Refined Face Enumeration in ν-Associahedra

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TU Dresden

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Discrete Geometry Seminar, Freie Universität Berlin

Outline

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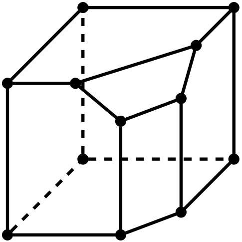
Faces of Polytopes

- $\mathcal{P} \subseteq \mathbb{R}^d$ .. polytope



Faces of Polytopes

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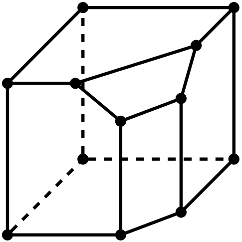
- **face numbers**: $f_k \overset{\mathsf{def}}{=} \big|\{F \in \mathcal{P} \mid \mathsf{dim}(F) = k\}\big|$



$f_3 = 1$
$f_2 = 8$
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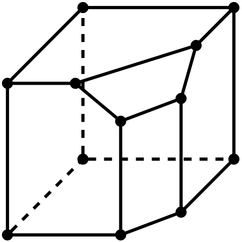

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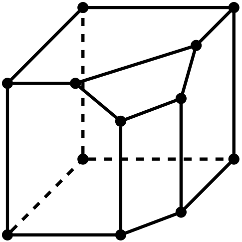

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$$\tilde{f}_{\mathcal{P}}(x) = 8x^3+18x^2+12x+1$$

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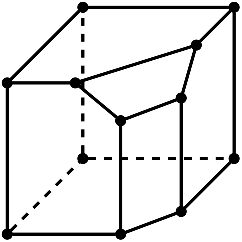

$h_3 = 3$
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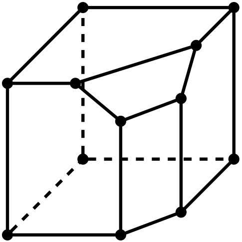

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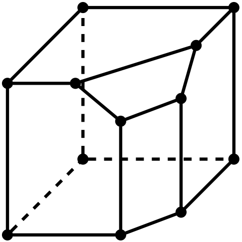

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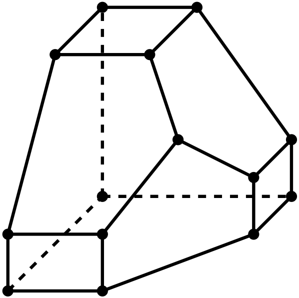
The (simple) Associahedron



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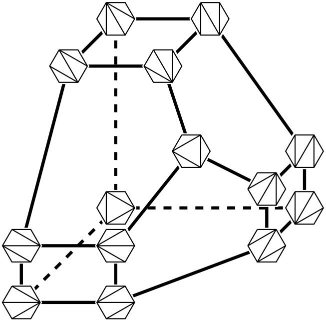
- **(simple) associahedron**: triangulations of a $d{+}3$-gon connected by diagonal flips $\rightsquigarrow \mathsf{Asso}(d)$



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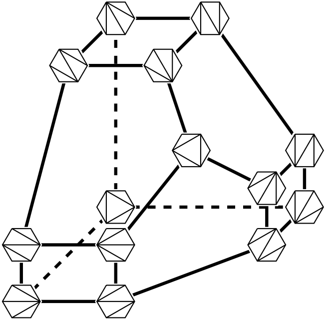

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Proposition (C. Lee, 1989)

For $d > 0$ and $0 \leq k \leq d$, we have

$$f_k = \frac{1}{d+2}\binom{d}{k}\binom{2(d+1)-k}{d+1}.$$

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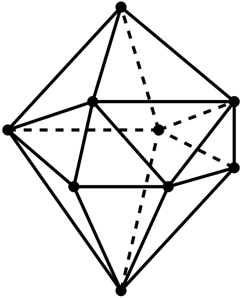

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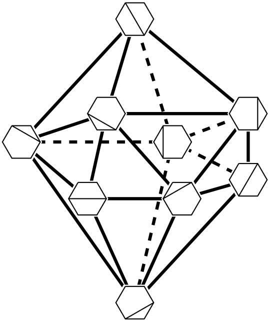

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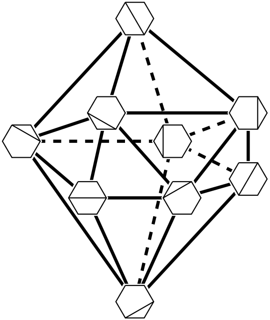

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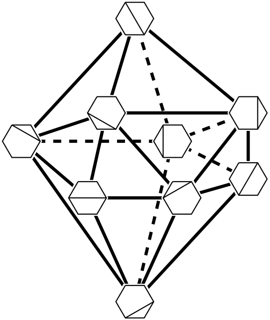

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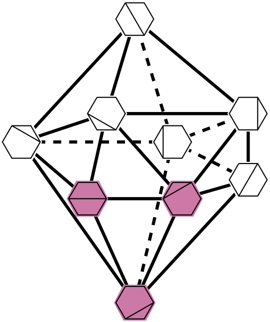

$$F(x,y) = 5x^3+5x^2y+3xy^2+y^3+10x^2+8xy+3y^2+6x+3y+1$$

Roots

- ε_i .. i^{th} unit vector in $\mathbb{R}^{d+1}$
- **(positive) roots**: $\alpha_{i,j} \stackrel{\text{def}}{=} \varepsilon_i - \varepsilon_j$ for $i < j$ $\quad\rightsquigarrow \Phi_+(d)$
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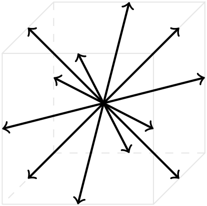

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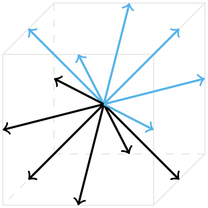

Clusters

- **almost positive roots**: $\Phi_{\geq -1}(d) \overset{\text{def}}{=} \Phi_+(d) \uplus -\Pi(d)$

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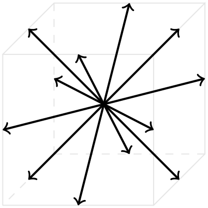

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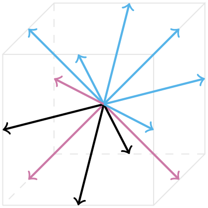

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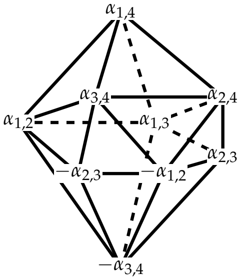

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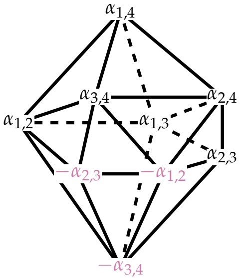

Chapoton's F-Triangle

- for $A \in \mathsf{Clus}(d)$ let
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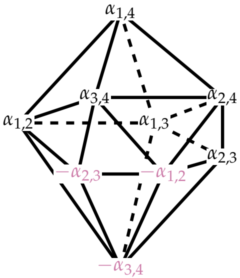

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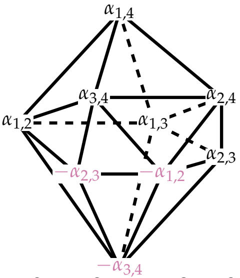

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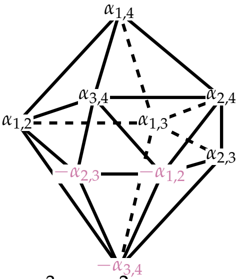

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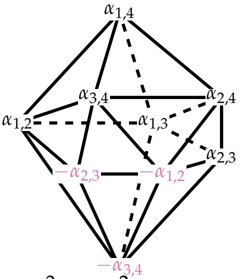

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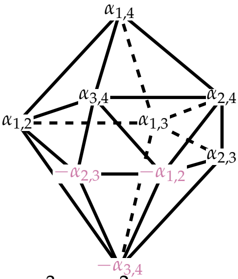

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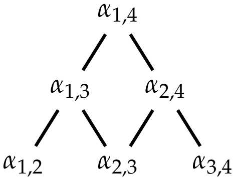

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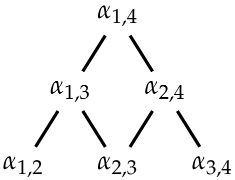

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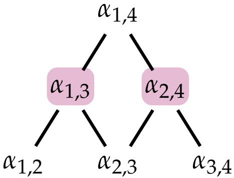

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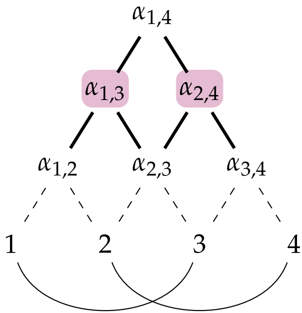

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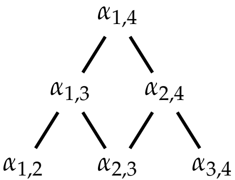

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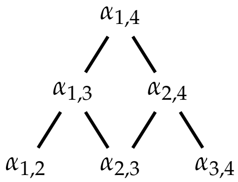

$$H_3(x,y) = x^3y^3 + 3x^2y^2 + 2x^2y + x^2 + 3xy + 3x + 1$$

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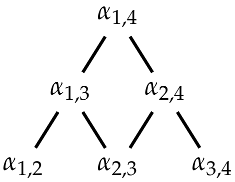
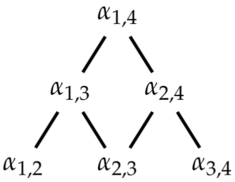


$$H_3(x,1) = x^3 + 6x^2 + 6x + 1$$

Chapoton's H-Triangle

- **root order**: $\alpha \preceq \beta$ if and only if $\beta - \alpha \in \mathsf{Span}_{\mathbb{N}}\, \Pi(d)$
- **nonnesting partition**: antichain in $\big(\Phi_+(d), \preceq\big)$ $\rightsquigarrow \mathsf{Nonn}(d)$
- ***H*-triangle**: $H_d(x,y) \overset{\mathsf{def}}{=} \sum_{A \in \mathsf{Nonn}(d)} x^{|A|} y^{|A \cap \Pi(d)|}$



$$H_3(x,1) = x^3 + 6x^2 + 6x + 1 = h_{\mathsf{Clus}(3)}(x)$$

The $F{=}H$-Correspondence

Conjecture (F. Chapoton, 2006)

For $d \geq 1$,

$$F_d(x,y) = x^d H_d\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

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$$\begin{aligned} h_{\mathsf{Clus}(d)}(x) &= f_{\mathsf{Clus}(d)}(x-1) \\ &= (x-1)^d \tilde{f}_{\mathsf{Clus}(d)}\left(\frac{1}{x-1}\right) \\ &= (x-1)^d F_d\left(\frac{1}{x-1}, \frac{1}{x-1}\right) \\ &= H_d(x,1) \end{aligned}$$

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- F. Chapoton: $\frac{\partial}{\partial y} F_d(x,y) = \sum_{\alpha\in\Pi(d)} F_{d\setminus\alpha}(x,y)$
- M. Thiel: $\frac{\partial}{\partial y} H_d(x,y) = x \sum_{\alpha\in\Pi(d)} H_{d\setminus\alpha}(x,y)$

The $F{=}H$-Correspondence

Theorem (M. Thiel, 2014)

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Outline

Dyck Paths

- **Dyck path**: northeast path from $(0,0)$ to (n,n) weakly above the diagonal $\rightsquigarrow \mathsf{Dyck}(n)$

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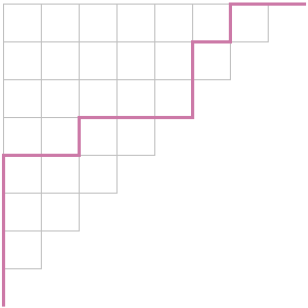

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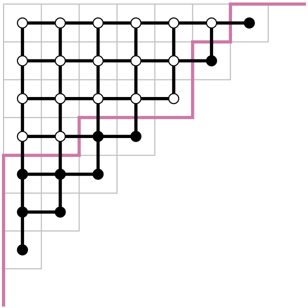

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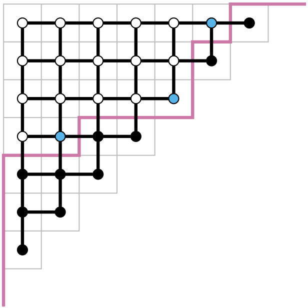
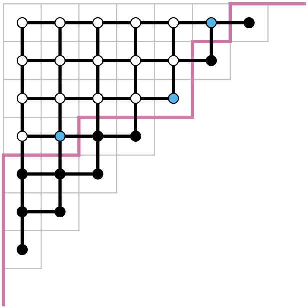


Dyck Paths

- **Dyck path**: northeast path from $(0,0)$ to (n,n) weakly above the diagonal $\rightsquigarrow \mathsf{Dyck}(n)$
- $A \in \mathsf{Nonn}(n)$ corresponds to **valleys** of $\mu \in \mathsf{Dyck}(n)$
- simple roots contained in A correspond to **returns**



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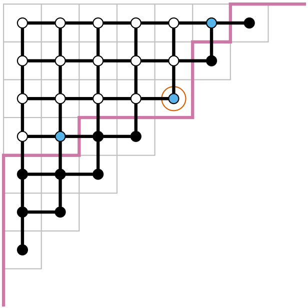

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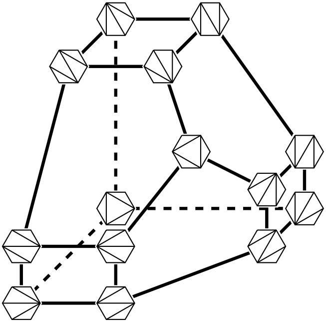
Corollary

For $n \geq 1$,

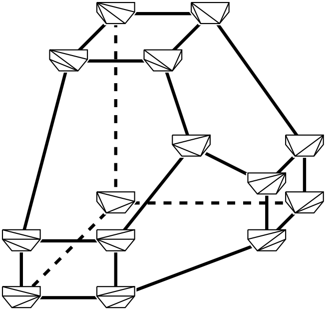
$$H_n(x,y) = \sum_{\mu \in \mathsf{Dyck}(n)} x^{\mathsf{val}(\mu)} y^{\mathsf{ret}(\mu)}.$$

The Tamari Lattice



The Tamari Lattice

- orient edges of Asso(n) according to slope



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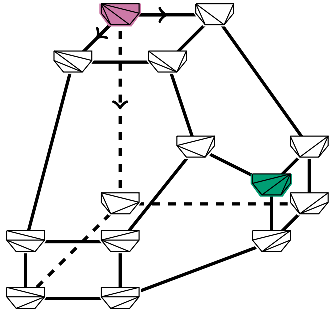

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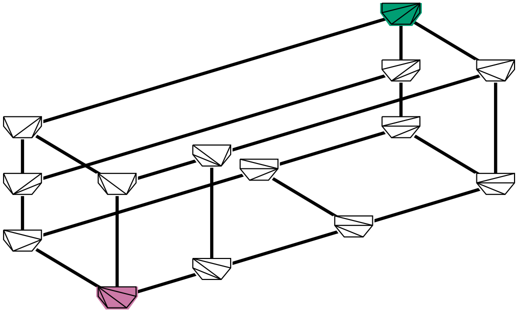
- orient edges of Asso(n) according to slope $\rightsquigarrow$ **Tam**(n)



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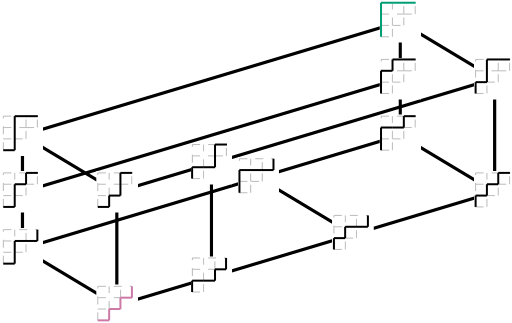

