Two Posets of Noncrossing Partitions

Henri Mühle

Institut für Algebra, TU Dresden

August 28, 2017

Eurocomb, Vienna

Outline

Outline

Noncrossing Partitions

- **noncrossing partition** $\rightsquigarrow NC_n$

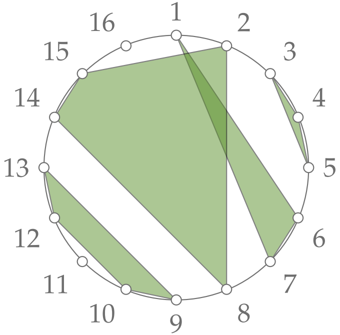
Noncrossing Partitions

- **noncrossing partition** $\rightsquigarrow NC_n$

$$\Big\{\{1,6,7\},\{2,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\Big\}$$

Noncrossing Partitions

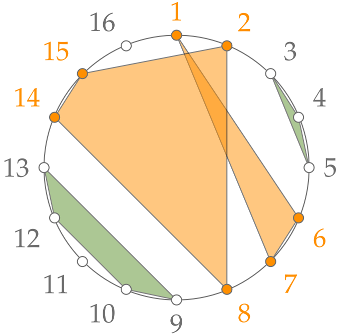
- **noncrossing partition** $\rightsquigarrow NC_n$



$$\Big\{\{1,6,7\},\{2,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\Big\}$$

Noncrossing Partitions

- **noncrossing partition** $\rightsquigarrow NC_n$

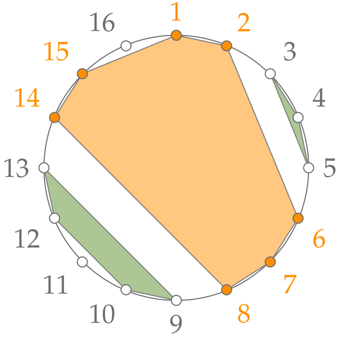


Nope!

$\Big\{\{1,6,7\},\{2,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\Big\}$

Noncrossing Partitions

- **noncrossing partition** $\rightsquigarrow NC_n$



$$\big\{\{1,2,6,7,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\big\}$$

First Properties

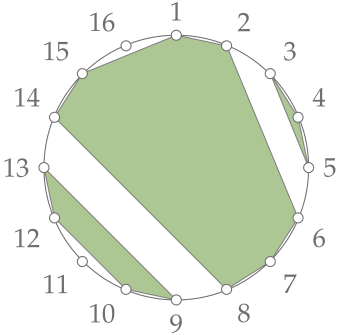
Theorem (G. Kreweras, 1972)

For $n \geq 0$, the cardinality of NC_n is

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Noncrossing Partitions

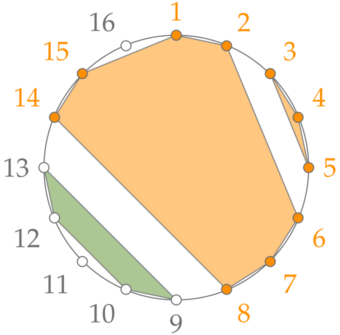
- **dual refinement order** $\rightsquigarrow \leq_{dref}$



$$\big\{\{1,2,6,7,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\big\}$$

Noncrossing Partitions

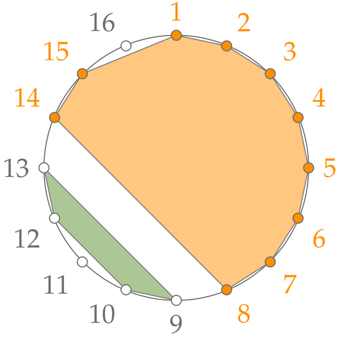
- **dual refinement order** $\rightsquigarrow \leq_{\mathrm{dref}}$



$$\big\{\{1,2,6,7,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\big\}$$

Noncrossing Partitions

- **dual refinement order** $\rightsquigarrow \leq_{\mathrm{dref}}$



$$\Big\{\{1,2,3,4,5,6,7,8,14,15\},\{9,10,12,13\},\{11\},\{16\}\Big\}$$

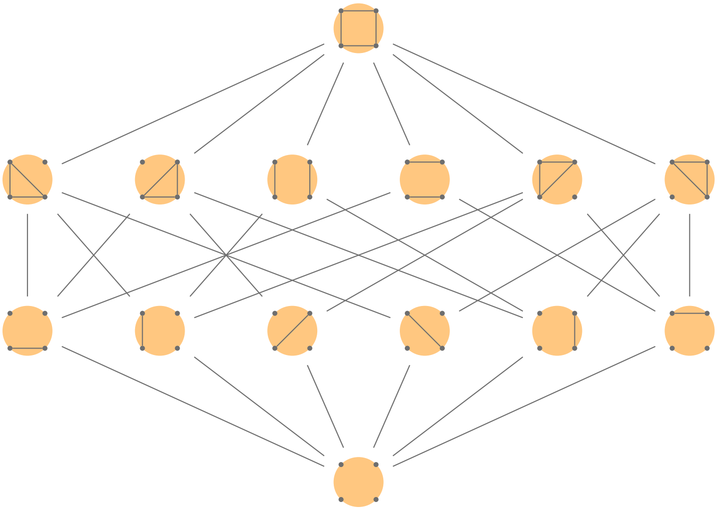
First Properties

- $\mathcal{NC}_n = (NC_n, \leq_{\text{dref}})$
- let $\mathbf{0} = 1|2|\cdots|n$ and $\mathbf{1} = 12\cdots n$
- let $[n] = \{1, 2, \ldots, n\}$

Theorem (G. Kreweras, 1972)

For $n \geq 0$, the poset $\mathcal{NC}_n$ is a lattice.

Example: $\mathcal{NC}_4$

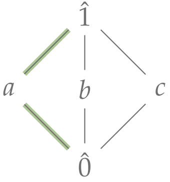


Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above

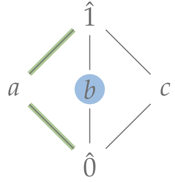
Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above



Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above

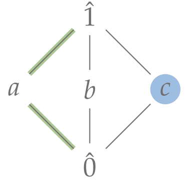


Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above

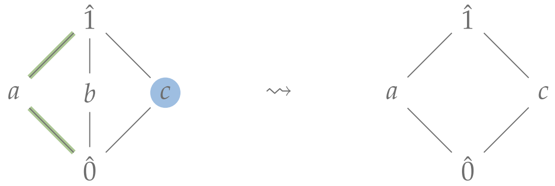
Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above



Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above



Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above

Theorem (P. Hersh, 1999)

For $n \geq 1$, the lattice $\mathcal{NC}_n$ is supersolvable.

Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above
- $\mathbf{x}_i$ .. noncrossing partition with only non-singleton block $[i-1] \cup \{n\}$

Supersolvability

- **supersolvable**: lattice $\mathcal{P}$ together with a maximal chain D of $\mathcal{P}$ such that D together with any other chain of $\mathcal{P}$ generates a distributive sublattice of $\mathcal{P}$
- ***M*-chain**: the chain D above
- $\mathbf{x}_i$ .. noncrossing partition with only non-singleton block $[i-1] \cup \{n\}$

Proposition (, 2017)

For $n \geq 1$, the chain $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an M-chain of $\mathcal{NC}_n$.

