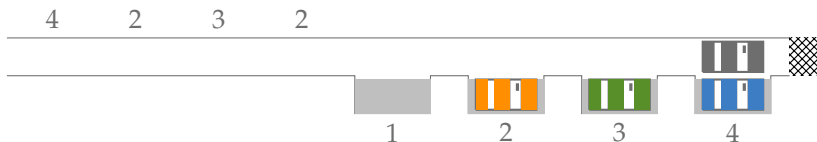
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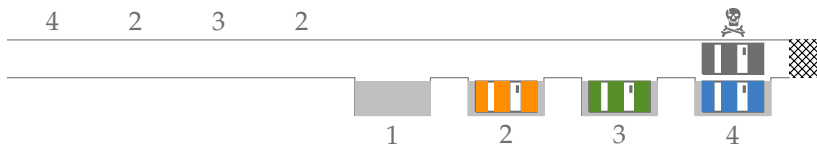
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Counting Parking Functions

Two Posets of
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Parking
Functions

Theorem (Folklore)

For $n \geq 0$, the cardinality of $\mathbb{PF}_n$ is $(n + 1)^{n-1}$.

Counting Parking Functions

Two Posets of
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Parking
Functions

Theorem (Folklore)

For $n \geq 0$, the cardinality of $\mathbb{PF}_n$ is $(n + 1)^{n-1}$.

Proof (H. Pollack, 1974).

- arrange $n + 1$ parking spaces on a circle
- now all n cars can park
- $(n + 1)^n$ possible assignments
- $\frac{1}{n+1}(n + 1)^n$ rotation classes
- parking function: space $n + 1$ remains empty
- one parking function per rotation class



The Shuffle Conjecture

Two Posets of
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Partitions

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Parking
Functions

- let $X = \{x_1, x_2, \dots, x_n\}, Y = \{y_1, y_2, \dots, y_n\}$
- $\mathfrak{S}_n$ acts diagonally on $\mathbb{Q}[X, Y]$ by

$$\begin{aligned}\sigma \cdot f(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \\ = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})\end{aligned}$$

- $\mathbb{Q}[X, Y]^{\mathfrak{S}_n}$ is generated by

$$p_{h,k}(X, Y) = \sum_{i=1}^n x_i^h y_i^k$$

The Shuffle Conjecture

Two Posets of
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Partitions

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Parking
Functions

- (bigraded) **ring of diagonal coinvariants**

$$DR_n = \mathbb{Q}[X, Y] / \langle p_{h,k}(X, Y) \mid h + k > 0 \rangle$$

The Shuffle Conjecture

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$$DR_n = \mathbb{Q}[X, Y] / \langle p_{h,k}(X, Y) \mid h + k > 0 \rangle$$

Theorem (M. Haiman, 2001)

For $n \geq 1$ we have

$$\dim DR_n = (n + 1)^{n-1}.$$

The Shuffle Conjecture

Two Posets of
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Functions

- (bigraded) **Hilbert series**

$$\mathcal{H}(DR_n; q, t) = \sum_{i,j \geq 0} t^i q^j \dim DR_n^{(i,j)}$$

The Shuffle Conjecture

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Conjecture (J. Haglund & N. Loehr, 2005)

For $n \geq 1$ we have

$$\mathcal{H}(DR_n; q, t) = \sum_{P \in \text{PF}_n} q^{\text{din}_v(P)} t^{\text{area}(P)}.$$

The Shuffle Conjecture

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- (bigraded) **Frobenius series**

$$\mathcal{F}(DR_n; q, t) = \sum_{i,j \geq 0} t^i q^j \sum_{\lambda \vdash n} \text{mult}(\chi^\lambda; DR_n^{(i,j)}) s_\lambda(X)$$

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Conjecture (The Shuffle Conjecture; J. Haglund, M. Haiman, N. Loehr, J. Remmel & A. Ulyanov, 2005)

For $n \geq 1$ we have

$$\mathcal{F}(\mathrm{DR}_n; q, t) = \sum_{P \in \mathrm{PF}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} s_P(X).$$

Undesired Parking Spaces

Two Posets of
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Parking
Functions

- generalizations of the Shuffle Conjecture involve sums over parking functions with undesired spaces



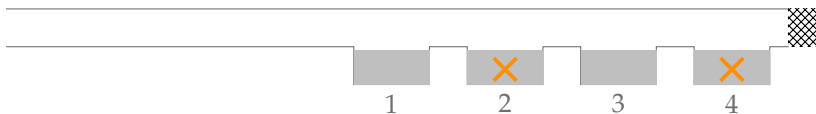
Undesired Parking Spaces

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Parking
Functions

- $I \subseteq [n]$
- **parking function avoiding I** : $f \in \text{PF}_n$ with $f \cap I = \emptyset$



Undesired Parking Spaces

Two Posets of
Noncrossing
Partitions

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Parking
Functions

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$



Undesired Parking Spaces

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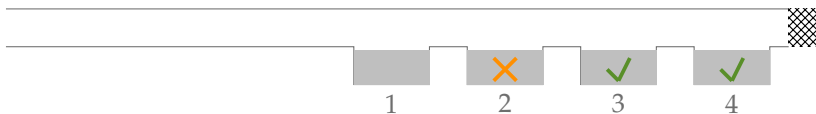
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- $\mathbb{PF}_3$:

(1, 1, 1)

(1, 1, 2) (1, 2, 1) (2, 1, 1)

(1, 2, 2) (2, 1, 2) (2, 2, 1)

(1, 1, 3) (1, 3, 1) (3, 1, 1)

(1, 2, 3) (1, 3, 2) (2, 1, 3) (2, 3, 1) (3, 1, 2) (3, 2, 1)

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Undesired Parking Spaces

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(1, 2, 3) (1, 3, 2) (2, 1, 3) (2, 3, 1) (3, 1, 2) (3, 2, 1)

Undesired Parking Spaces

Two Posets of
Noncrossing
Partitions

Henri Mühle

Parking
Functions

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathfrak{S}_3$:

(1, 1, 1)

(1, 1, 2) (1, 2, 1) (2, 1, 1)

(1, 2, 2) (2, 1, 2) (2, 2, 1)

(1, 1, 3) (1, 3, 1) (3, 1, 1)

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For $n \geq 0$ and $k \in [n]$, the cardinality of $\mathbb{PF}_{n,k}$ is

$$\frac{n!}{k!} \left| \mathbb{PF}_{k,k} \right|.$$

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$$\frac{n!}{k!} \left((k+1)^{k-1} - k^{k-1} \right).$$