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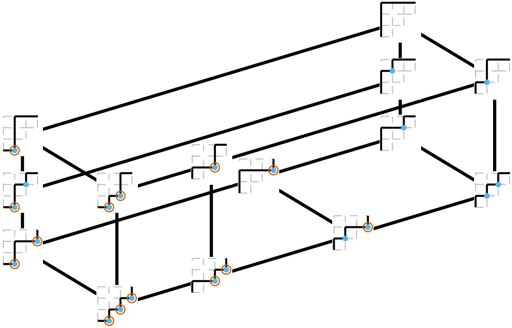

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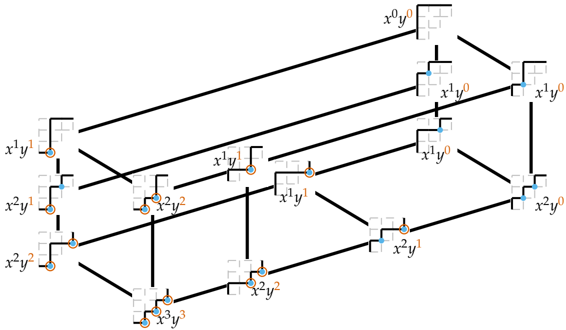

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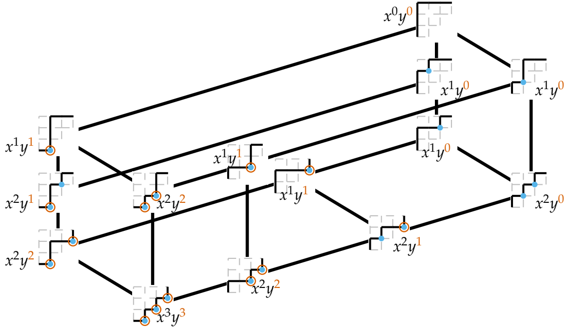

$$H_3(x,y) = x^3y^3+3x^2y^2+2x^2y+x^2+3xy+3x+1$$

The ν-Tamari Lattice via Paths

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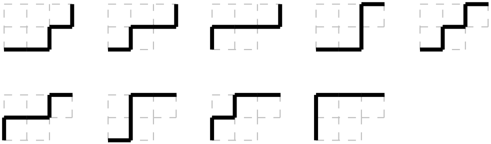

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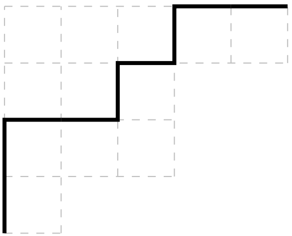
- **rotating** ν-paths by valleys



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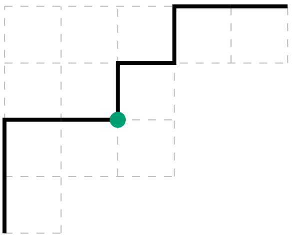

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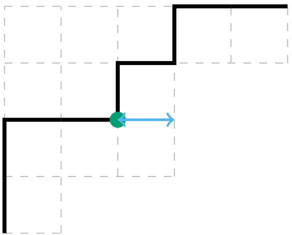

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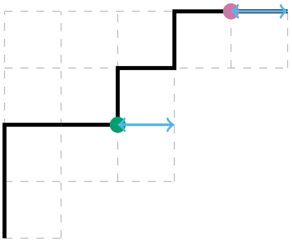

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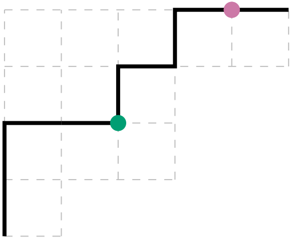

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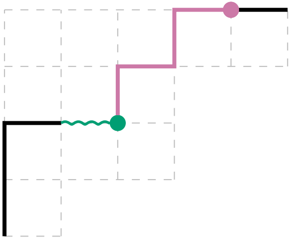

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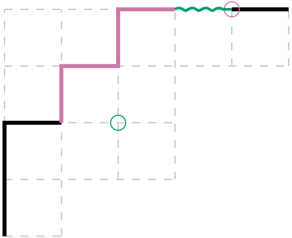

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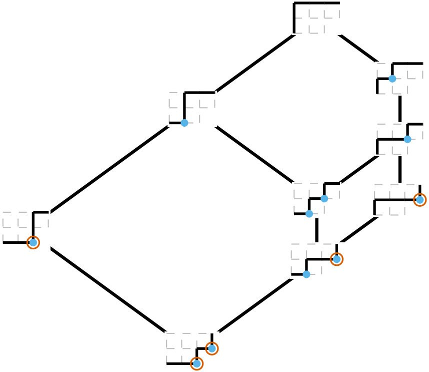

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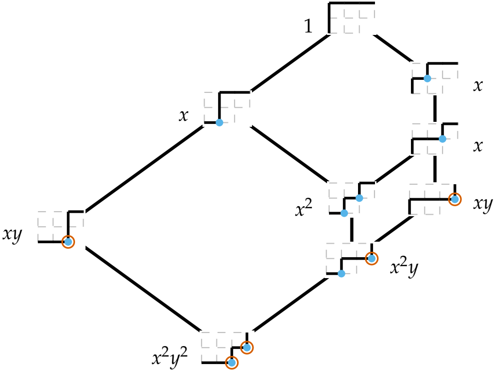

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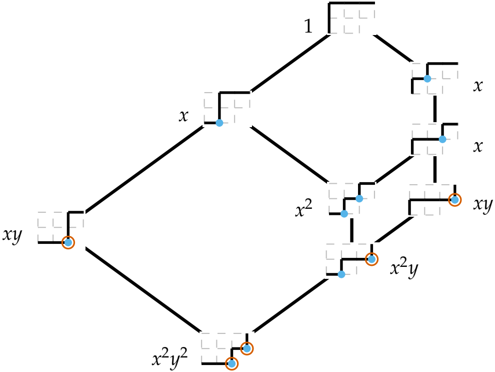

$$H_{EENEN}(x,y) = x^2y^2 + x^2y + x^2 + 2xy + 3x + 1$$

ν-Trees

- **ν-incompatible** nodes

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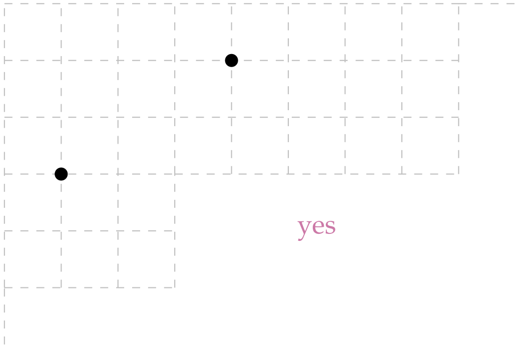

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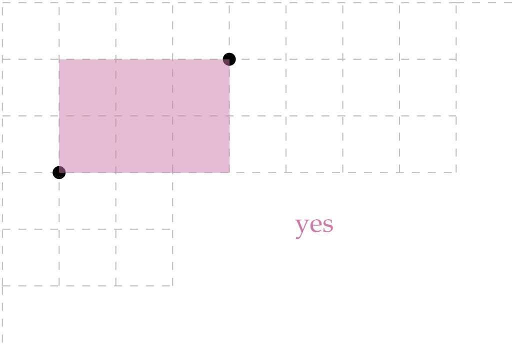

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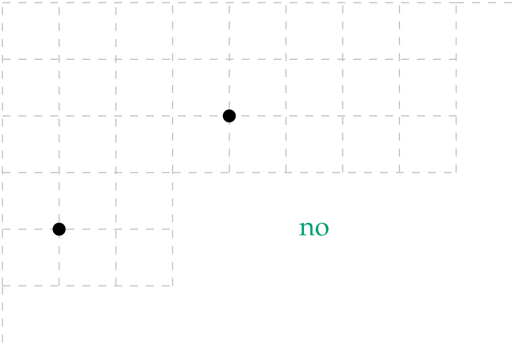

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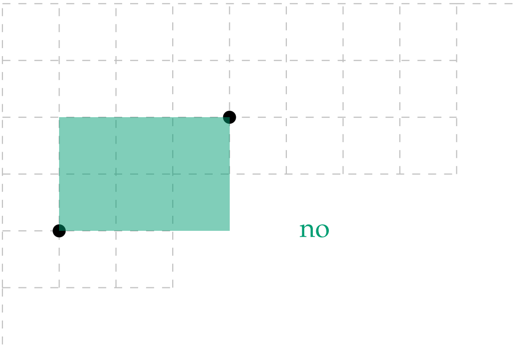
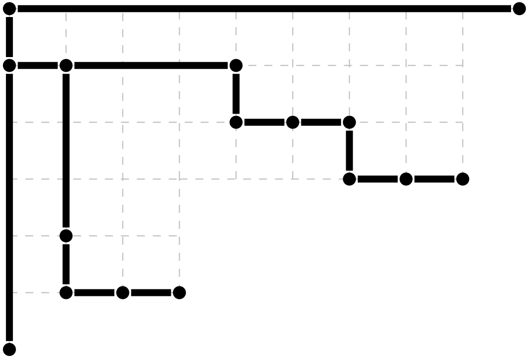


ν-Trees

- **ν-incompatible** nodes
- **ν-tree**: maximal collection of ν-compatible nodes

$\rightsquigarrow \mathsf{Tree}(\nu)$



From ν-Paths to ν-Trees

- the **right-flushing** bijection

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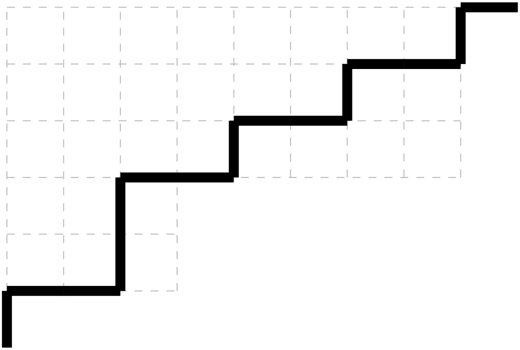

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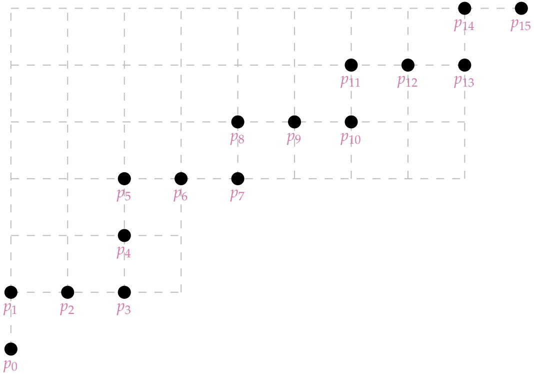

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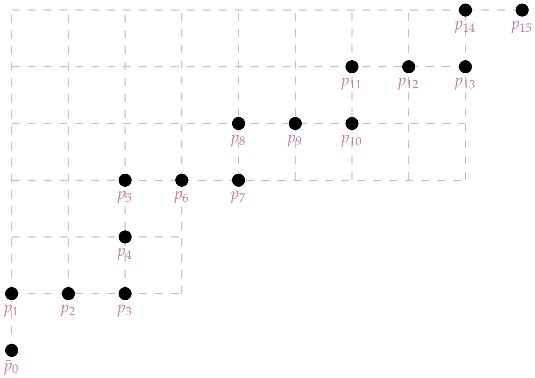

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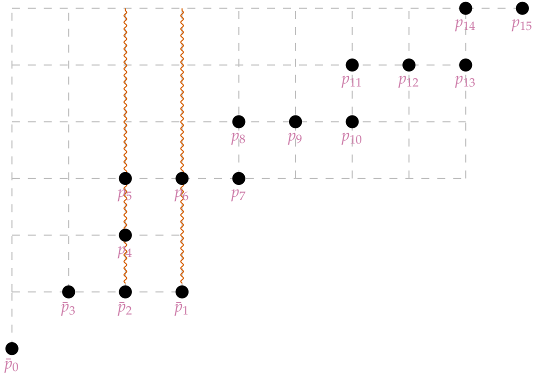

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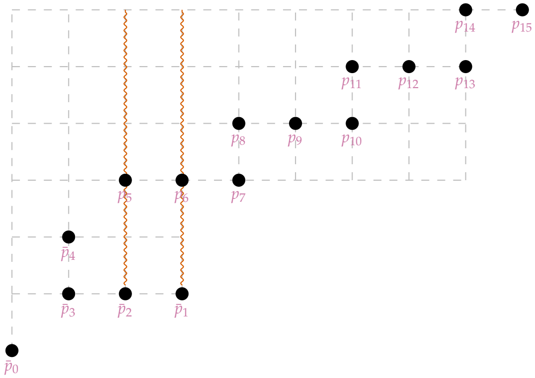

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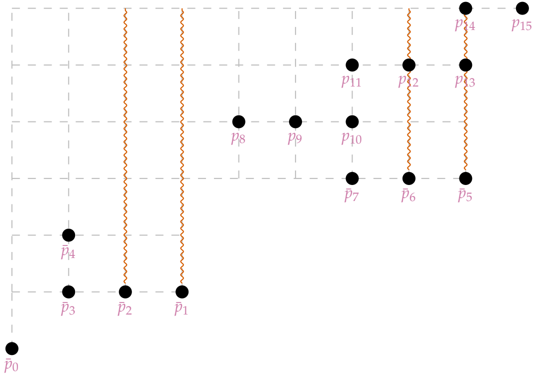

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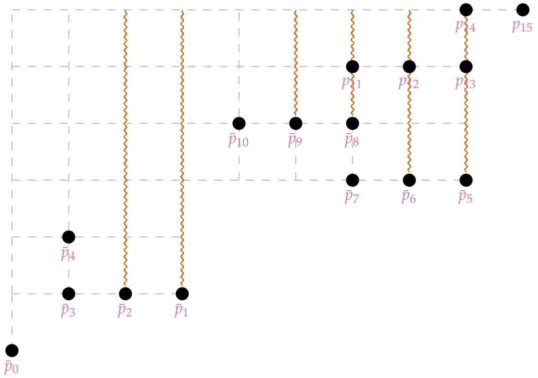

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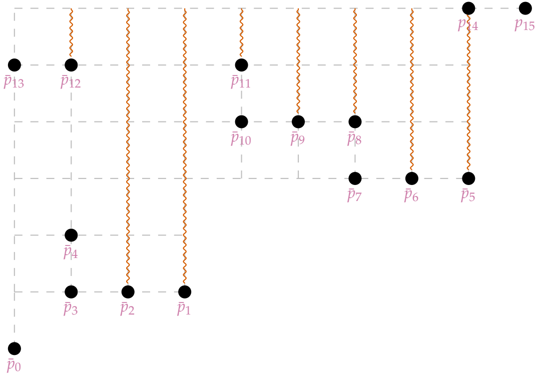

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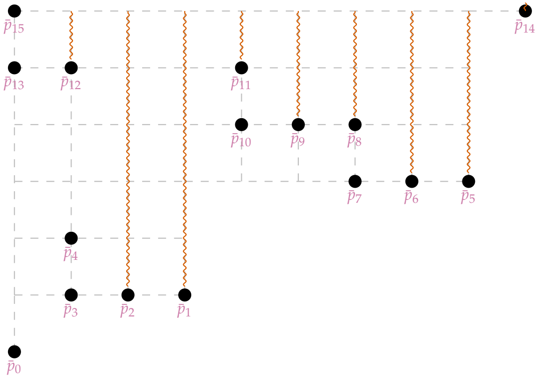

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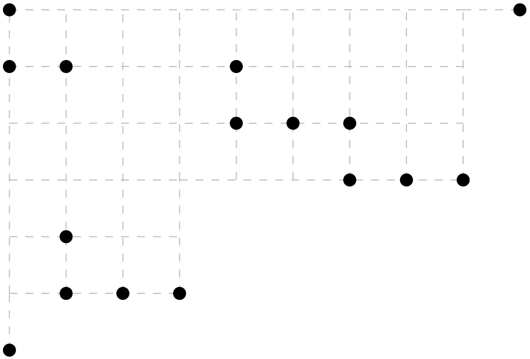

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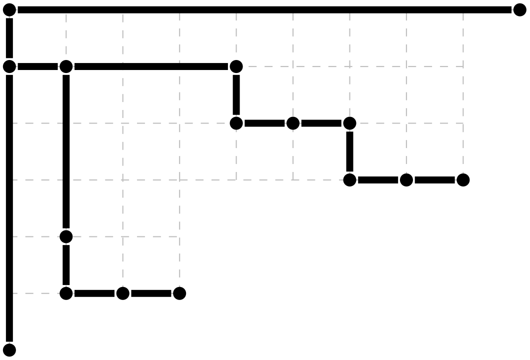


From ν-Paths to ν-Trees

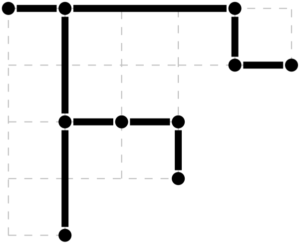
- the **right-flushing** bijection

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2020)

Right-flushing is a bijection from $\mathsf{Dyck}(\nu)$ *to* $\mathsf{Tree}(\nu)$.