Möbius Function

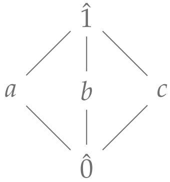
- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y,\\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y,\\ 0 & \text{otherwise} \end{cases}$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

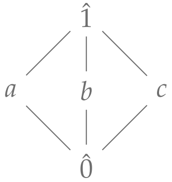
$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$



Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

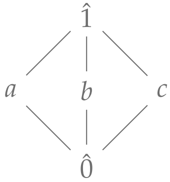


$$-\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \mu_{\mathcal{P}}(\hat{0},\hat{0}) + \mu_{\mathcal{P}}(\hat{0},a) + \mu_{\mathcal{P}}(\hat{0},b) + \mu_{\mathcal{P}}(\hat{0},c)$$

Möbius Function

- $\mathcal{P}=(P,\leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}}:P\times P\to\mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y)=\begin{cases}1, & \text{if } x=y,\\ -\sum_{x\leq z<y}\mu_{\mathcal{P}}(x,z), & \text{if } x<y,\\ 0 & \text{otherwise}\end{cases}$$

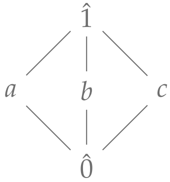


$$-\mu_{\mathcal{P}}(\hat{0},\hat{1})=\mu_{\mathcal{P}}(\hat{0},\hat{0})-\mu_{\mathcal{P}}(\hat{0},\hat{0})+\mu_{\mathcal{P}}(\hat{0},b)+\mu_{\mathcal{P}}(\hat{0},c)$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum\limits_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

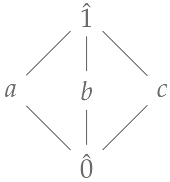


$$-\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \mu_{\mathcal{P}}(\hat{0},\hat{0}) - \mu_{\mathcal{P}}(\hat{0},\hat{0}) - \mu_{\mathcal{P}}(\hat{0},\hat{0}) + \mu_{\mathcal{P}}(\hat{0},c)$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum\limits_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

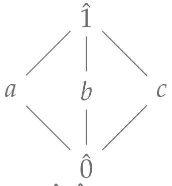


$$-\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \mu_{\mathcal{P}}(\hat{0},\hat{0}) - \mu_{\mathcal{P}}(\hat{0},\hat{0}) - \mu_{\mathcal{P}}(\hat{0},\hat{0}) - \mu_{\mathcal{P}}(\hat{0},\hat{0})$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y,\\ -\sum\limits_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y,\\ 0 & \text{otherwise} \end{cases}$$

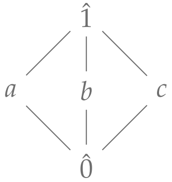


$$-\mu_{\mathcal{P}}(\hat{0},\hat{1}) = -2\mu_{\mathcal{P}}(\hat{0},\hat{0})$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum\limits_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

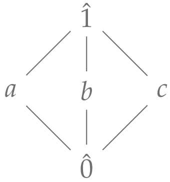


$$\mu_{\mathcal{P}}(\hat{0}, \hat{1}) = 2\mu_{\mathcal{P}}(\hat{0}, \hat{0})$$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$



$\mu_{\mathcal{P}}(\hat{0}, \hat{1}) = 2$

Möbius Function

- $\mathcal{P} = (P, \leq)$ .. a (finite) poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

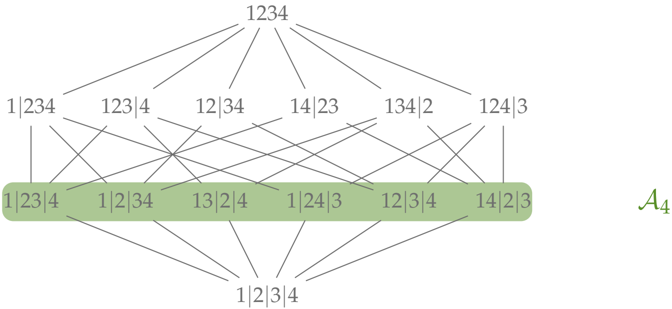
Theorem (G. Kreweras, 1972)

For $n \geq 1$ we have

$$\mu_{\mathcal{NC}_n}(\mathbf{0},\mathbf{1}) = (-1)^{n-1}\operatorname{Cat}(n-1).$$

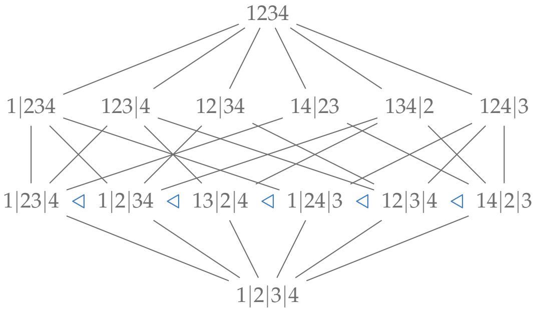
NBB-Bases of $\mathcal{NC}_n$

- $\mathbf{a}_{i,j}$ .. noncrossing partition with only non-singleton block $\{i,j\}$
- $\mathcal{A}_n = \{\mathbf{a}_{i,j} \mid 1 \leq i < j \leq n\}$
- let $\trianglelefteq$ be any partial order on $\mathcal{A}_n$; $X \subseteq \mathcal{A}_n$



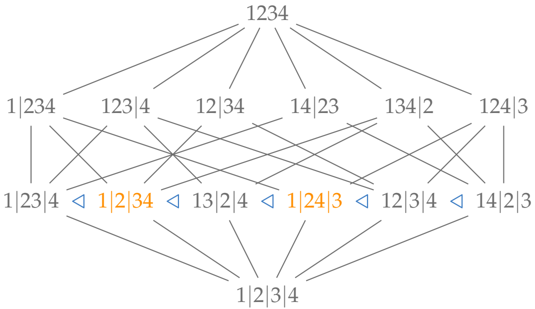
NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$



NBB-Bases of $\mathcal{NC}_n$

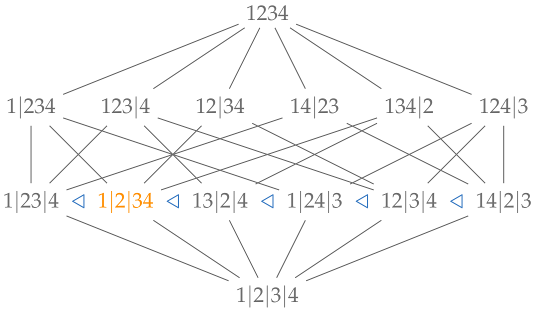
- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\mathrm{dref}} \bigvee X$