The ν-Tamari Lattice via Trees

- **rotating** ν-trees by ascent nodes



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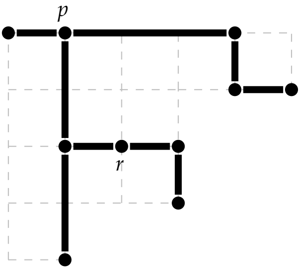

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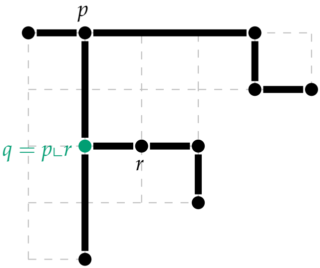

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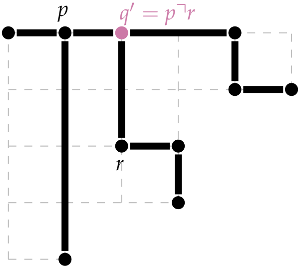
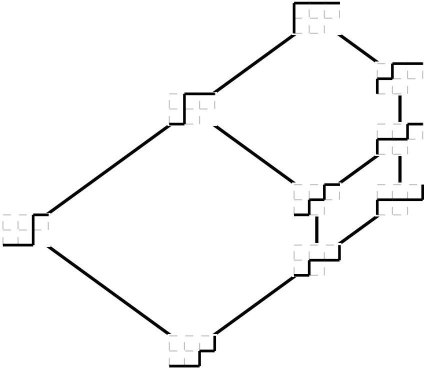


The ν-Tamari Lattice via Trees

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2020)

Right-flushing converts rotation on ν-paths into rotation on ν-trees.

The ν-Tamari Lattice via Trees



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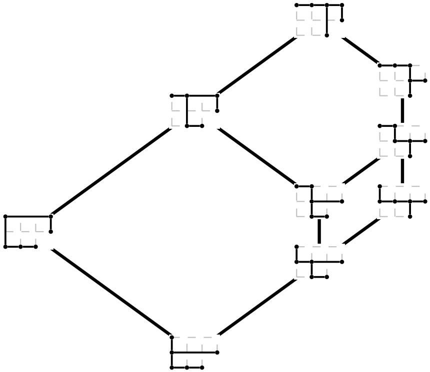

The ν-Associahedron

- **ν-face**: collection of pairwise ν-compatible nodes
- **ν-Tamari complex**: simplicial complex of ν-faces

$\rightsquigarrow \mathcal{TC}(\nu)$

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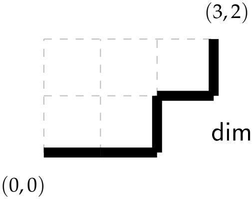

$\dim \mathcal{TC}(\nu) = 5$

The ν-Associahedron

- **covering ν-face**: ν-face containing top-left corner and at least one node per row and column

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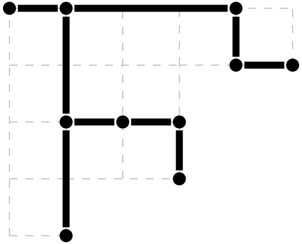

The ν-Associahedron

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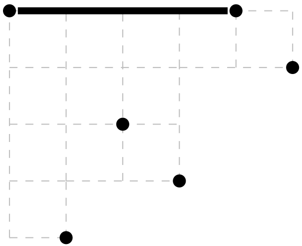

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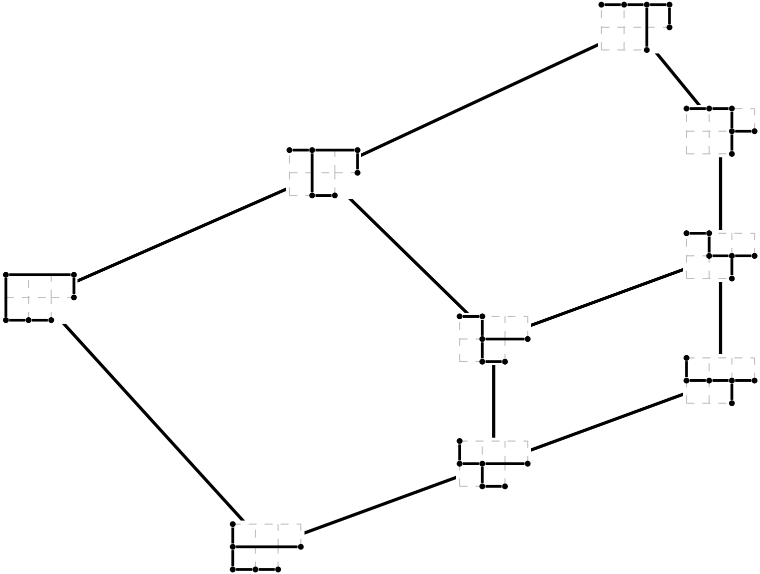
- **ν-associahedron**: polytopal complex of covering ν-faces $\rightsquigarrow \mathsf{Asso}(\nu)$

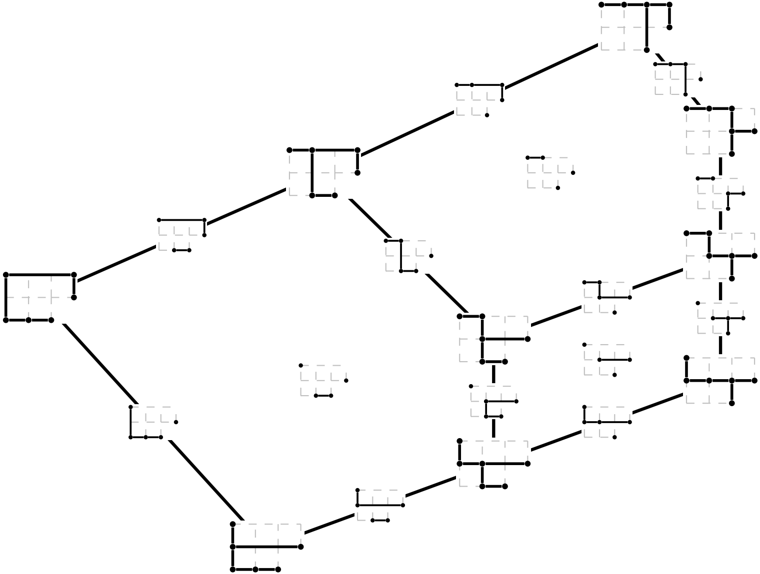
Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2019)

$\mathsf{Asso}(\nu)$ *is a polytopal complex dual to the complex of interior faces of* $\mathcal{TC}(\nu)$.

The ν-Associahedron



The ν-Associahedron



Outline

Some Statistics

- **relevant** node: node in first column and in row of a valley of ν

Some Statistics

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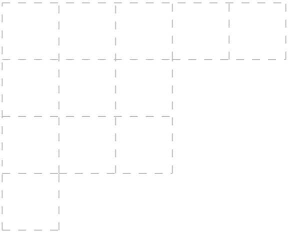
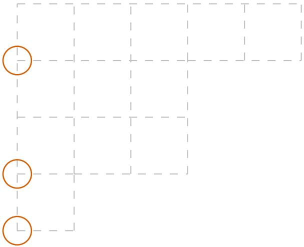


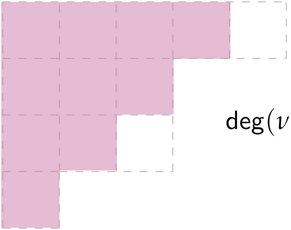
Some Statistics

- **relevant** node: node in first column and in row of a valley of ν



Some Statistics

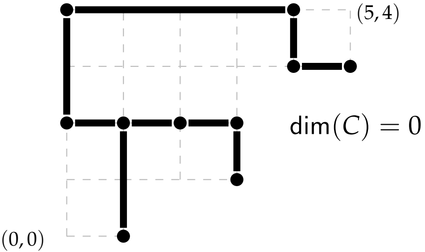
- $\mathsf{deg}(\nu) \overset{\mathsf{def}}{=} \max\bigl\{\mathsf{val}(\mu) \mid \mu \in \mathsf{Dyck}(\nu)\bigr\}$



$\mathsf{deg}(\nu) = 4$

Some Statistics

- ν goes from $(0,0)$ to (m,n); $C \in \mathsf{Asso}(\nu)$
- $\mathsf{deg}(\nu) \overset{\mathsf{def}}{=} \max\{\mathsf{val}(\mu) \mid \mu \in \mathsf{Dyck}(\nu)\}$
- $\mathsf{dim}(C) \overset{\mathsf{def}}{=} m + n + 1 - |C|$



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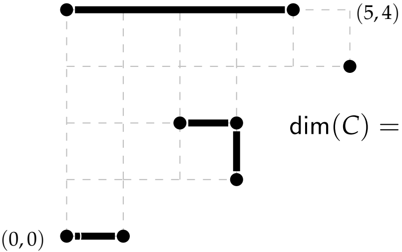

$\mathsf{dim}(C) = 2$

Some Statistics

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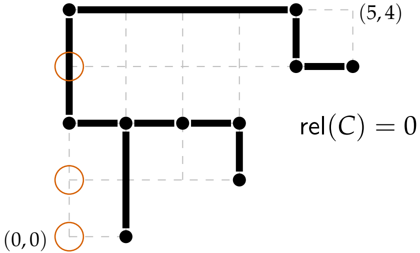
- $\mathsf{deg}(\nu) \overset{\mathsf{def}}{=} \mathsf{max}\{\mathsf{val}(\mu) \mid \mu \in \mathsf{Dyck}(\nu)\}$
- $\mathsf{dim}(C) \overset{\mathsf{def}}{=} m+n+1-|C|$
- $\mathsf{rel}(C)$ is number of relevant nodes contained in C



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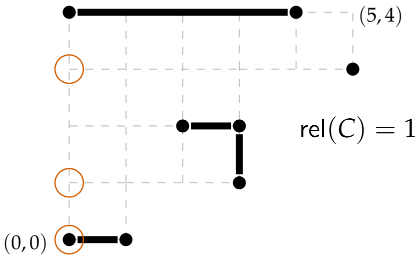

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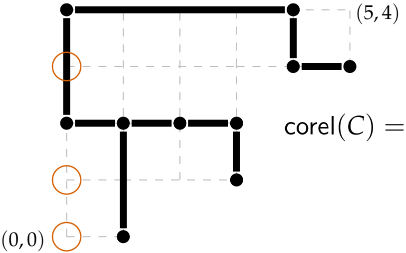
- $\mathsf{rel}(C)$ is number of relevant nodes contained in C
- $\mathsf{corel}(C) \overset{\mathsf{def}}{=} \mathsf{deg}(\nu) - \mathsf{dim}(C) - \mathsf{rel}(C)$



$\mathsf{corel}(C) = 4$

Some Statistics

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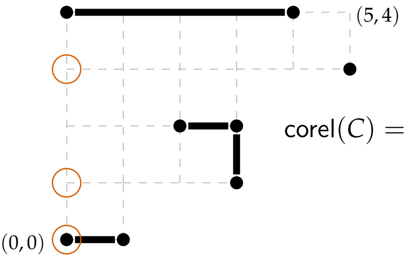

$\mathsf{corel}(C) = 1$

Some Statistics

- $C \in \mathsf{Asso}(\nu)$
- $\mathsf{rel}(C)$ is number of relevant nodes contained in C
- $\mathsf{asc}(C)$ is number of ascent nodes of C

Some Statistics

- $C \in \mathsf{Asso}(\nu)$
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Lemma (C. Ceballos & ✇, 2020)

Right-flushing sends val *to* asc *and* ret *to* rel.

Some Statistics

- recall: $H_\nu(x,y) = \sum_{\mu\in\mathsf{Dyck}(\nu)} x^{\mathsf{val}(\mu)}y^{\mathsf{ret}(\mu)}$

Corollary (C. Ceballos & ✠, 2020)

For any northeast path ν,

$$H_\nu(x,y) = \sum_{T\in\mathsf{Tree}(\nu)} x^{\mathsf{asc}(T)}y^{\mathsf{rel}(T)}.$$

The F-Triangle

recall:

$$F_d(x,y) = x^d H_d\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

The F-Triangle

$$F_\nu(x,y) = x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

The F-Triangle

$$F_\nu(x,y) = x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

$$= x^{\mathsf{deg}(\nu)} \sum_{T\in\mathsf{Tree}(\nu)} \left(\frac{x+1}{x}\right)^{\mathsf{asc}(T)} \left(\frac{y+1}{x+1}\right)^{\mathsf{rel}(T)}$$

The F-Triangle

$$\begin{aligned} F_\nu(x,y) &= x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)\\ &= x^{\mathsf{deg}(\nu)} \sum_{T\in\mathsf{Tree}(\nu)} \left(\frac{x+1}{x}\right)^{\mathsf{asc}(T)} \left(\frac{y+1}{x+1}\right)^{\mathsf{rel}(T)}\\ &= \sum_{T\in\mathsf{Tree}(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T)-\mathsf{rel}(T)}(y+1)^{\mathsf{rel}(T)} \end{aligned}$$

The F-Triangle

- $\mathsf{Asc}(T)$ is set of ascent nodes of T
- $\mathsf{Rel}(T)$ is set of relevant nodes of T

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$$
\begin{aligned}
F_\nu(x,y) &= \sum_{T\in\mathsf{Tree}(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T)-\mathsf{rel}(T)}(y+1)^{\mathsf{rel}(T)}\\
&= \sum_{T\in\mathsf{Tree}(\nu)} \Biggl(\sum_{A''\subseteq\mathsf{Asc}(T)\setminus\mathsf{Rel}(T)} x^{\mathsf{deg}(\nu)-\mathsf{rel}(T)-\lvert A''\rvert}\\
&\qquad\qquad \times \sum_{A'\subseteq\mathsf{Rel}(T)} y^{\mathsf{rel}(T)-\lvert A'\rvert}\Biggr)
\end{aligned}
$$

The F-Triangle

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The F-Triangle

Proposition (C. Ceballos & V. Pons, 2019)

The map $(T,A)\mapsto T\setminus A$ is a bijection from $\bigl\{(T,A)\mid T\in\mathsf{Tree}(\nu), A\subseteq\mathsf{Asc}(T)\bigr\}$ to $\mathsf{Asso}(\nu)$.

$$\begin{aligned}F_\nu(x,y) &= \sum_{T\in\mathsf{Tree}(\nu)}\Biggl(\sum_{A''\subseteq\mathsf{Asc}(T)\setminus\mathsf{Rel}(T)} x^{\mathsf{deg}(\nu)-\mathsf{rel}(T)-\lvert A''\rvert}\\ &\qquad\times\sum_{A'\subseteq\mathsf{Rel}(T)} y^{\mathsf{rel}(T)-\lvert A'\rvert}\Biggr)\\ &= \sum_{T\in\mathsf{Tree}(\nu)}\sum_{\substack{A\subseteq\mathsf{Asc}(T)\\ A'=A\cap\mathsf{Rel}(T)\\ A''=A\setminus\mathsf{Rel}(T)}} x^{\mathsf{deg}(\nu)-\mathsf{rel}(T)-\lvert A''\rvert}y^{\mathsf{rel}(T)-\lvert A'\rvert}\end{aligned}$$

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$$= \sum_{C\in\mathsf{Asso}(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{dim}(C)-\mathsf{rel}(C)}y^{\mathsf{rel}(C)}$$

$$= \sum_{C\in\mathsf{Asso}(\nu)} x^{\mathsf{corel}(C)}y^{\mathsf{rel}(C)}$$

The F-Triangle

$$F_\nu(x,y) \overset{\mathsf{def}}{=} \sum_{C \in \mathsf{Asso}(\nu)} x^{\mathsf{corel}(C)} y^{\mathsf{rel}(C)}$$

The F-Triangle

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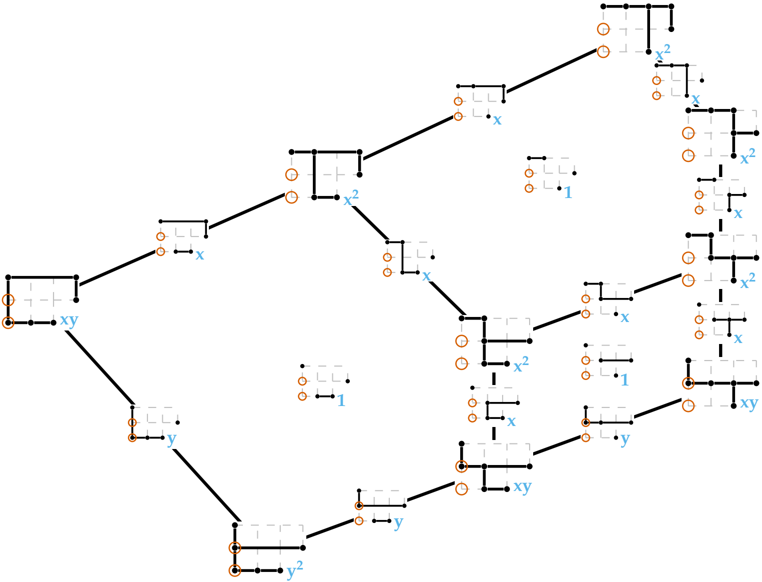
For every northeast path ν,

$$F_\nu(x,y) = x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

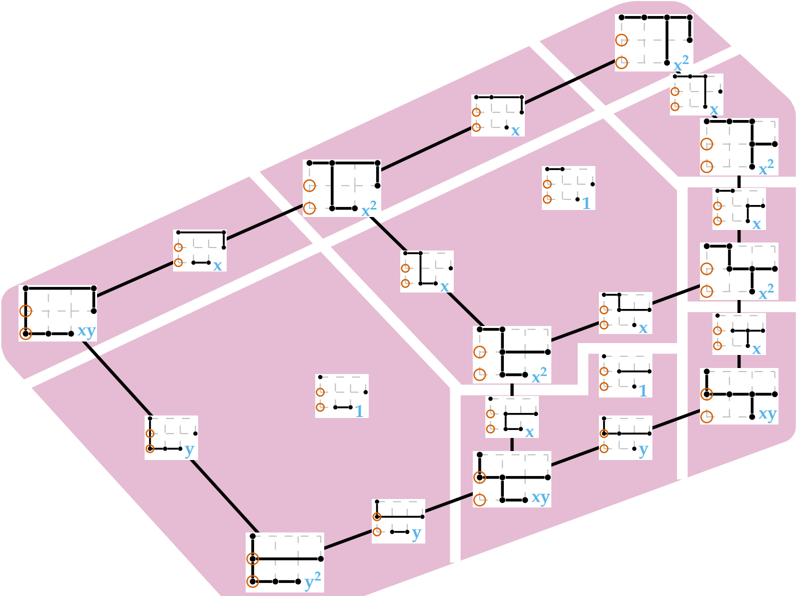
$$F_\nu(x,y) \overset{\mathsf{def}}{=} \sum_{C\in\mathsf{Asso}(\nu)} x^{\mathsf{corel}(C)} y^{\mathsf{rel}(C)}$$

$$= \sum_{T\in\mathsf{Tree}(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{asc}(T)} (x+1)^{\mathsf{asc}(T)-\mathsf{rel}(T)} (y+1)^{\mathsf{rel}(T)}$$

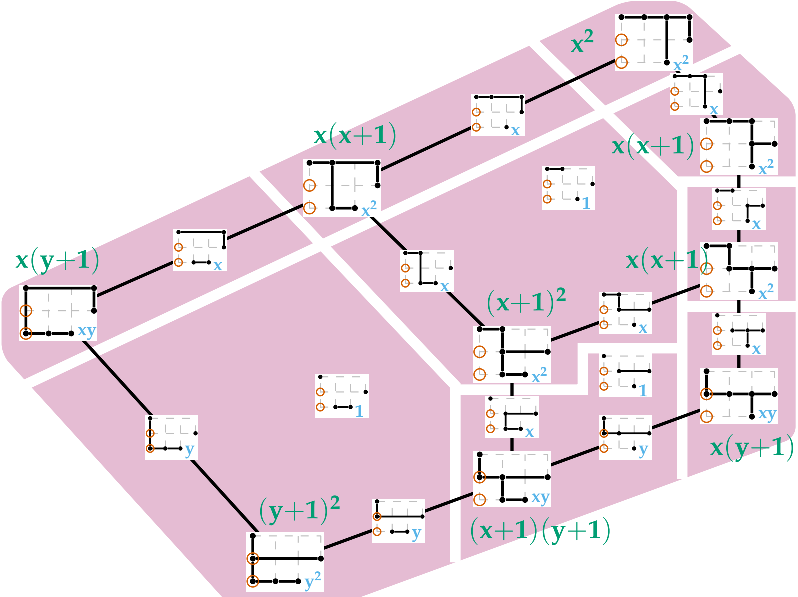
The F-Triangle



The F-Triangle

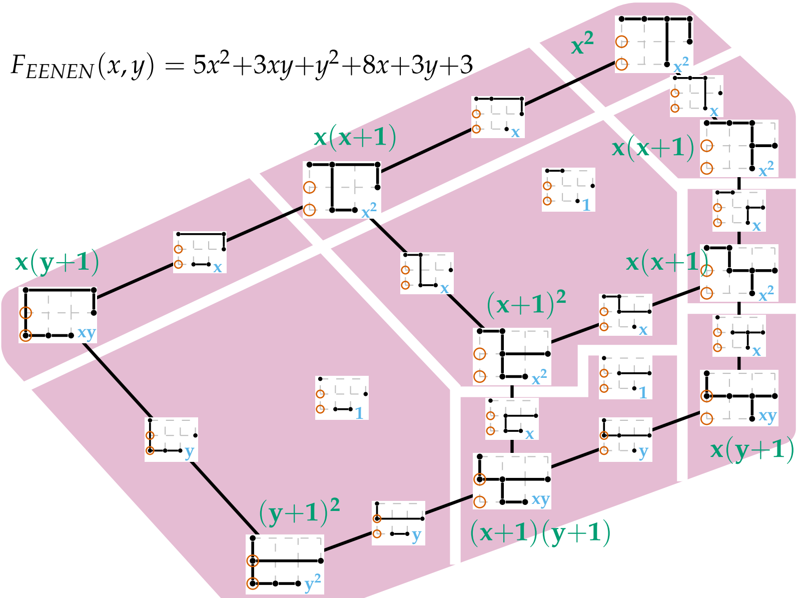


The *F*-Triangle



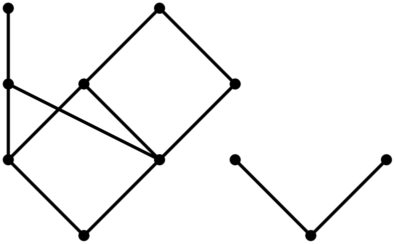
The F-Triangle

$$F_{EENEN}(x,y) = 5x^2+3xy+y^2+8x+3y+3$$



F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq); p \in P$
- $\mathsf{Succ}(p) \overset{\mathsf{def}}{=} \{p' \in P \mid p \lessdot p'\}$
- $\mathsf{out}(p) \overset{\mathsf{def}}{=} \big|\mathsf{Succ}(p)\big|$



F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq)$; $p \in P$; λ .. 01-labeling
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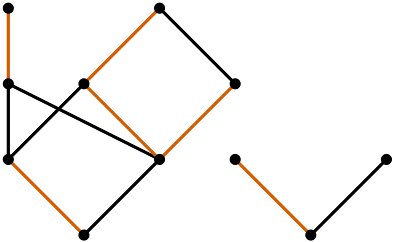

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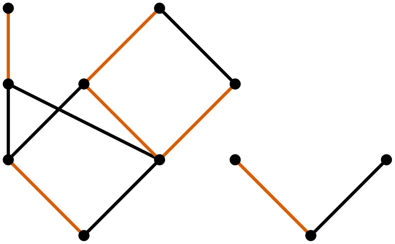
- $\mathsf{mrk}(p) \overset{\mathsf{def}}{=} \big|\{p' \in \mathsf{Succ}(p) \mid \lambda(p,p') = 1\}\big|$



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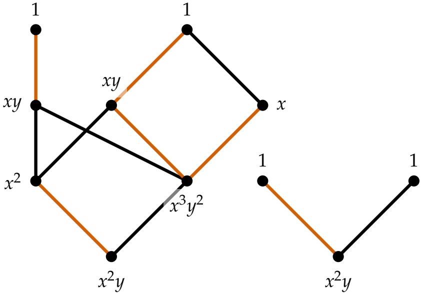
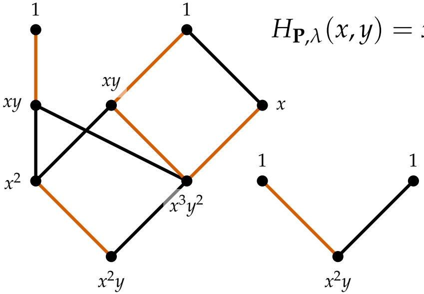


F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq)$; $p \in P$; λ .. 01-labeling
- $H_{\mathbf{P},\lambda}(x,y) \stackrel{\mathsf{def}}{=} \sum_{p \in P} x^{\mathsf{out}(p)} y^{\mathsf{mrk}(p)}$



$$H_{\mathbf{P},\lambda}(x,y) = x^3y^2 + 2x^2y + x^2 + 2xy + x + 4$$

F- and H-Triangles for Posets

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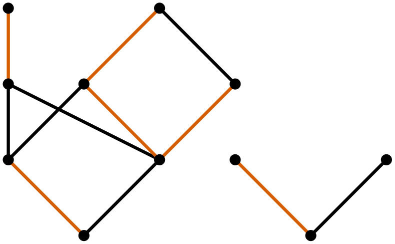

F- and H-Triangles for Posets

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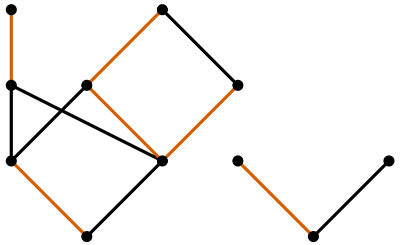

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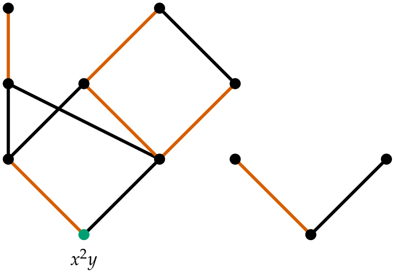

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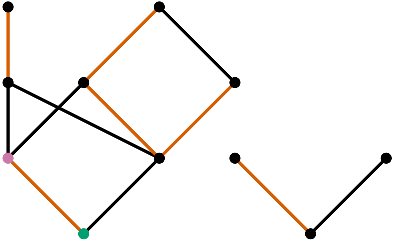

x^2y+x^2

F- and H-Triangles for Posets

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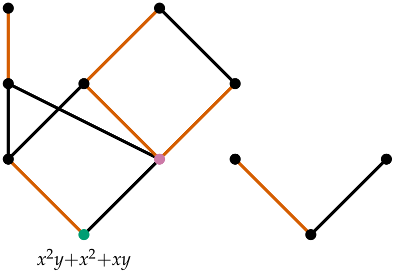

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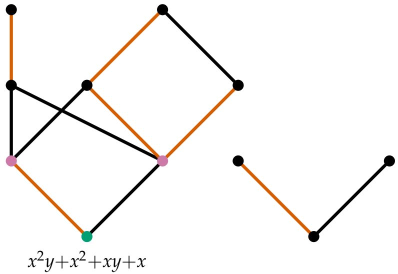

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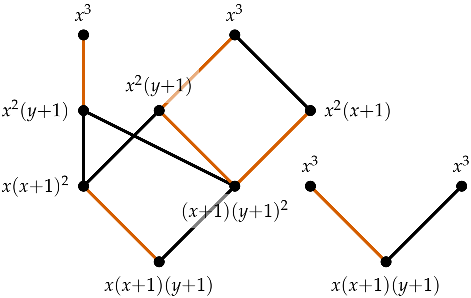
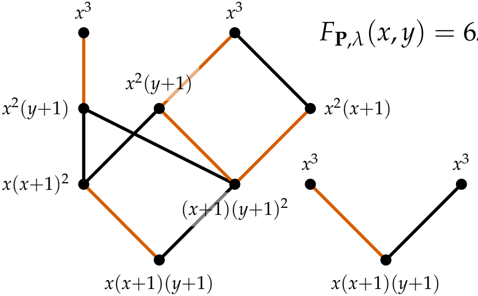
- $\mathsf{neg}(p, S) \overset{\mathsf{def}}{=} \mathsf{mrk}(p) - \Bigl|\bigl\{s \in S \mid \lambda(p, s) = 1\bigr\}\Bigr|$
- $\mathsf{pos}(p, S) \overset{\mathsf{def}}{=} \mathsf{deg}(\mathbf{P}) - |S| - \mathsf{neg}(p, S)$



F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq)$; $p \in P$; $S \subseteq \mathsf{Succ}(p)$
- $F_{\mathbf{P},\lambda}(x,y) \overset{\mathsf{def}}{=} \sum_{p\in P} \sum_{S\subseteq \mathsf{Succ}(p)} x^{\mathsf{pos}(p,S)} y^{\mathsf{neg}(p,S)}$



$$F_{\mathbf{P},\lambda}(x,y) = 6x^3+4x^2y+xy^2+7x^2 +4xy+y^2+4x+2y+1$$

F- and H-Triangles for Posets

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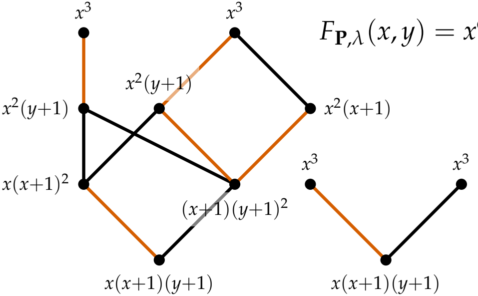


$$F_{\mathbf{P},\lambda}(x,y) = x^{\mathsf{deg}(\mathbf{P})} H_{\mathbf{P},\lambda}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq)$; $p \in P$; $S \subseteq \mathsf{Succ}(p)$
- $H_{\mathbf{P},\lambda}(x,y) \overset{\mathsf{def}}{=} \sum_{p \in P} x^{\mathsf{out}(p)} y^{\mathsf{mrk}(p)}$
- $F_{\mathbf{P},\lambda}(x,y) \overset{\mathsf{def}}{=} \sum_{p \in P} \sum_{S \subseteq \mathsf{Succ}(p)} x^{\mathsf{pos}(p,S)} y^{\mathsf{neg}(p,S)}$