BB

NBB-Bases of $\mathcal{NC}_n$

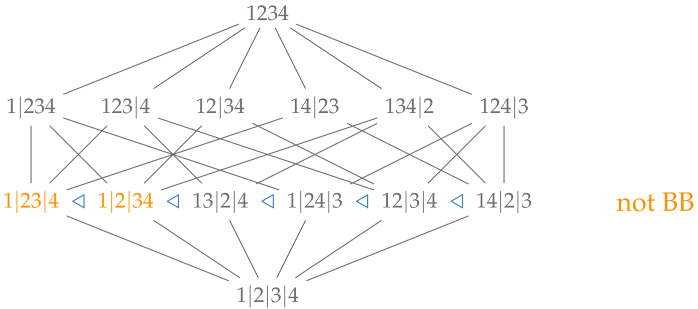
- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$



not BB

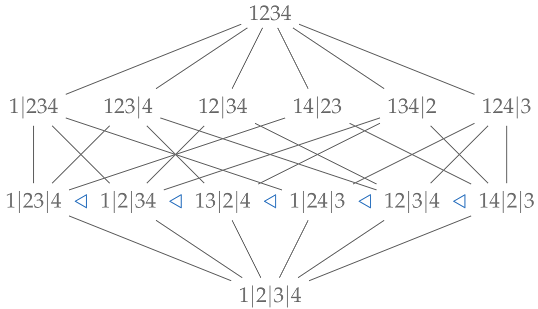
NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lhd \mathbf{x}$ and $\mathbf{a} <_{\mathrm{dref}} \bigvee X$



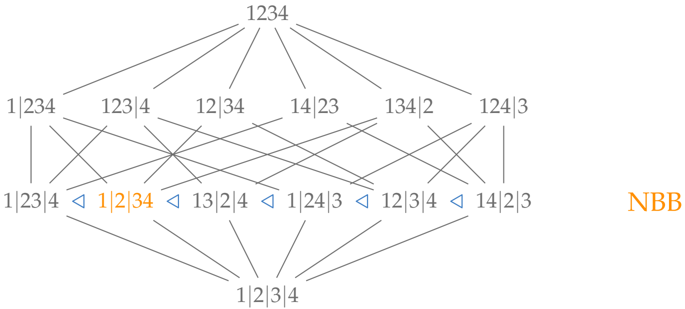
NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lhd \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB



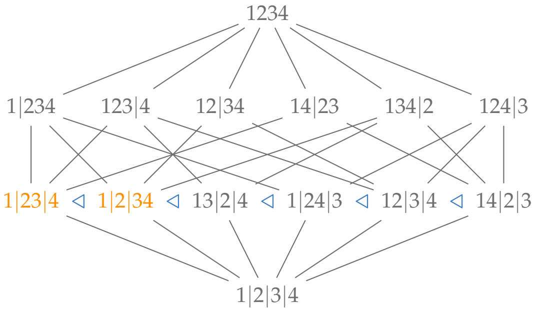
NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB



NBB-Bases of $\mathcal{NC}_n$

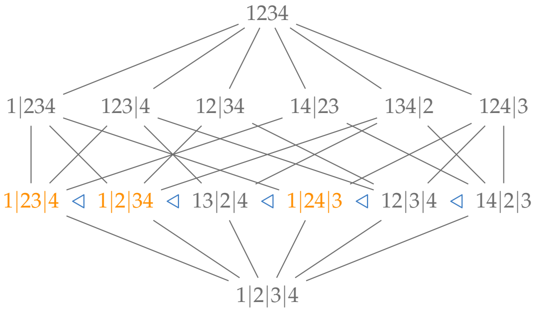
- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB



NBB

NBB-Bases of $\mathcal{NC}_n$

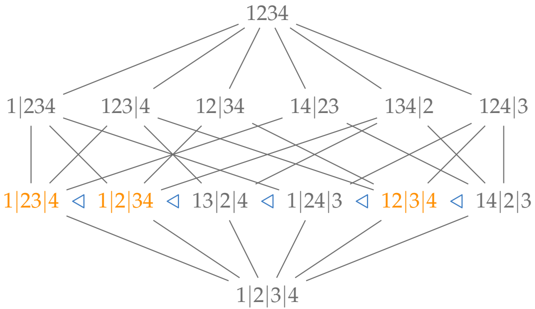
- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB



not NBB

NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\mathrm{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB
- **NBB-base** for $\mathbf{x}$: X is NBB and $\bigvee X = \mathbf{x}$



NBB-base for **1**

NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lhd \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB
- **NBB-base** for $\mathbf{x}$: X is NBB and $\bigvee X = \mathbf{x}$

Theorem (A. Blass, B. Sagan, 1997)

Let $\mathcal{P} = (P, \leq)$ be a finite lattice and $\trianglelefteq$ any partial order on the atoms of $\mathcal{P}$. For $x \in P$ we have

$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_X (-1)^{|X|},$$

where the sum runs over the NBB-bases for x.

NBB-Bases of $\mathcal{NC}_n$

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \mathcal{A}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$
- **NBB**: no nonempty subset of X is BB
- **NBB-base** for $\mathbf{x}$: X is NBB and $\bigvee X = \mathbf{x}$

Corollary (The Crosscut Theorem; G.-C. Rota, 1964)

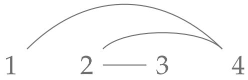
Let $\mathcal{P} = (P, \leq)$ be a finite lattice. For $x \in P$ we have

$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_X (-1)^{|X|},$$

where the sum runs over all subsets of atoms of $\mathcal{P}$ whose join is x.

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

- subsets of $\mathcal{A}_n$ correspond to certain graphs on $[n]$

$\{\mathbf{a}_{1,4}, \mathbf{a}_{2,3}, \mathbf{a}_{2,4}\} \quad \leftrightarrow$ 

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

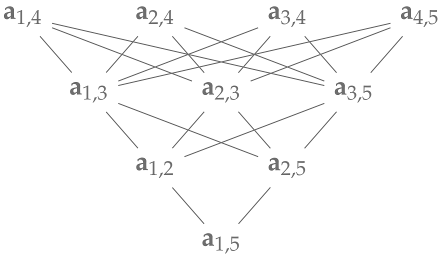
- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the M-chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\mathrm{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\mathrm{dref}} \mathbf{x}_{i+1}\}$

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the M-chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\mathrm{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\mathrm{dref}} \mathbf{x}_{i+1}\}$
- let $\mathbf{a} \trianglelefteq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the M-chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\text{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\text{dref}} \mathbf{x}_{i+1}\}$
- let $\mathbf{a} \trianglelefteq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$



NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the M-chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\mathrm{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\mathrm{dref}} \mathbf{x}_{i+1}\}$
- let $\mathbf{a} \trianglelefteq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$

Proposition (, 2017)

For $n \geq 1$ the NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$ are precisely those maximal chains of $(\mathcal{A}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \ldots, n\}$ for some $k \in [n-1]$.

NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_n$

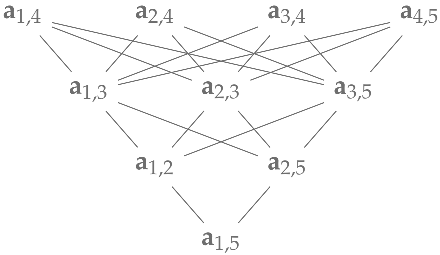
- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the M-chain from before
- let $A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \not\leq_{\mathrm{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\mathrm{dref}} \mathbf{x}_{i+1}\}$
- let $\mathbf{a} \trianglelefteq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i, \mathbf{a}' \in A_j$ and $i \leq j$

Corollary

For $n \geq 1$ we have

$$\mu_{\mathcal{NC}_n}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1}\operatorname{Cat}(n-1).$$

Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_5$

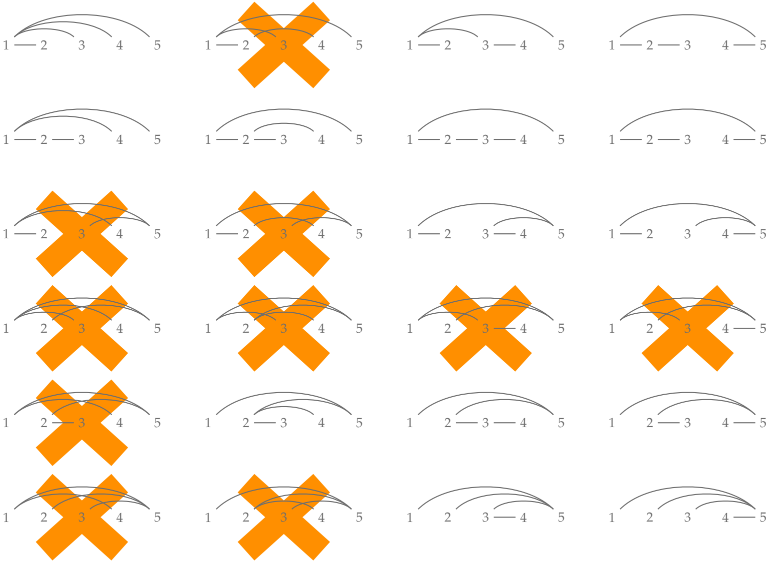


Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_5$

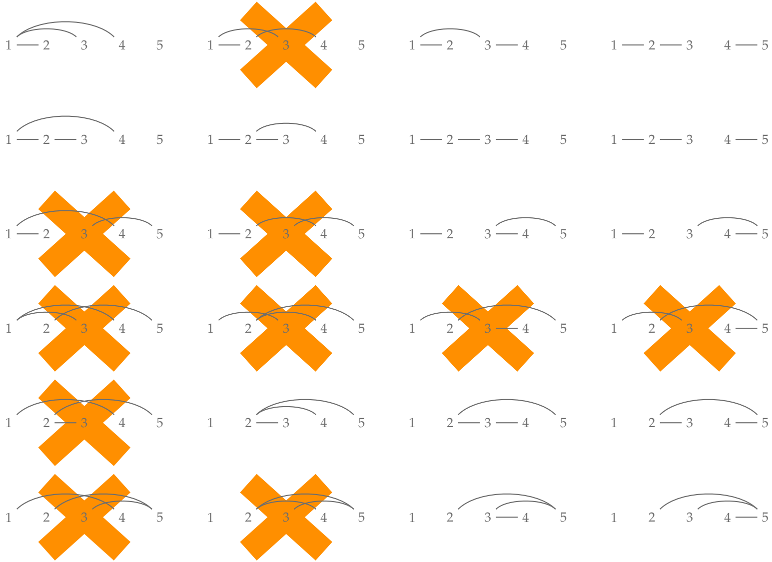
$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{1,3},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{1,3},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{1,3},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{1,3},\mathbf{a}_{4,5}\}$
$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{2,3},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{2,3},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{2,3},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{2,3},\mathbf{a}_{4,5}\}$
$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{3,5},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{3,5},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{3,5},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{1,2},\mathbf{a}_{3,5},\mathbf{a}_{4,5}\}$
$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{1,3},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{1,3},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{1,3},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{1,3},\mathbf{a}_{4,5}\}$
$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{2,3},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{2,3},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{2,3},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{2,3},\mathbf{a}_{4,5}\}$
$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{3,5},\mathbf{a}_{1,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{3,5},\mathbf{a}_{2,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{3,5},\mathbf{a}_{3,4}\}$	$\{\mathbf{a}_{1,5},\mathbf{a}_{2,5},\mathbf{a}_{3,5},\mathbf{a}_{4,5}\}$

Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_5$

Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_5$



Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{NC}_5$



Outline

Patterns in Noncrossing Partitions

- for $\mathbf{x} \in NC_n$ write $i \sim_{\mathbf{x}} j$ if $\{i, j\} \subseteq B \in \mathbf{x}$

Patterns in Noncrossing Partitions

- for $\mathbf{x} \in NC_n$ write $i \sim_{\mathbf{x}} j$ if $\{i,j\} \subseteq B \in \mathbf{x}$
- **incidence pattern**: $(i,j) \in X \subseteq [n] \times [n]$
- **block pattern**: $Y \in \mathcal{Y} \subseteq \wp\bigl([n]\bigr)$
- for $Z_n \subseteq NC_n$ define

$$Z_n[X;\mathcal{Y}] = \bigl\{\mathbf{x} \in Z_n \mid i \sim_{\mathbf{x}} j \text{ for } (i,j) \in X, \\ \text{and } B \in \mathbf{x} \text{ for } B \in \mathcal{Y}\bigr\}$$

Patterns in Noncrossing Partitions

Idea

For $X_1, X_2, \ldots, X_s \subseteq [n] \times [n]$ *and* $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s \subseteq \wp([n])$ *let*

$$\mathbb{P}_n = \bigcup_{i=1}^{s} NC_n[X_s; \mathcal{Y}_s].$$

Study the poset $(NC_n \setminus \mathbb{P}_n, \leq_{\text{dref}})$ *of noncrossing partitions avoiding the "patterns"* $\mathbb{P}_n$*.*

Patterns in Noncrossing Partitions

Idea

For $X_1, X_2, \ldots, X_s \subseteq [n] \times [n]$ and $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s \subseteq \wp([n])$ let

$$\mathbb{P}_n = \bigcup_{i=1}^{s} NC_n[X_s; \mathcal{Y}_s].$$

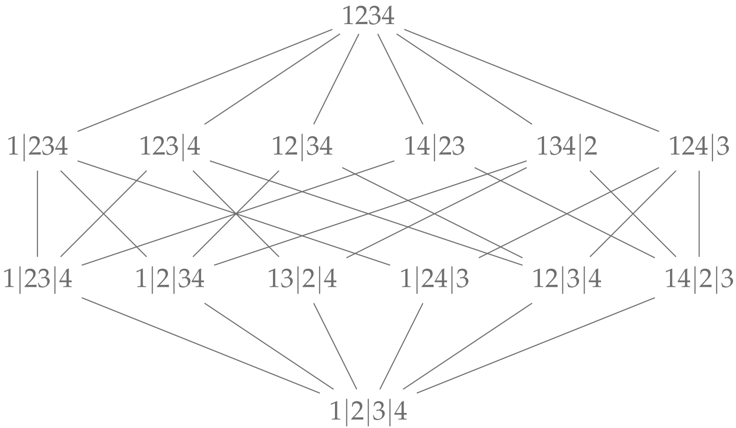
Study the poset $\left(NC_n \setminus \mathbb{P}_n, \leq_{\mathrm{dref}}\right)$ of noncrossing partitions avoiding the "patterns" $\mathbb{P}_n$.