Theorem (C. Ceballos & , 2021)

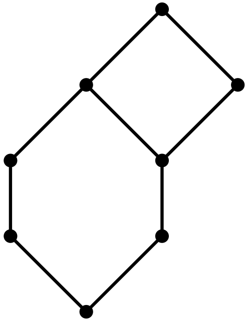
For every finite poset $\mathbf{P}$ *and every* 01*-labeling* λ*,*

$$F_{\mathbf{P},\lambda}(x,y) = x^{\mathsf{deg}(\mathbf{P})} H_{\mathbf{P},\lambda}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

Outline

The M-Triangle

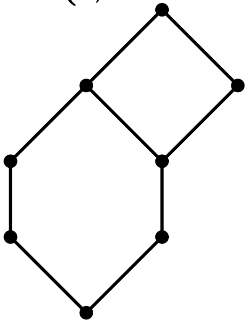
- $\mathbf{P} = (P, \leq)$ .. poset



The M-Triangle

- $\mathbf{P} = (P, \leq)$ .. poset
- **Möbius function**:

$$\mu_{\mathbf{P}}(p,q) \overset{\mathsf{def}}{=} \begin{cases} 1, & p = q,\\ -\sum_{p\leq r<q} \mu_{\mathbf{P}}(p,r), & p < q,\\ 0, & \text{otherwise}\end{cases}$$



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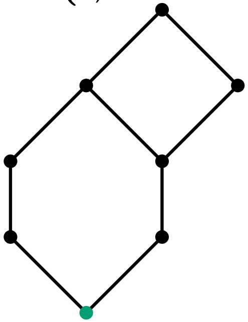

$\mu(\bullet, \bullet) = 1$

The M-Triangle

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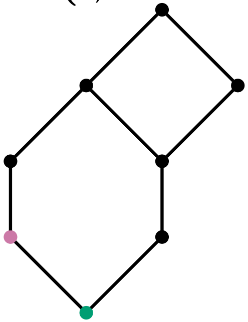

$\mu(\bullet, \bullet) = -1$

The M-Triangle

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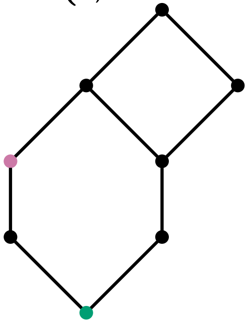

$\mu(\bullet, \bullet) = 0$

The M-Triangle

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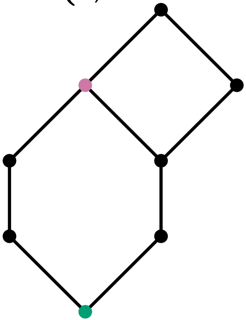
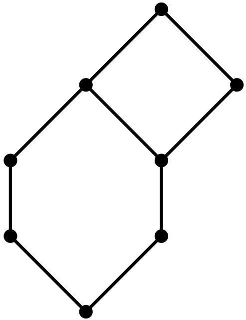


$\mu(\bullet, \bullet) = 1$

The M-Triangle

- $\mathbf{P} = (P, \leq)$ .. ranked poset
- ***M*-triangle**: $M_{\mathbf{P}}(x,y) \stackrel{\text{def}}{=} \sum_{p,q \in P} \mu_{\mathbf{P}}(p,q) x^{\mathsf{rk}(p)} y^{\mathsf{rk}(q)}$



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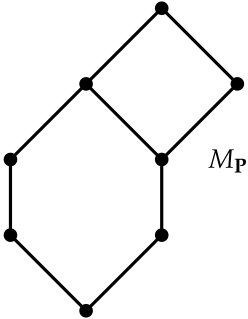
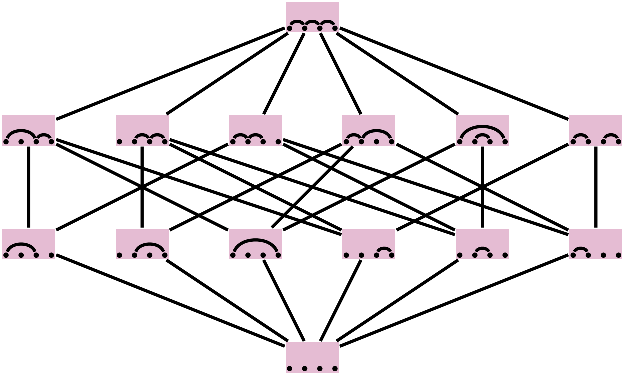


$$M_{\mathbf{P}}(x,y) = x^4y^4 - 2x^3y^4 + 2x^3y^3 + x^2y^4 - 3x^2y^3 + 2x^2y^2 - 2xy^2 + y^3 + 2xy - 2y + 1$$

The M-Triangle

- $M_n(x,y) \overset{\text{def}}{=} M_{\mathbf{Nonc}(n)}(x,y)$



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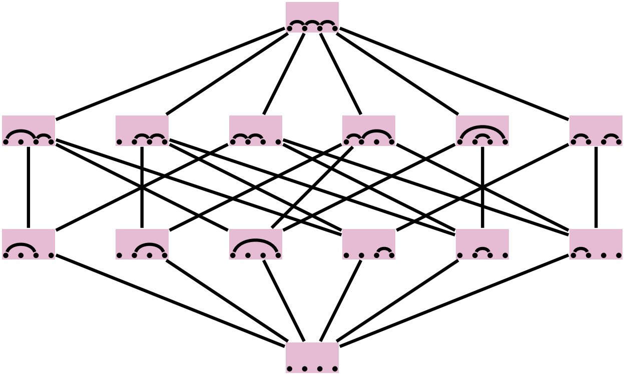
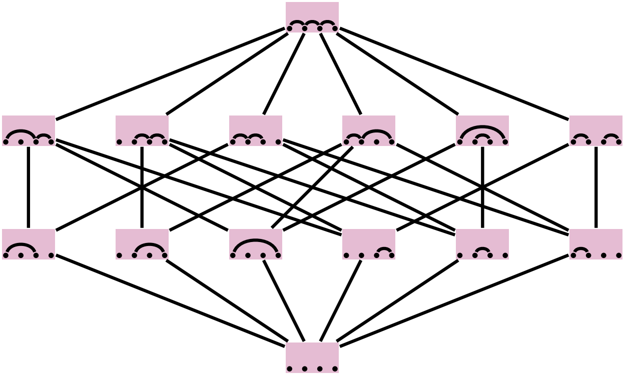


$$M_3(x,y) = x^3y^3 - 6x^2y^3 + 6x^2y^2 + 10xy^3 - 16xy^2 - 5y^3 + 6xy + 10y^2 - 6y + 1$$

The M-Triangle

- $F_3(x,y) =$
$$5x^3+5x^2y+3xy^2+y^3+10x^2+8xy+3y^2+6x+3y+1$$



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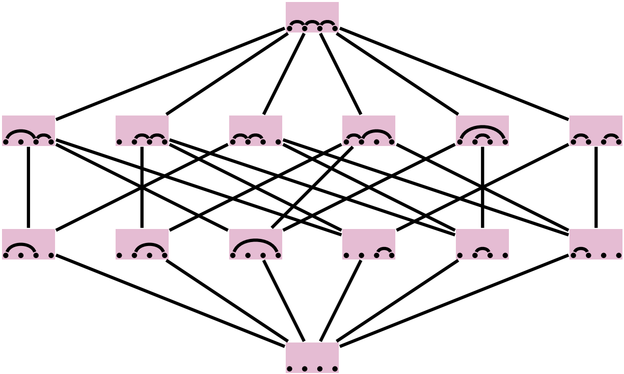

$$M_3(x,y) = (xy-1)^3 F_3\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$$

The M-Triangle

- $$F_n(x,y) \overset{\mathsf{def}}{=} \sum_{A\in\mathsf{Clus}(n)} x^{\mathsf{pos}(A)} y^{\mathsf{neg}(A)}$$
- $$M_n(x,y) \overset{\mathsf{def}}{=} \sum_{p,q\in\mathbf{Nonc}(n)} \mu_{\mathbf{Nonc}(n)}(p,q) x^{\mathsf{rk}(p)} y^{\mathsf{rk}(q)}$$

Conjecture (F. Chapoton, 2004)

For $n \geq 1$,

$$M_n(x,y) = (xy-1)^n F_n\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right).$$

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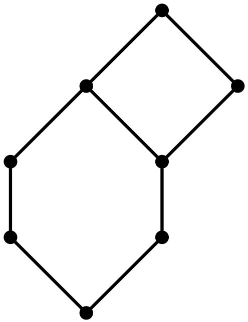
Theorem (C. Athanasiadis, 2007)

For $n \geq 1$,

$$M_n(x,y) = (xy-1)^n F_n\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right).$$

The Core Label Order

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice



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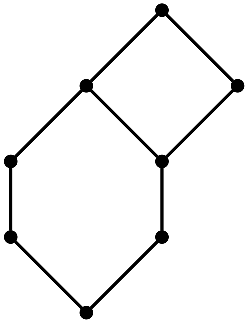
- $\mathbf{L} = (L, \leq)$ .. (finite) lattice, $p \in L$
- **nucleus**: $p^{\uparrow} \overset{\mathsf{def}}{=} p \vee \bigvee \mathsf{Succ}(p)$



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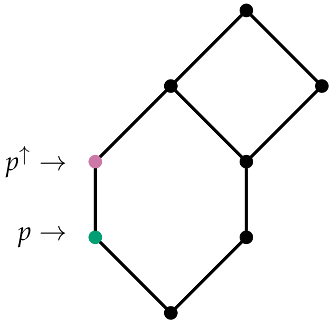

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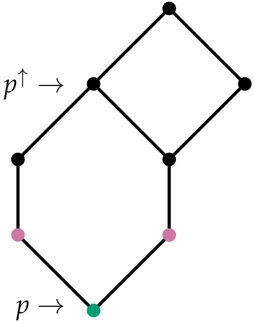

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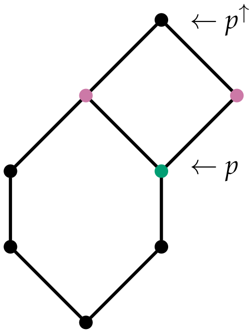

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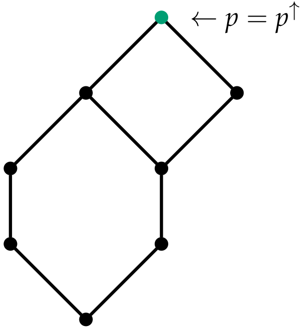

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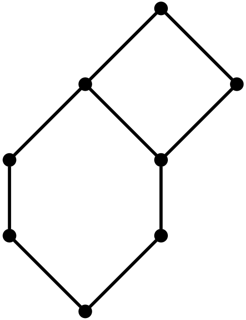
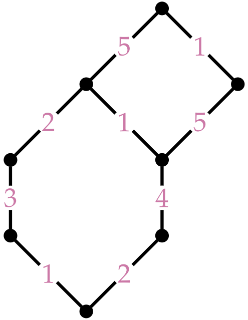


The Core Label Order

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice, λ .. edge labeling



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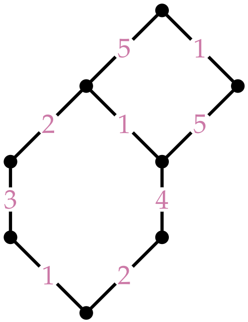

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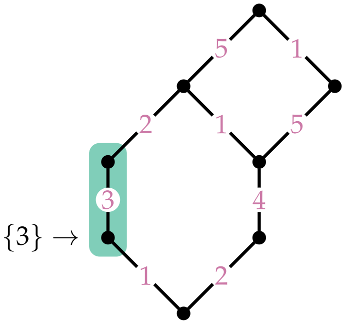

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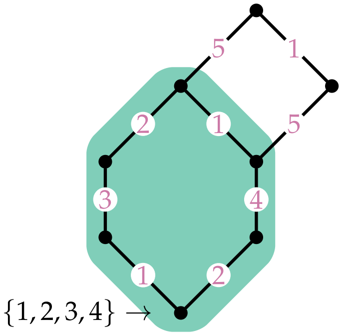

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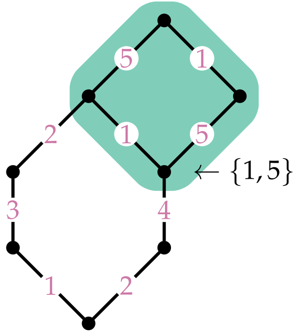

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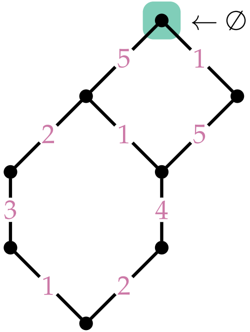

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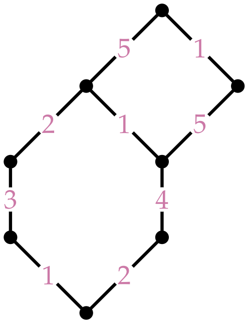

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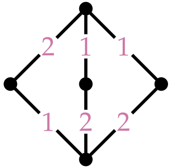

not a core labeling

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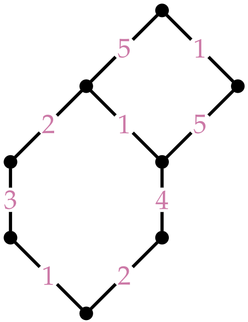

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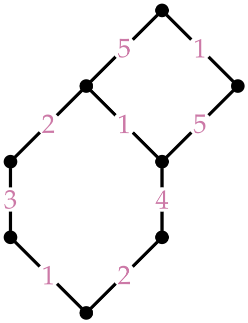
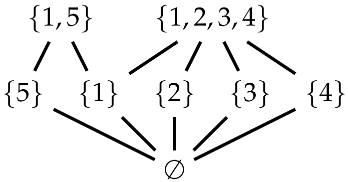
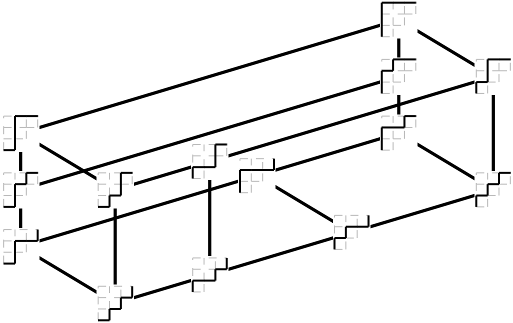




The Core Label Order of the Tamari Lattice



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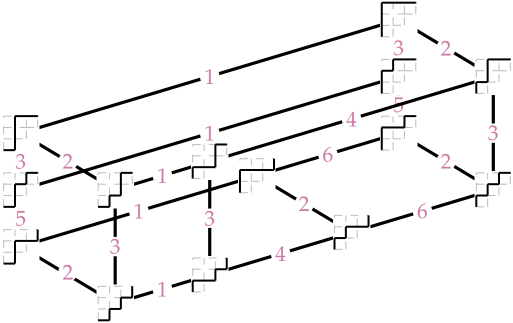

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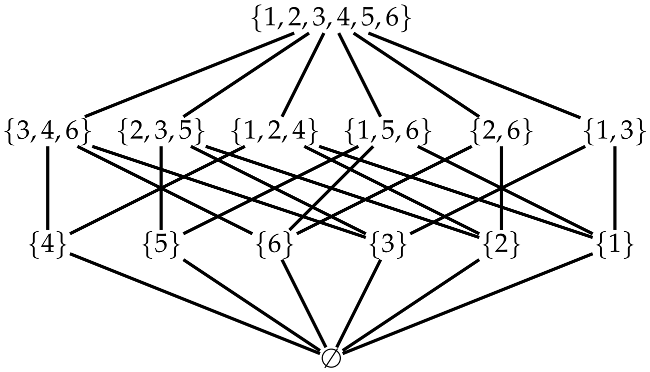

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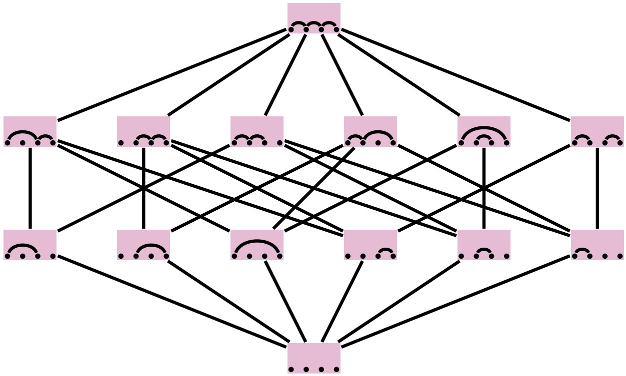


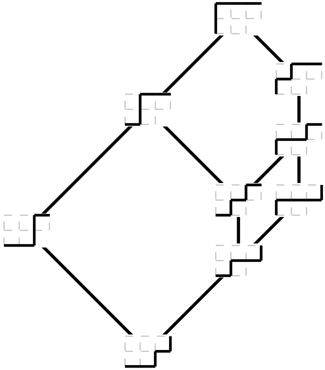
The Core Label Order of the Tamari Lattice

Theorem (N. Reading, 2011)

For $n > 0$, the core label order of $\mathsf{Tam}(n)$ *is isomorphic to the noncrossing partition lattice* $\mathsf{Nonc}(n)$.

The M-Triangle of ν-Tamari Lattices

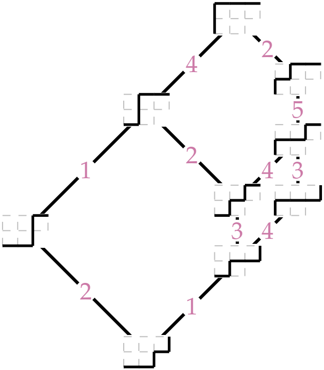
- labeling edges of $\mathsf{Tam}(\nu)$ by **perspectivity**
  $\rightsquigarrow \mathsf{CLO}\big(\mathsf{Tam}(\nu)\big)$



The M-Triangle of ν-Tamari Lattices

Perspectivity Irreducibility

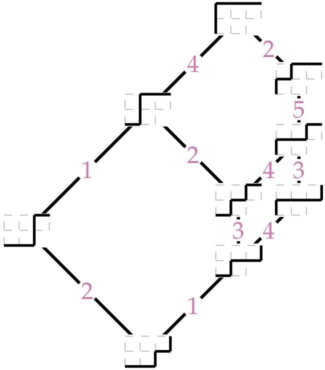
- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity** $\rightsquigarrow \mathbf{CLO}\big(\mathbf{Tam}(\nu)\big)$

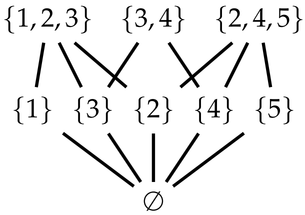


The M-Triangle of ν-Tamari Lattices

Perspectivity Irreducibility

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity** $\rightsquigarrow \mathbf{CLO}\big(\mathbf{Tam}(\nu)\big)$





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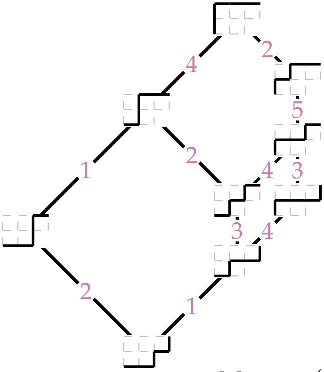

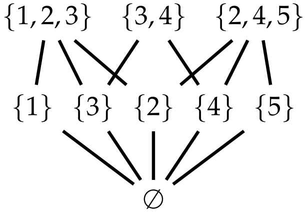

$$M_{EENEN}(x,y) = 3x^2y^2 - 8xy^2 + 5xy + 5y^2 - 5y + 1$$

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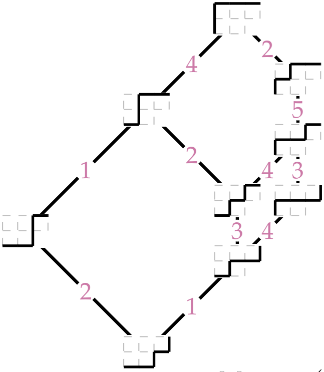
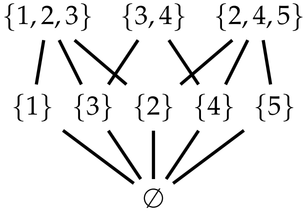
- $F_{EENEN}(x,y) = 5x^2 + 3xy + y^2 + 8x + 3y + 3$





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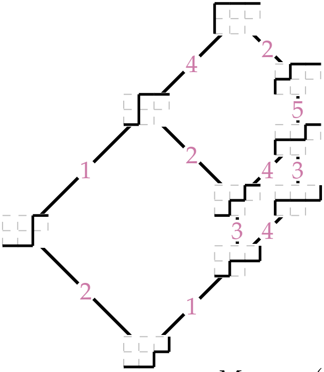

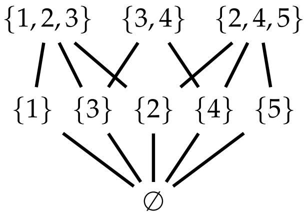

$$M_{EENEN}(x,y) = (xy-1)^2 F_{EENEN}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$$

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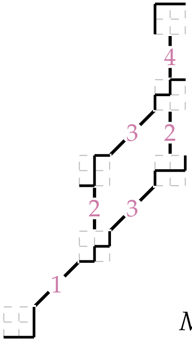

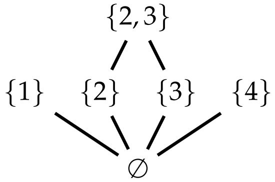

$$M_{EENN}(x,y) = x^2y^2 - 2xy^2 + 4xy + y^2 - 4y + 1$$

$$F_{EENN}(x,y) = 5x^2 + xy + 6x + 1$$

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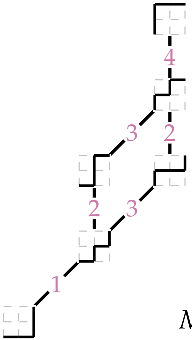

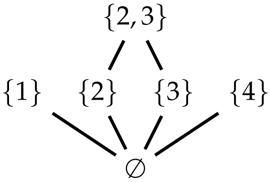

$$M_{EENN}(x,y) \neq (xy-1)^2 F_{EENN}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$$

$$F_{EENN}(x,y) = 5x^2+xy+6x+1$$

The M-Triangle of ν-Tamari Lattices

Perspectivity Irreducibility

Conjecture (, 2021)

Let ν be a northeast path. Then,
$M_\nu(x,y) = (xy-1)^{\mathsf{deg}(\nu)} F_\nu\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$ *if and only if ν does not contain two consecutive east steps before two consecutive north steps.*

The M-Triangle of ν-Tamari Lattices

Perspectivity Irreducibility

Conjecture (, 2021)

Let ν be a northeast path. Then, $\mathsf{Asso}(\nu)$ *is pure if and only if ν does not contain two consecutive east steps before two consecutive north steps.*

Open Questions

- can we find other (geometric) intepretations of the F- and H-triangles?
- what is the (geometric) nature of the M-triangle?
- (why) is pureness important?
- we need more examples!

Examples

Thank You.

Perspectivity

M-triangle

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice

Perspectivity

M-triangle

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- **edge**: (p,q) such that $p < q$ and no $p < r < q$ $\quad\rightsquigarrow \mathcal{E}(\mathbf{L})$
- **perspective**: $(p,q) \mathrel{\overset{=}{\wedge}} (r,s)$ such that $q \wedge r = p$ and $q \vee r = s$ (or $s \wedge p = r$ and $s \vee p = q$)

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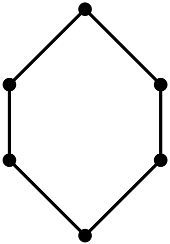

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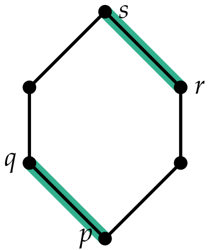

perspective

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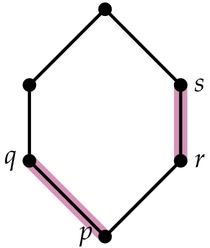

not perspective

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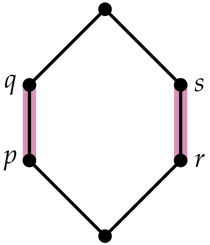

not perspective

Irreducibility

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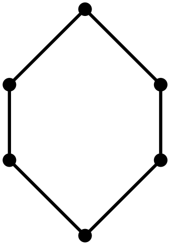

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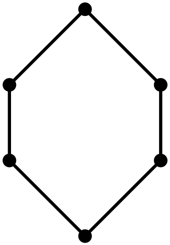

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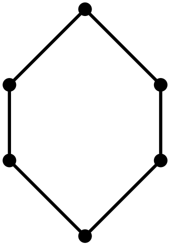

$\leftarrow$ meet irreducible

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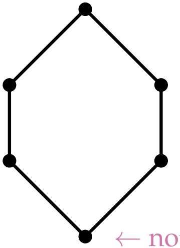

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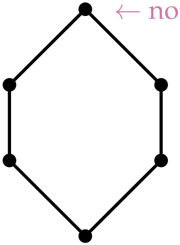

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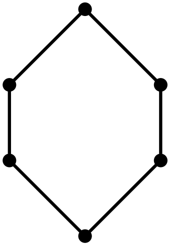

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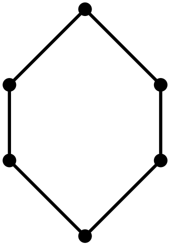

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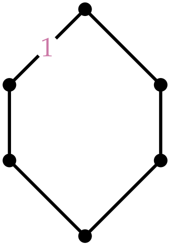

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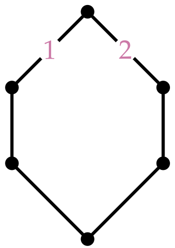

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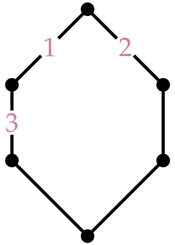

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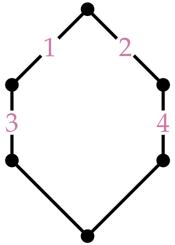

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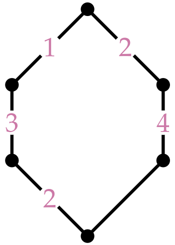

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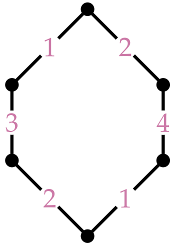

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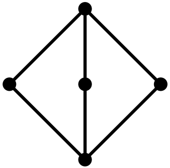

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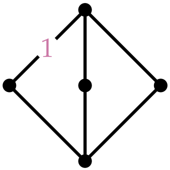

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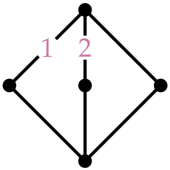

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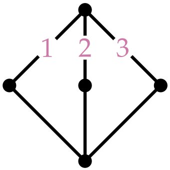

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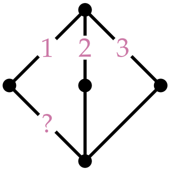

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- **perspectivity labeling**:

$$\lambda \colon \mathcal{E}(\mathbf{L}) \to \mathcal{M}(\mathbf{L}), \quad (p, q) \mapsto m$$

such that $(p, q) \overline{\overline{\wedge}} (m, m^*)$

Meet-Semidistributive Lattices

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice
- **meet-semidistributive**:

$$p \wedge q = p \wedge r \quad \text{implies} \quad p \wedge q = p \wedge (q \vee r)$$

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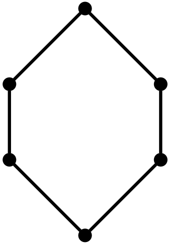

yes

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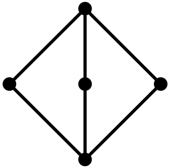

no

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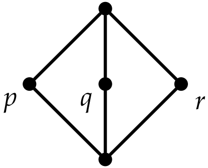

no

Meet-Semidistributive Lattices

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice
- **meet-semidistributive**:

$$p \wedge q = p \wedge r \quad \text{implies} \quad p \wedge q = p \wedge (q \vee r)$$

- if $\mathbf{L}$ is meet-semidistributive, then

$$\lambda(p,q) \overset{\text{def}}{=} \max\{r \mid q \wedge r = p\}$$

is a perspectivity labeling

Proposition (, 2021)

Every meet-semidistributive lattice is M-determined.

Meet-Semidistributive Lattices

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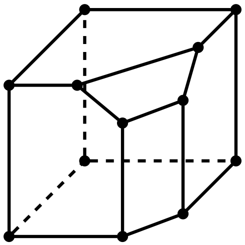
is a perspectivity labeling

Conjecture (, 2021)

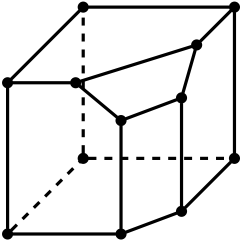
A lattice is M-determined if and only if it is meet semidistributive.