- easy examples
  - $NC_n\big[\emptyset; \{B\}\big] \cong \prod_{i=1}^{s} NC_{n_i}$, where the n_i depend on B
  - $NC_n\big[\{(i,j)\}; \emptyset\big] = \{\mathbf{x} \in NC_n \mid \mathbf{a}_{i,j} \leq_{\mathrm{dref}} \mathbf{x}\}$

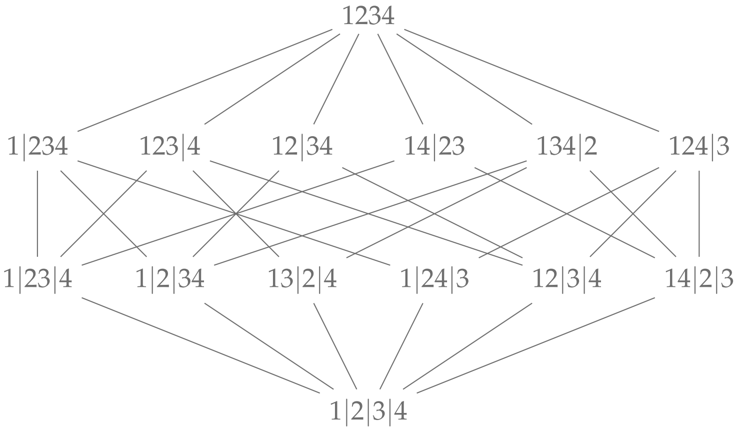
Another Example

- consider the following patterns:
  - let $X_1 = \emptyset, \mathcal{Y}_1 = \big\{\{n-1, n\}\big\}$
  - let $X_2 = \big\{(1, n-1)\big\}, \mathcal{Y}_2 = \big\{\{n\}\big\}$
- straightforward:
  - $\Big|NC_n[X_1;\mathcal{Y}_1]\Big| = \mathrm{Cat}(n-2) = \Big|NC_n[X_2;\mathcal{Y}_2]\Big|$
- define $PE_n = NC_n \setminus \Big(NC_n[X_1;\mathcal{Y}_1] \cup NC_n[X_2;\mathcal{Y}_2]\Big)$
- let $\mathcal{PE}_n = \big(PE_n, \leq_{\mathrm{dref}}\big)$

Example: $\mathcal{NC}_4$

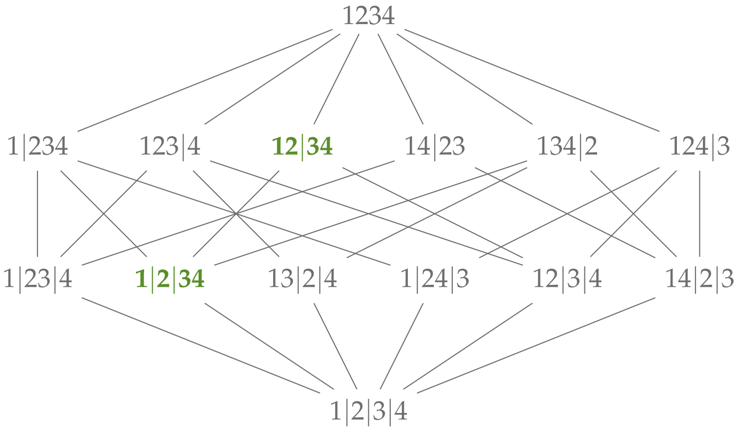


Example: $\mathcal{NC}_4$



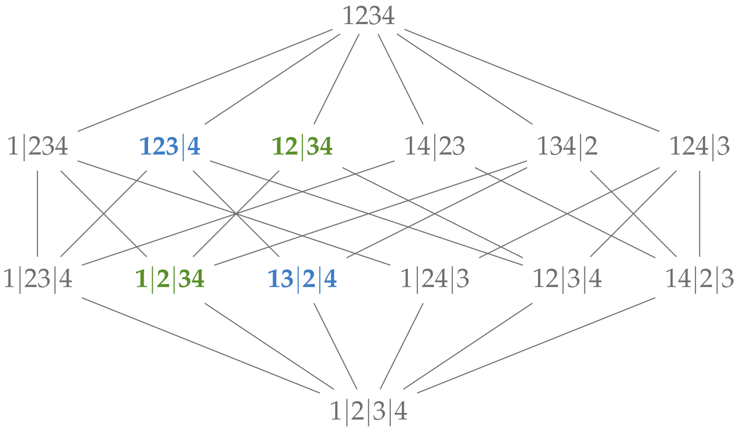
$NC_4\big[\emptyset;\{\{3,4\}\}\big]$ $NC_4\big[\{(1,3)\};\{4\}\big]$

Example: $\mathcal{NC}_4$



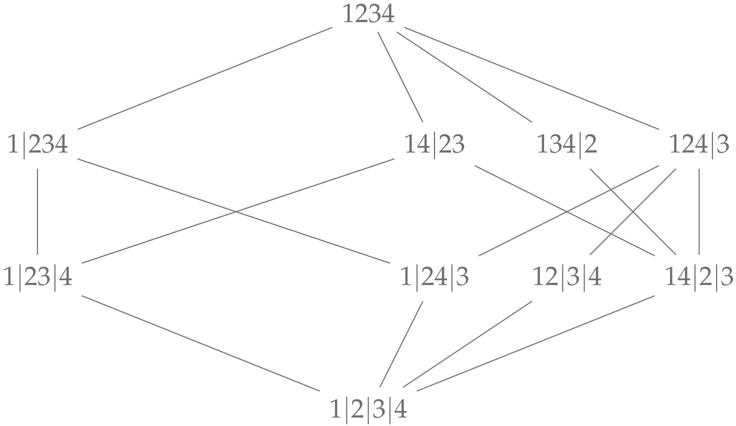
$\mathbf{NC_4}\Big[\emptyset;\{\{\mathbf{3},\mathbf{4}\}\}\Big]$ $\qquad$ $NC_4\Big[\{(1,3)\};\{4\}\Big]$

Example: $\mathcal{NC}_4$



$\mathbf{NC_4}\Big[\emptyset; \big\{\{\mathbf{3}, \mathbf{4}\}\big\}\Big]$ $\mathbf{NC_4}\Big[\big\{(\mathbf{1}, \mathbf{3})\big\}; \{\mathbf{4}\}\Big]$

Example: $\mathcal{PE}_4$



$\mathbf{NC_4}\big[\emptyset; \{\{\mathbf{3},\mathbf{4}\}\}\big]$ $\mathbf{NC_4}\big[\{(\mathbf{1},\mathbf{3})\}; \{\mathbf{4}\}\big]$

Properties of $\mathcal{PE}_n$

- $\lvert PE_n \rvert = \mathrm{Cat}(n) - 2\,\mathrm{Cat}(n-2)$

Lemma (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo, I. Nicolas, 2016)

We have $\lvert PE_3 \rvert = 3$ and for $n \geq 4$ we have

$$\lvert PE_n \rvert = \left(\frac{5}{n+1} + \frac{9}{n-3}\right)\binom{2n-4}{n-4}.$$

Properties of $\mathcal{PE}_n$

- recall: $\mathbf{x}_i$ has non-singleton block $[i-1] \cup \{n\}$

Theorem (, 2017)

For $n \geq 3$, the poset $\mathcal{PE}_n$ is a supersolvable lattice with M-chain $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$.