Posets from Polytopes



Posets from Polytopes

- cost function on vertices



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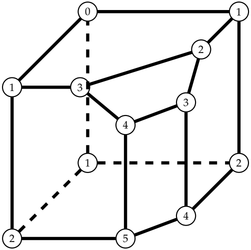

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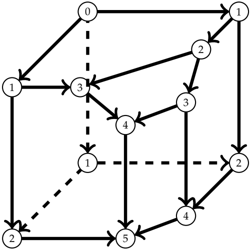

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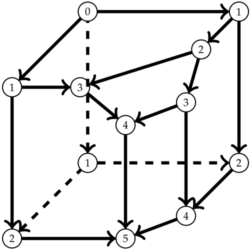
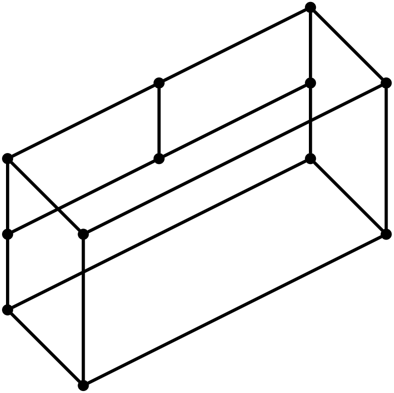
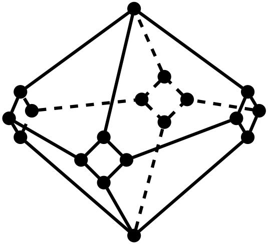




Posets from Polytopes

- counterexample to lattice property by F. Santos



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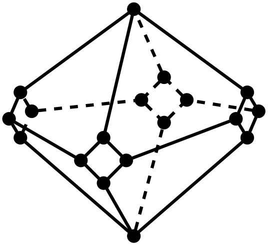

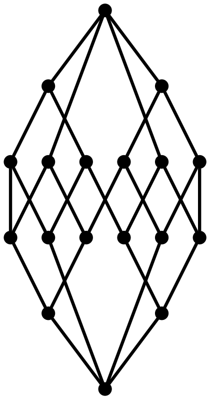

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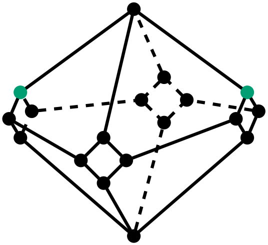

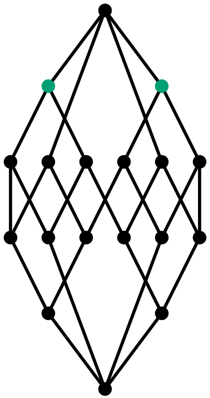

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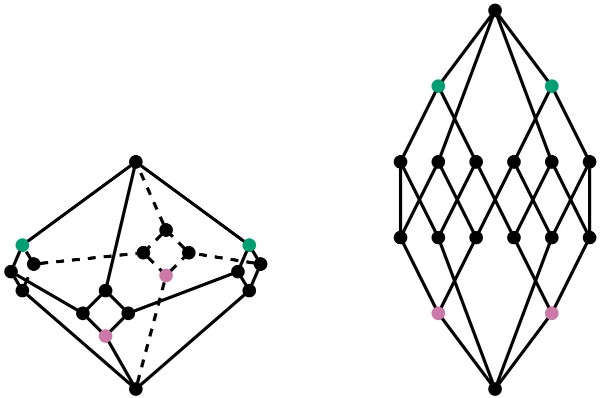


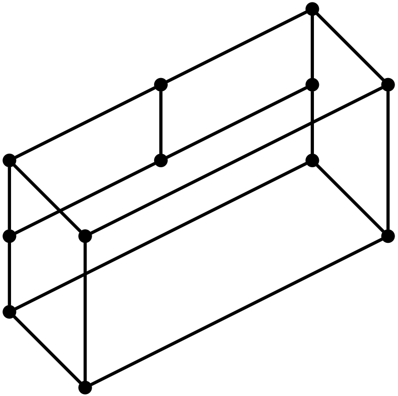
Posets from Polytopes

Question (, 2021)

Let $\mathbf{L}$ *be a lattice arising from an orientation of the line graph of a polytope* $\mathcal{P}$*. Is* $\mathbf{L}$ *always meet semidistributive?*

Facial Intervals of Lattices

- $\mathbf{L} = (L, \leq)$ .. lattice; $p \in L$; $S \subseteq \mathsf{Succ}(p)$



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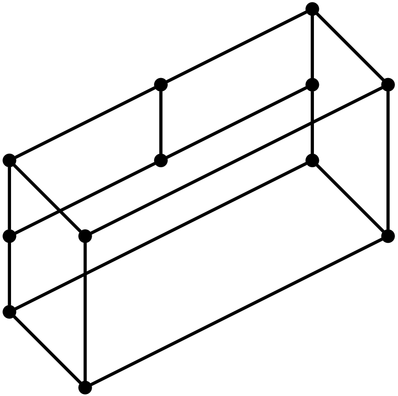
$p \rightarrow$



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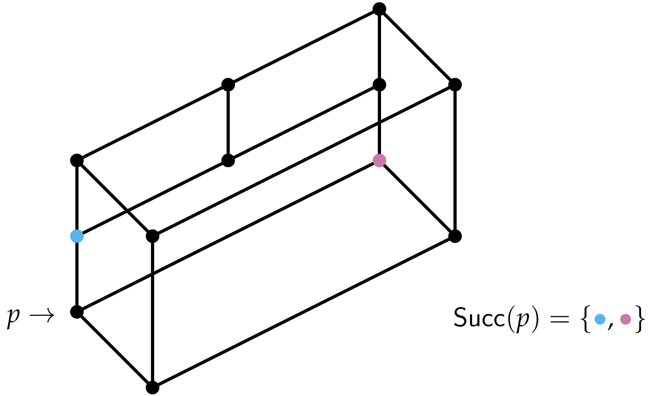

Facial Intervals of Lattices

- $\mathbf{L} = (L, \leq)$ .. lattice; $p \in L$; $S \subseteq \mathsf{Succ}(p)$
- $p_{\uparrow S} \overset{\mathsf{def}}{=} p \vee \bigvee S$
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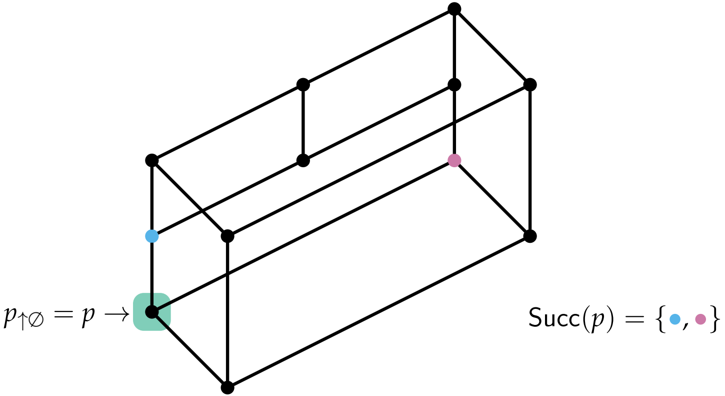

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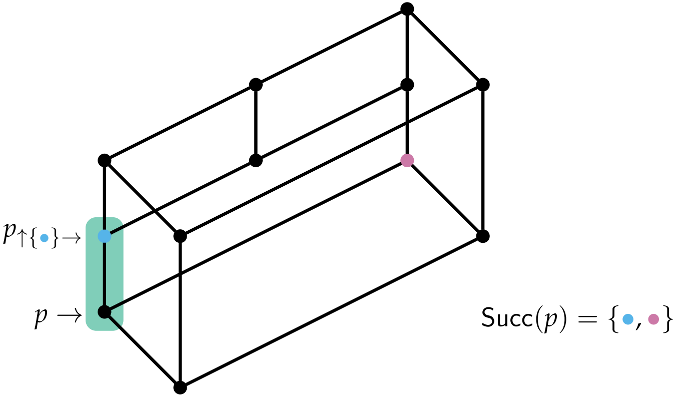

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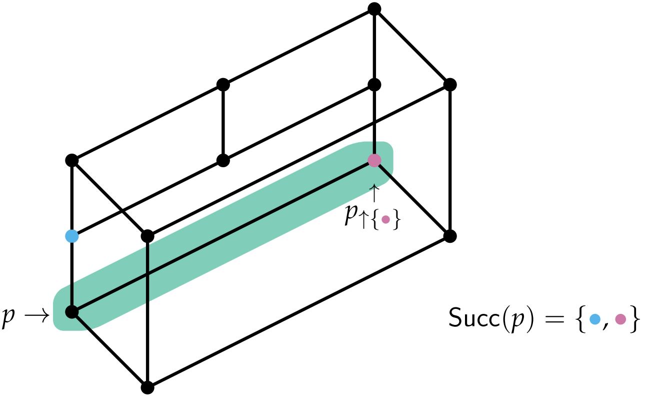

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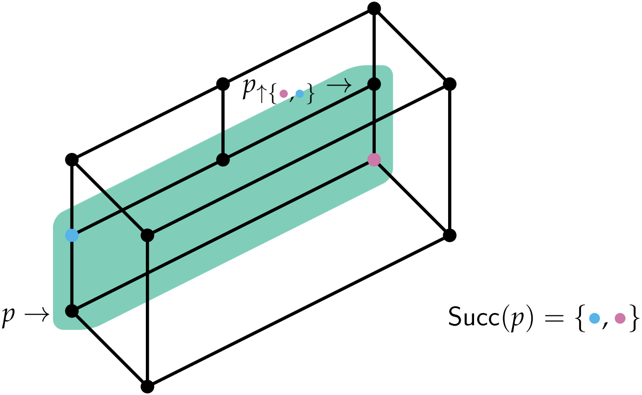

Facial Intervals of Lattices

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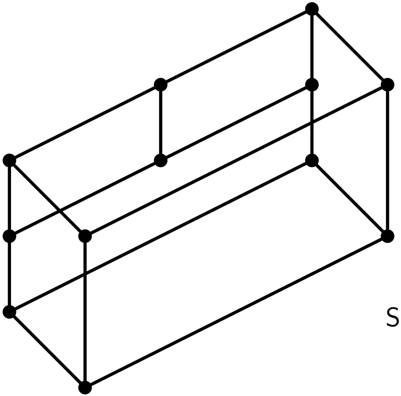

$\mathsf{Succ}(p) = \{\bullet, \bullet\}$

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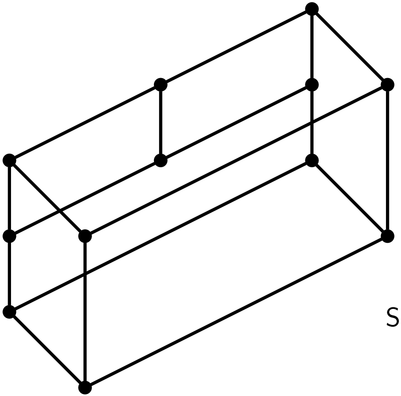

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Proposition (, 2021)

If $\mathbf{L}$ is meet semidistributive, then the assignment $(p, S) \mapsto \langle p, S \rangle$ is injective.

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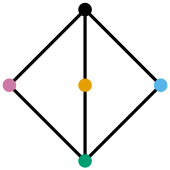

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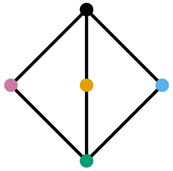

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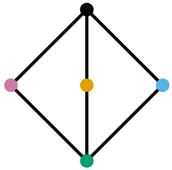

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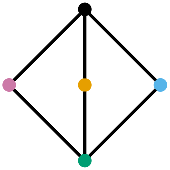

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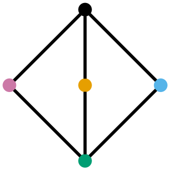

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Question

Can we define an "intersection" operation on intervals of a meet-semidistributive lattice $\mathbf{L}$ *that equips* $\mathsf{CP}(\mathbf{L})$ *with the structure of a polytopal complex?*

Facial Intervals of Lattices

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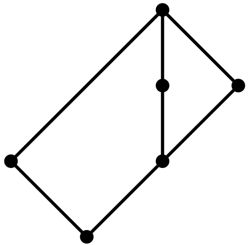

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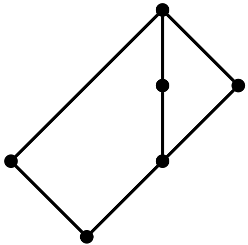
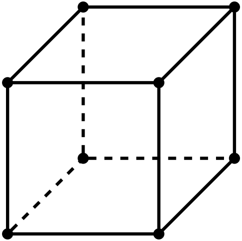
not meet semidistributive

$(p, S) \mapsto \langle p, S \rangle$ injective

not "polytopal"

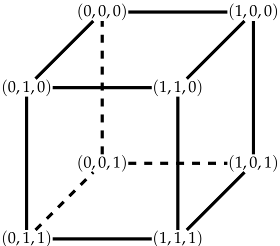


Cubes



Cubes

Open questions



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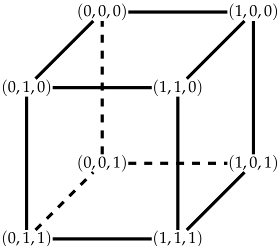
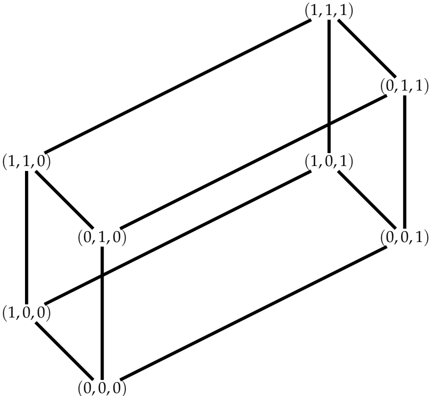
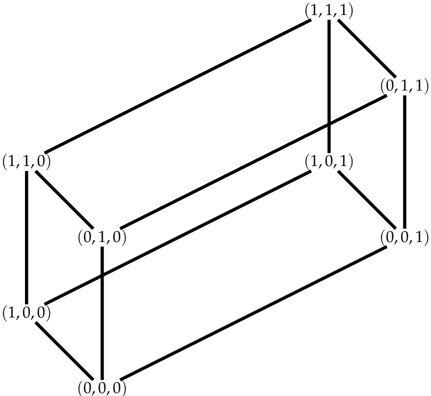




Cubes

Open questions

- **binary tuple**: integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1\}$ $\rightsquigarrow \mathsf{Bin}(n)$



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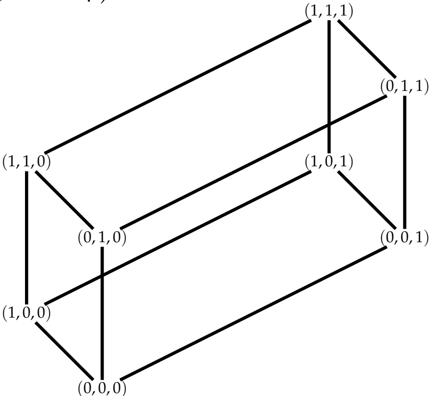

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Theorem (Folklore)

For $n > 0$, $\mathbf{Hoch}(n)$ is meet semidistributive.

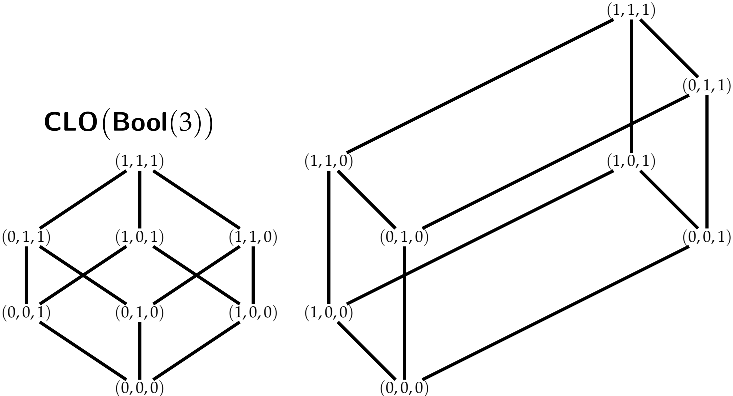
Cubes

Refined Face Enumeration in ν-Associahedra

Henri Mühle

Open questions

$\mathsf{CLO}\big(\mathsf{Bool}(3)\big)$



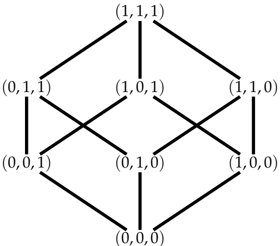
Cubes

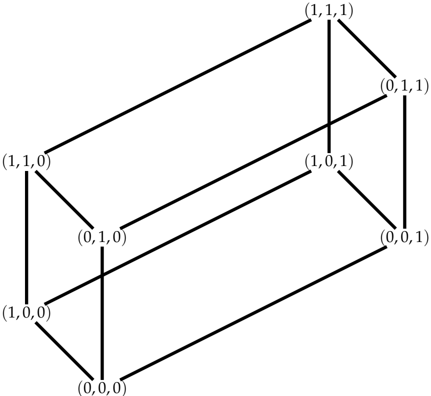
Open questions

Theorem (, 2019)

For $n > 0$, $\mathbf{CLO}\big(\mathbf{Bool}(n)\big)$ is isomorphic to $\mathbf{Bool}(n)$.