Properties of $\mathcal{PE}_n$

- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$
- let $\bar{\mathcal{A}}_n = \mathcal{A}_n \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$

Proposition (, 2017)

For $n \geq 3$ the NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are precisely those maximal chains of $(\bar{\mathcal{A}}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \ldots, n\}$ for some $k \in [n-2]$.

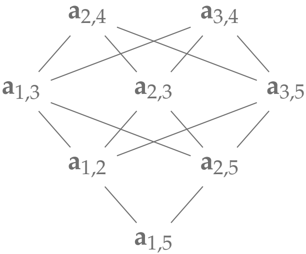
Properties of $\mathcal{PE}_n$

- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$
- let $\bar{\mathcal{A}}_n = \mathcal{A}_n \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$

Proposition (, 2017)

For $n \geq 3$ the NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are precisely those maximal chains of $(\bar{\mathcal{A}}_n, \trianglelefteq)$, whose associated graph is a noncrossing tree with an edge between 1 and n such that the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \ldots, n\}$ for some $k \in [n-2]$.

Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{PE}_5$



Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{PE}_5$

$$\{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{1,3}, \mathbf{a}_{3,4}\}$$

$$\{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{2,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{2,3}, \mathbf{a}_{3,4}\}$$

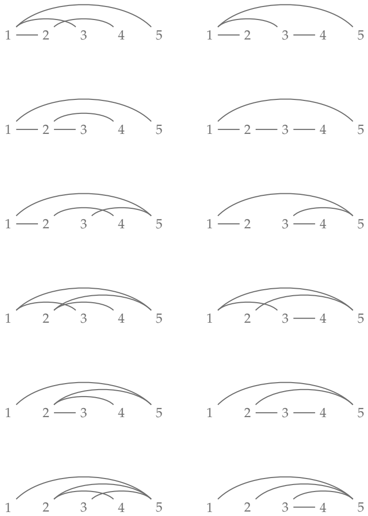
$$\{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{3,5}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{1,2}, \mathbf{a}_{3,5}, \mathbf{a}_{3,4}\}$$

$$\{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{1,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{1,3}, \mathbf{a}_{3,4}\}$$

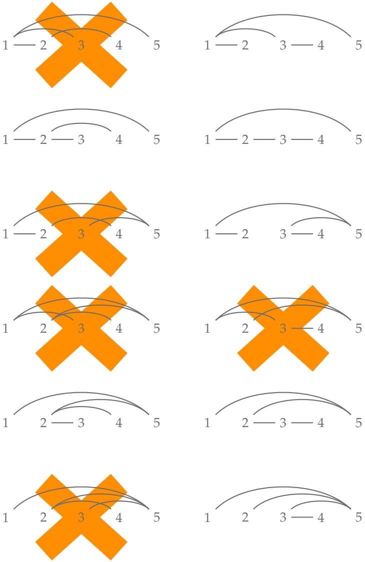
$$\{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{2,3}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{2,3}, \mathbf{a}_{3,4}\}$$

$$\{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{3,5}, \mathbf{a}_{2,4}\} \quad \{\mathbf{a}_{1,5}, \mathbf{a}_{2,5}, \mathbf{a}_{3,5}, \mathbf{a}_{3,4}\}$$

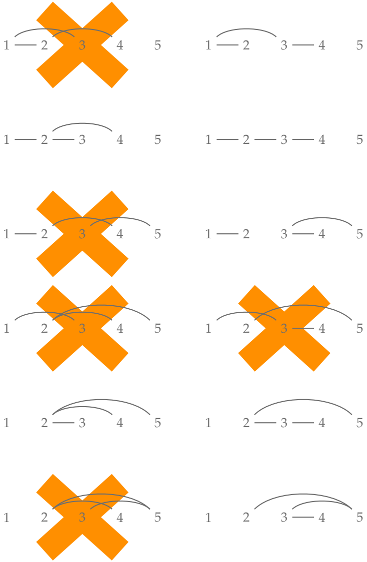
Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{PE}_5$



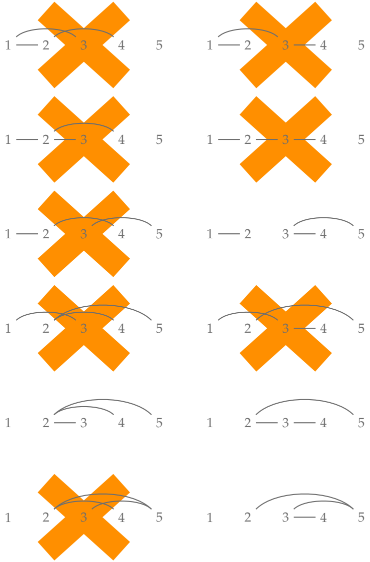
Example: NBB-Bases for $\mathbf{1}$ in $\mathcal{PE}_5$



Example: NBB-Bases for **1** in $\mathcal{PE}_5$



Example: NBB-Bases for **1** in $\mathcal{PE}_5$



Properties of $\mathcal{PE}_n$

- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$

Theorem (, 2017)

For $n \geq 3$, we have

$$\mu_{\mathcal{PE}_n}(\mathbf{0},\mathbf{1}) = (-1)^{n-1}\Bigl(\operatorname{Cat}(n-1) - 2\operatorname{Cat}(n-2)\Bigr).$$

Properties of $\mathcal{PE}_n$

- NBB-bases for $\mathbf{1}$ in $\mathcal{PE}_n$ are NBB-bases for $\mathbf{1}$ in $\mathcal{NC}_n$

Theorem (, 2017)

For $n \geq 3$, we have

$$\mu_{\mathcal{PE}_n}(\mathbf{0},\mathbf{1}) = (-1)^{n-1}\frac{4}{n}\binom{2n-5}{n-4}.$$

Thank You.

Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n

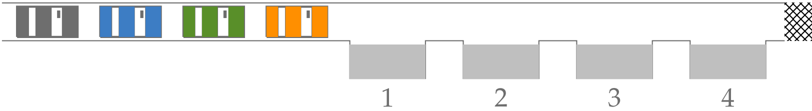
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n
- $\mathbb{PF}_3$:

$$
\begin{array}{llllll}
(1,1,1) \\
(1,1,2) & (1,2,1) & (2,1,1) \\
(1,2,2) & (2,1,2) & (2,2,1) \\
(1,1,3) & (1,3,1) & (3,1,1) \\
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1)
\end{array}
$$

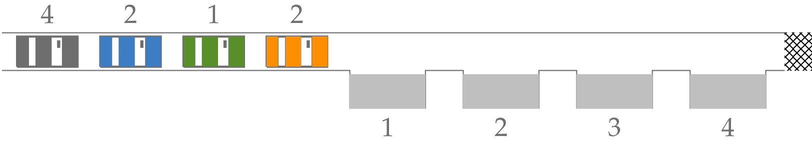
Parking Functions

- $[n]=\{1,2,\ldots,n\}$
- **parking function**: a map $f:[n]\to[n]$ such that for all $k\in[n]$ the set $f^{-1}\bigl([k]\bigr)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



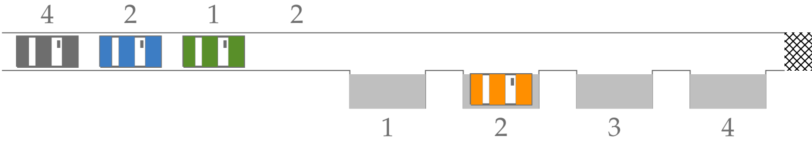
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



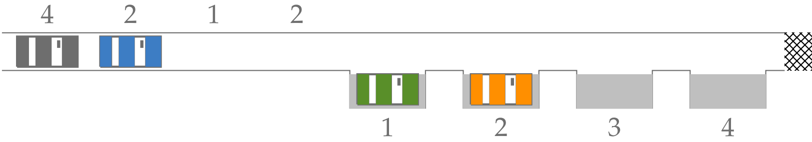
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



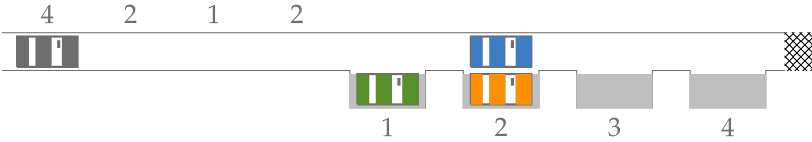
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



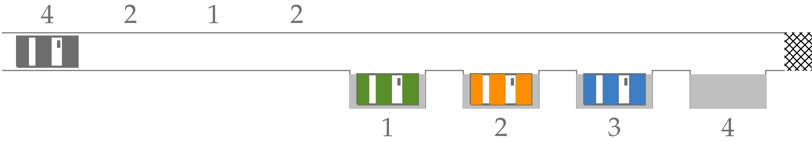
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



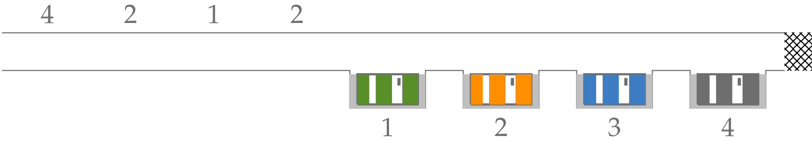
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



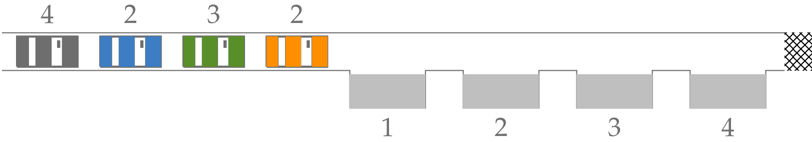
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



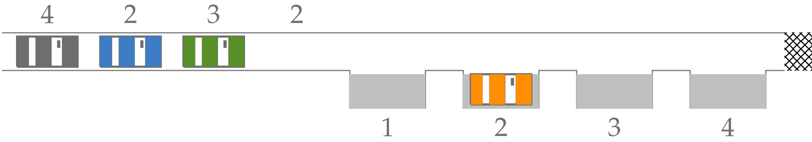
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



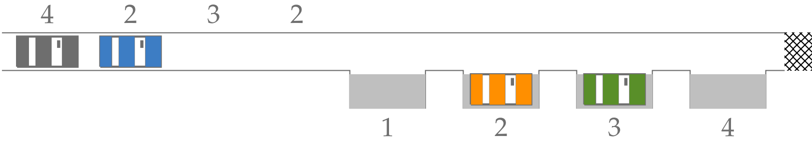
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



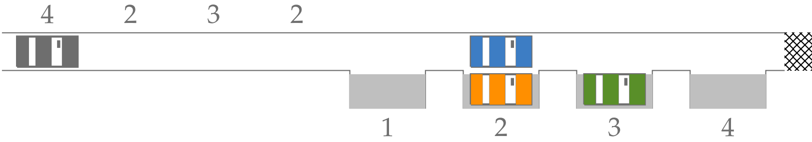
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



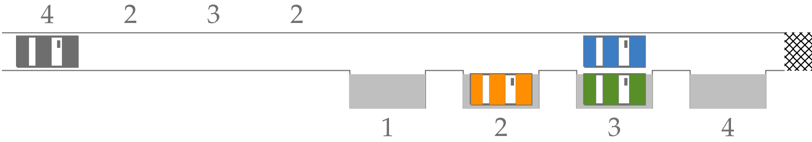
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



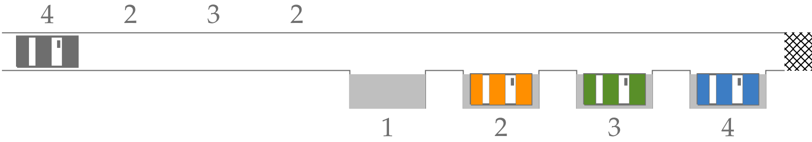
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



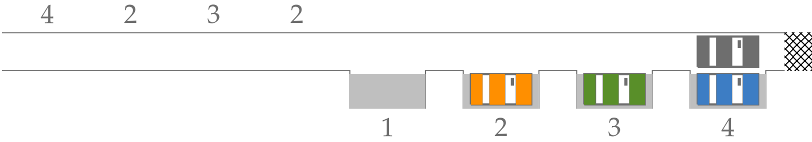
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



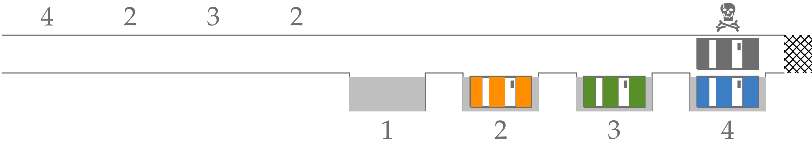
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\bigl([k]\bigr)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}\big([k]\big)$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n



Counting Parking Functions

Theorem (Folklore)

For $n \geq 0$*, the cardinality of* $\mathbb{PF}_n$ *is* $(n+1)^{n-1}$.

Counting Parking Functions

Theorem (Folklore)

For $n \geq 0$, the cardinality of $\mathbb{PF}_n$ is $(n+1)^{n-1}$.

Proof (H. Pollack, 1974).

- arrange $n+1$ parking spaces on a circle
- now all n cars can park
- $(n+1)^n$ possible assignments
- $\frac{1}{n+1}(n+1)^n$ rotation classes
- parking function: space $n+1$ remains empty
- one parking function per rotation class

□

The Shuffle Conjecture

- let $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_n\}$
- $\mathfrak{S}_n$ acts diagonally on $\mathbb{Q}[X, Y]$ by

$$\begin{aligned}\sigma \cdot f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\\ = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})\end{aligned}$$

- $\mathbb{Q}[X, Y]^{\mathfrak{S}_n}$ is generated by

$$p_{h,k}(X, Y) = \sum_{i=1}^{n} x_i^h y_i^k$$

The Shuffle Conjecture

- (bigraded) **ring of diagonal coinvariants**

$$DR_n = \mathbb{Q}[X,Y]/\langle p_{h,k}(X,Y) \mid h+k>0\rangle$$

The Shuffle Conjecture

- (bigraded) **ring of diagonal coinvariants**

$$DR_n = \mathbb{Q}[X,Y]/\langle p_{h,k}(X,Y) \mid h+k>0\rangle$$

Theorem (M. Haiman, 2001)

For $n \geq 1$ we have

$$\dim DR_n = (n+1)^{n-1}.$$

The Shuffle Conjecture

- (bigraded) **Hilbert series**

$$\mathcal{H}(DR_n; q, t) = \sum_{i,j \geq 0} t^i q^j \dim DR_n^{(i,j)}$$

The Shuffle Conjecture

- (bigraded) **Hilbert series**

$$\mathcal{H}(DR_n;q,t) = \sum_{i,j\geq 0} t^i q^j \dim DR_n^{(i,j)}$$

Conjecture (J. Haglund & N. Loehr, 2005)

For $n \geq 1$ we have

$$\mathcal{H}(DR_n;q,t) = \sum_{P\in\mathbb{PF}_n} q^{dinv(P)} t^{area(P)}.$$

The Shuffle Conjecture

- (bigraded) **Frobenius series**

$$\mathcal{F}(DR_n;q,t) = \sum_{i,j\geq 0} t^i q^j \sum_{\lambda\vdash n} \operatorname{mult}\bigl(\chi^{\lambda}; DR_n^{(i,j)}\bigr) s_{\lambda}(X)$$

The Shuffle Conjecture

- (bigraded) **Frobenius series**

$$\mathcal{F}(DR_n;q,t) = \sum_{i,j\geq 0} t^i q^j \sum_{\lambda\vdash n} \operatorname{mult}\bigl(\chi^{\lambda}; DR_n^{(i,j)}\bigr) s_{\lambda}(X)$$

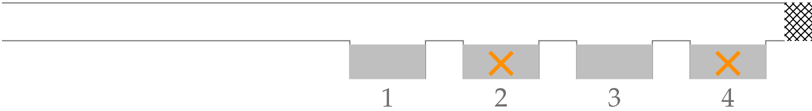
Conjecture (The Shuffle Conjecture; J. Haglund, M. Haiman, N. Loehr, J. Remmel & A. Ulyanov, 2005)

For $n \geq 1$ we have

$$\mathcal{F}(DR_n;q,t) = \sum_{P\in\mathbb{PF}_n} q^{dinv(P)} t^{area(P)} s_P(X).$$

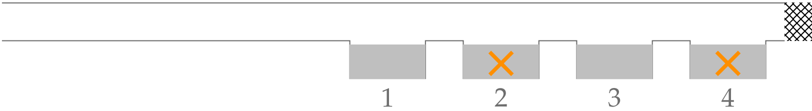
Undesired Parking Spaces

- generalizations of the Shuffle Conjecture involve sums over parking functions with undesired spaces



Undesired Parking Spaces

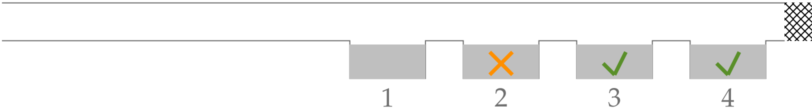
- $I \subseteq [n]$
- **parking function avoiding I**: $f \in \mathbb{PF}_n$ with $f \cap I = \emptyset$



Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

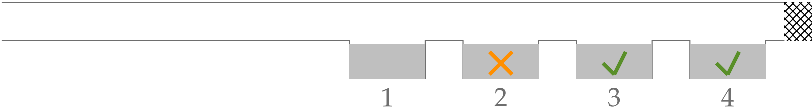


Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$



Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathbb{PF}_3$:

$(1,1,1)$

$(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$(1,2,2)$ $(2,1,2)$ $(2,2,1)$

$(1,1,3)$ $(1,3,1)$ $(3,1,1)$

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathbb{PF}_{3,1}$:

$(1,1,1)$

$(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$(1,2,2)$ $(2,1,2)$ $(2,2,1)$

$(1,1,3)$ $(1,3,1)$ $(3,1,1)$

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f,\ \text{but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathbb{PF}_{3,2}$:

$(1,1,1)$

$(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$(1,2,2)$ $(2,1,2)$ $(2,2,1)$

$(1,1,3)$ $(1,3,1)$ $(3,1,1)$

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathbb{PF}_{3,3}$:

$(1,1,1)$

$(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$(1,2,2)$ $(2,1,2)$ $(2,2,1)$

$(1,1,3)$ $(1,3,1)$ $(3,1,1)$

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \bigl\{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\bigr\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

- $\mathfrak{S}_3$:

$(1,1,1)$

$(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$(1,2,2)$ $(2,1,2)$ $(2,2,1)$

$(1,1,3)$ $(1,3,1)$ $(3,1,1)$

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \{f \in \mathbb{PF}_n \mid k \notin f, \text{ but } l \in f \text{ for all } l > k\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

Proposition (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $n \geq 0$ and $k \in [n]$, the cardinality of $\mathbb{PF}_{n,k}$ is

$$\frac{n!}{k!}\left|\mathbb{PF}_{k,k}\right|.$$

Undesired Parking Spaces

- for simplicity:

$$\mathbb{PF}_{n,k} = \bigl\{f \in \mathbb{PF}_n \mid k \notin f,\ \text{but } l \in f \text{ for all } l > k\bigr\}$$

- it follows: $\mathbb{PF}_n = \mathfrak{S}_n \uplus \mathbb{PF}_{n,1} \uplus \cdots \uplus \mathbb{PF}_{n,n}$

Proposition (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $n \geq 0$ and $k \in [n]$, the cardinality of $\mathbb{PF}_{n,k}$ is

$$\frac{n!}{k!}\Bigl((k+1)^{k-1} - k^{k-1}\Bigr).$$