$\mathbf{CLO}\big(\mathbf{Bool}(3)\big)$





Cubes

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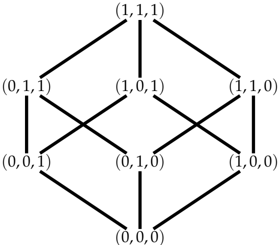

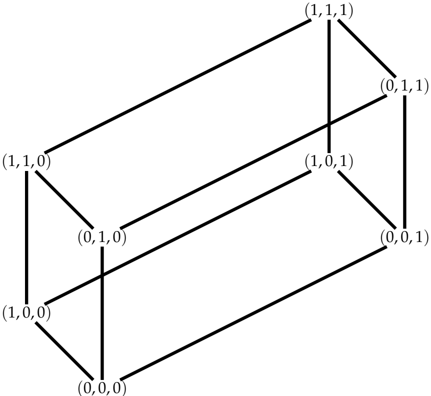

Cubes

Open questions

- mark edges if perspective to **atoms**

Theorem (🗙, 2020)

For $n > 0$, we have

$$F_{\mathsf{Bool}(n)}(x,y) = (x+y+1)^n,$$
$$H_{\mathsf{Bool}(n)}(x,y) = (xy+1)^n,$$
$$M_{\mathsf{Bool}(n)}(x,y) = (xy-y+1)^n.$$

Cubes

Open questions

- mark all edges

Theorem (, 2020)

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Stacked Cubes

Open questions

- **chain poset**: $\mathsf{Chain}(n) \stackrel{\text{def}}{=} ([n], \leq)$
- let $\mathbf{L}(n_1, n_2, \ldots, n_k) \stackrel{\text{def}}{=} \prod_{i=1}^{k} \mathsf{Chain}(n_i)$

Stacked Cubes

Open questions

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Theorem (, 2020)

For $n > 0$, we have

$$F_{\mathbf{L}(n_1,n_2,\ldots,n_k)}(x,y) = \prod_{i=1}^{k}\Bigl((n_i-1)x+y+(n_i-1)\Bigr),$$

$$H_{\mathbf{L}(n_1,n_2,\ldots,n_k)}(x,y) = \prod_{i=1}^{k}\Bigl(xy+(n_i-2)x+1\Bigr),$$

$$M_{\mathbf{L}(n_1,n_2,\ldots,n_k)}(x,y) = \prod_{i=1}^{k}\Bigl((n_i-1)xy-(n_i-1)y+1\Bigr).$$

Stacked Cubes

Open questions

- **chain poset**: $\mathbf{Chain}(n) \overset{\text{def}}{=} ([n], \leq)$
- $\mathbf{L}(2,2,\ldots,2) \cong \mathbf{Bool}(k)$

Corollary (, 2020)

For $n > 0$, we have

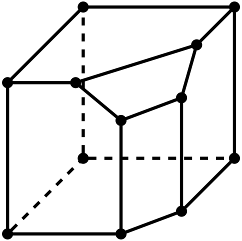
$$F_{\mathbf{L}(2,2,\ldots,2)}(x,y) = \prod_{i=1}^{k}\Bigl(x+y+1\Bigr),$$

$$H_{\mathbf{L}(2,2,\ldots,2)}(x,y) = \prod_{i=1}^{k}\Bigl(xy+1\Bigr),$$

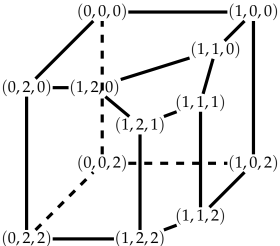
$$M_{\mathbf{L}(2,2,\ldots,2)}(x,y) = \prod_{i=1}^{k}\Bigl(xy-y+1\Bigr).$$

Freehedra

Open questions

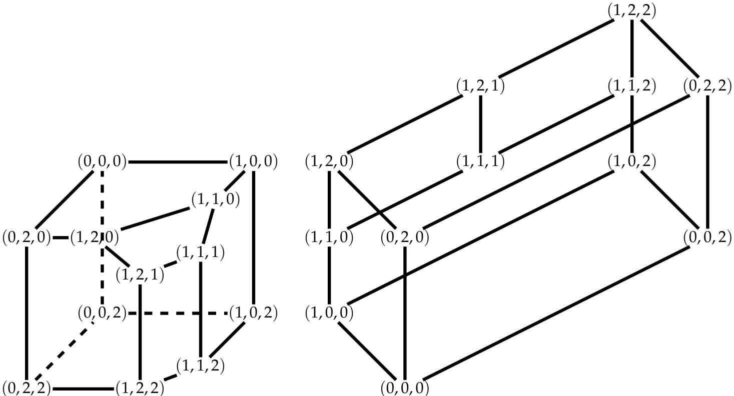


Freehedra



Freehedra

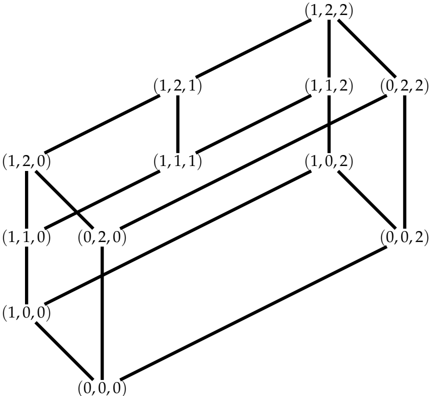
Open questions



Freehedra

Open questions

- **triword**: integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1, 2\}$ $\rightsquigarrow \mathsf{Tri}(n)$
  - $u_1 \neq 2$
  - $u_i = 0$ implies $u_j \neq 1$ for all $j > i$



Freehedra

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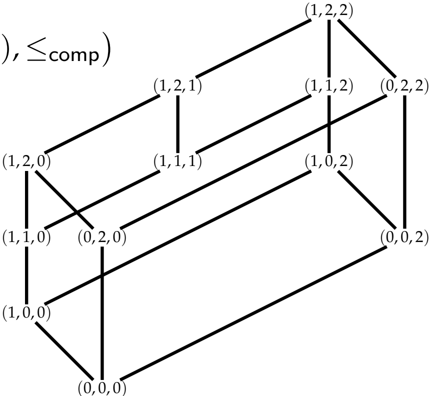
- **Hochschild lattice**: $\mathbf{Hoch}(n) \overset{\text{def}}{=} \big(\mathsf{Tri}(n), \leq_{\mathsf{comp}}\big)$



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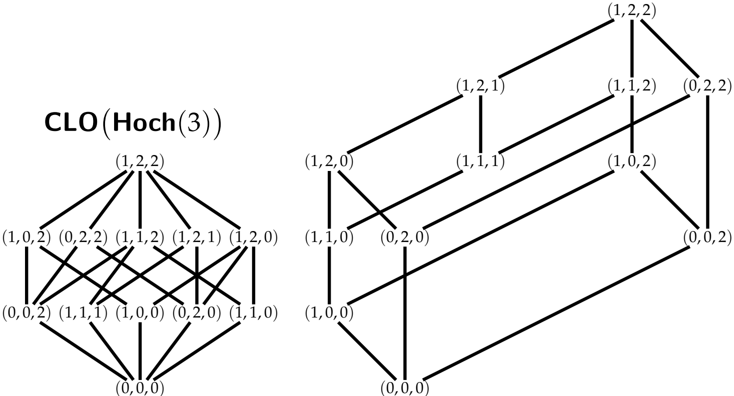
Theorem (C. Combe, 2020)

For $n > 0$, $\mathbf{Hoch}(n)$ is meet semidistributive.

Freehedra

Open questions

CLO(Hoch(3))



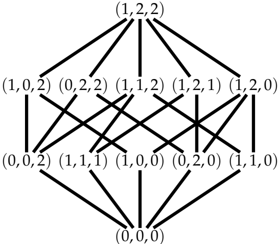
Freehedra

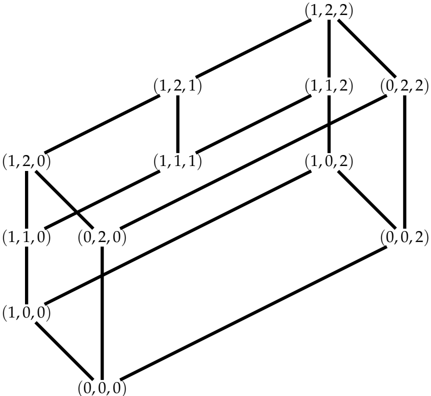
Open questions

Theorem (, 2020)

For $n > 0$, $\mathsf{CLO}\big(\mathsf{Hoch}(n)\big)$ is isomorphic to $\mathsf{Shuf}(n-1, 1)$.

$\mathsf{CLO}\big(\mathsf{Hoch}(3)\big)$





Freehedra

Theorem (, 2020)

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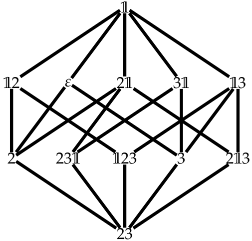
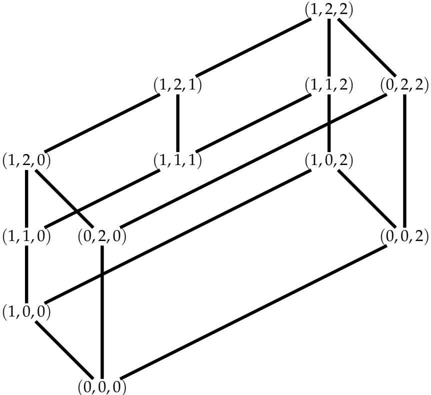
$\mathbf{Shuf}(2,1)$





Freehedra

- mark edges if perspective to **atoms**

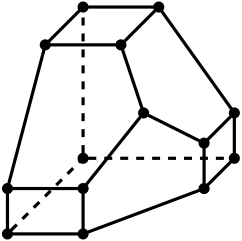
Theorem (, 2020)

For $n > 0$, we have

$$F_{\mathsf{Hoch}(n)}(x,y) = (x+y+1)^{n-2}\bigl(nx^2+2xy+(n+1)x+(y+1)^2\bigr),$$
$$H_{\mathsf{Hoch}(n)}(x,y) = (xy+1)^{n-2}\bigl(x^2y^2+2xy+(n-1)x+1\bigr),$$
$$M_{\mathsf{Hoch}(n)}(x,y) = (xy-y+1)^{n-2} \times \Bigl((n+1)\bigl((x-1)y-xy^2\bigr)+(n+x^2)y^2+1\Bigr).$$

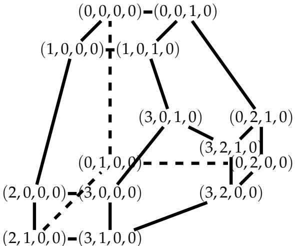
Associahedra

Open questions



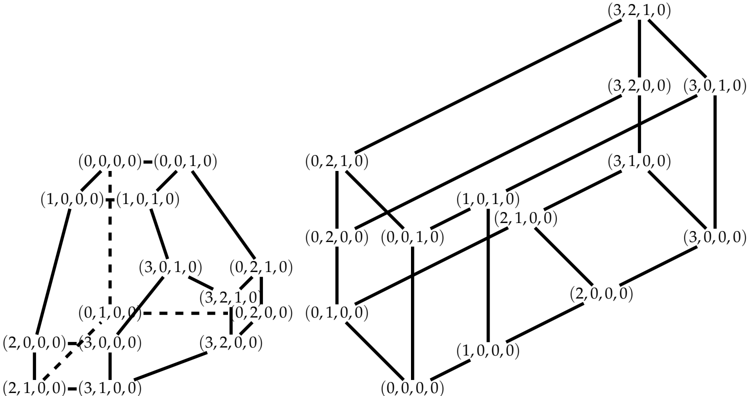
Associahedra

Open questions



Associahedra

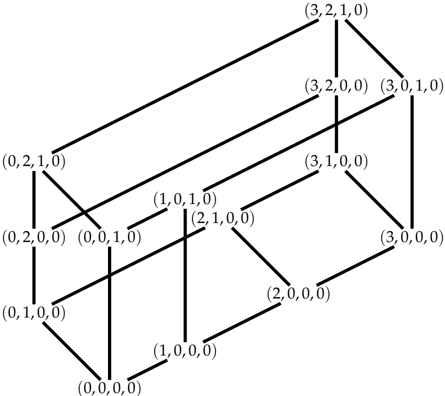
Open questions



Associahedra

Open questions

- **bracket vector**: integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1, \ldots, n-i\}$ $\rightsquigarrow \mathsf{Brac}(n)$
  - $u_{i+j} \leq u_i - j$ for all $j \in \{0, 1, \ldots, u_i\}$



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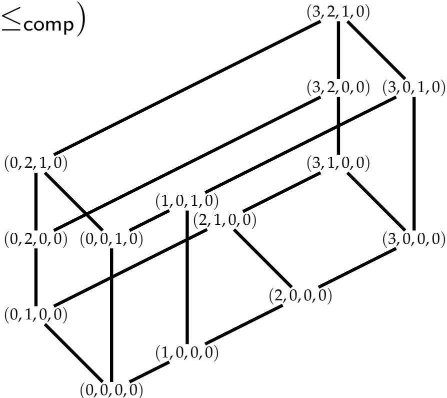
  $\mathbf{Tam}(n) \overset{\text{def}}{=} \big(\mathsf{Brac}(n), \leq_{\mathsf{comp}}\big)$



Associahedra

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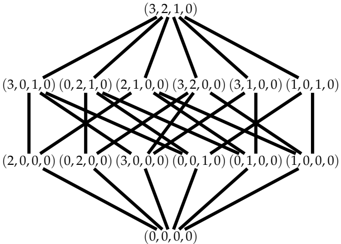
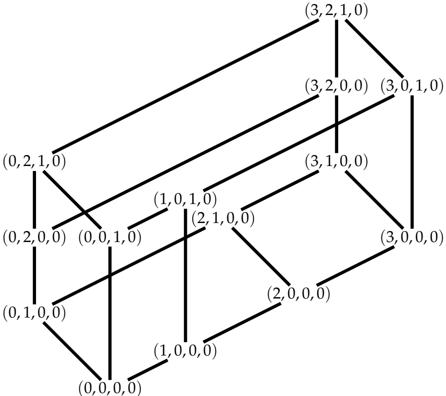
Theorem (A. Urquhart, 1978)

For $n > 0$, $\mathbf{Tam}(n)$ is meet semidistributive.

Associahedra

$\mathbf{CLO}\big(\mathbf{Tam}(4)\big)$





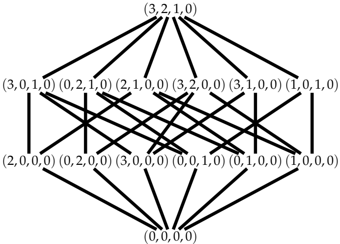
Associahedra

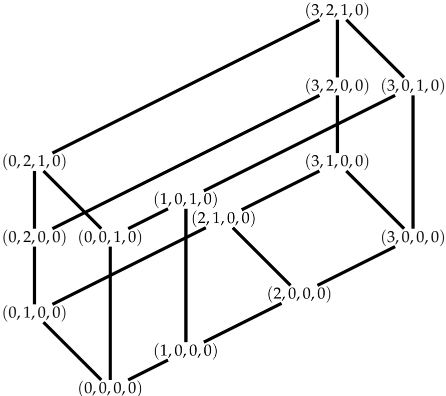
Open questions

Theorem (N. Reading, 2011)

For $n > 0$, $\mathsf{CLO}\big(\mathsf{Tam}(n)\big)$ is isomorphic to $\mathsf{Nonc}(n)$.

$\mathsf{CLO}\big(\mathsf{Tam}(4)\big)$





Associahedra

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