Poset-Topological Aspects of Noncrossing Partition Lattices

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Outline

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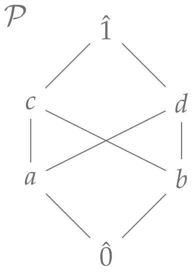
Poset Topology

- main task: investigate the (topological) structure of the order complex of a poset
- in particular:
  - determine the homotopy type
  - compute the homology
  - compute bases for the homology
- helpful tools: poset labelings

Posets

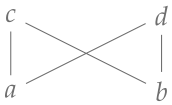
- bounded poset



Posets

- proper part

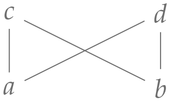
$\overline{\mathcal{P}}$



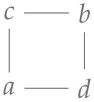
Posets

- order complex $\Delta(\mathcal{P})$

$\overline{\mathcal{P}}$

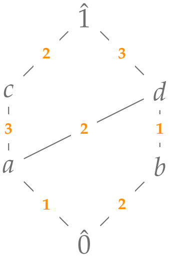


$\Delta(\overline{\mathcal{P}})$



Edge-Labelings

- EL-labeling



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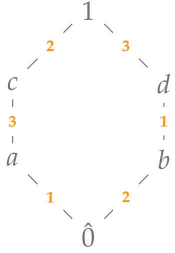

not an EL-labeling!

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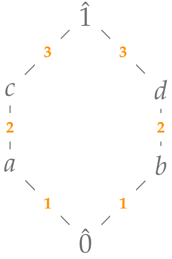

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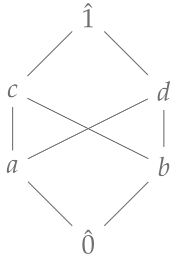
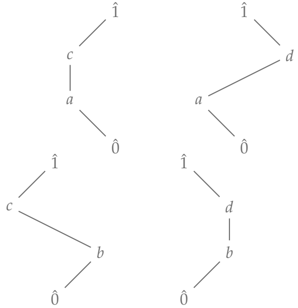
Chain-Edge-Labelings

- CL-labeling





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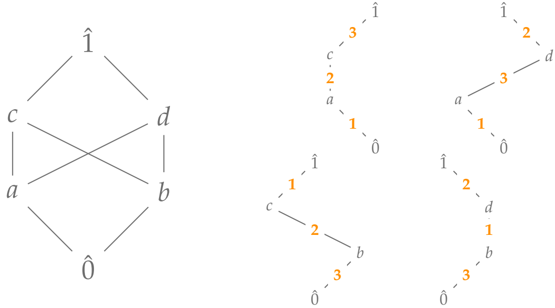

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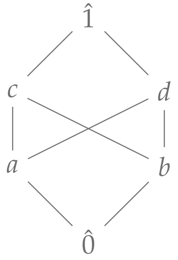

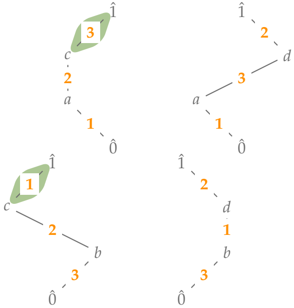

Recursive Atom Orders

- recursive atom order: total order $a_1 \prec a_2 \prec \cdots \prec a_s$ such that
  - there exists a recursive atom order of $[a_j, \hat{1}]$ such that the first elements of this order are those that cover some $a_i \prec a_j$
  - if $i < j$ and $a_i, a_j \leq y$, then there is some $k < j$ and some $z \leq y$ such that $a_k, a_j \lessdot z$

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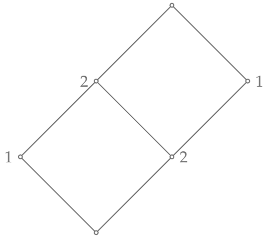

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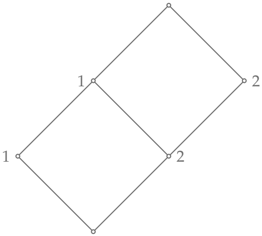

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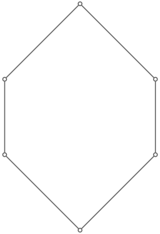

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Theorem (Björner & Wachs, 1983)

A bounded poset admits an CL-labeling if and only if it admits a recursive atom order.

Lexicographically Shellable Posets

- lexicographically shellable poset: admits an EL-labeling or a CL-labeling
- if $\mathcal{P}$ is lexicographically shellable, then
  - $\Delta(\overline{\mathcal{P}})$ is shellable,
  - it is homotopic to a wedge of spheres,
  - the dimension of its i-th homology group is given by the number of falling maximal chains of length $i-2$

Outline

Well-Generated Reflection Groups

- complex reflection: unitary transformation that fixes a hyperplane pointwise $\rightsquigarrow T$
- complex reflection group: group generated by complex reflections $\rightsquigarrow W$
- rank: codimension of fixed space
- irreducible: no nontrivial factors
- well-generated: irreducible, rank equals minimal number of generators

Well-Generated Reflection Groups

- monomial matrix: one non-zero entry per row and per column
- $G(d,e,n)$ .. $(n \times n)$-monomial matrices, non-zero entries are d-th roots of unity, product is $\frac{d}{e}$-th root of unity

Theorem (Shephard & Todd, 1954)

A finite group W is a well-generated reflection group if and only if $W \cong G(d,e,n)$ for $d \geq 1$, $e \in \{1,d\}$, or W is one of 26 exceptional groups.

Well-Generated Reflection Groups

- $G(1,1,n) \cong A_{n-1} \cong \mathfrak{S}_n$
- $G(2,1,n) \cong B_n$
- $G(2,2,n) \cong D_n$
- $G(d,d,2) \cong I_2(d) \cong \mathfrak{D}_d$
- $G_{23} \cong H_3$
- $G_{28} \cong F_4$
- $G_{30} \cong H_4$
- $G_{35} \cong E_6$
- $G_{36} \cong E_7$
- $G_{37} \cong E_8$

Coxeter Elements

- **degrees**: certain invariants of W $\rightsquigarrow d_1 \leq \cdots \leq d_n$
- **Coxeter number**: highest degree $\rightsquigarrow h$
- **regular element**: has eigenvector that does not lie in any reflection hyperplane
- **Coxeter element**: regular element of order h $\rightsquigarrow c$

Theorem (Lehrer & Springer, 1999)

Coxeter elements exist in well-generated reflection groups.

Absolute Order

- absolute length: length of a minimal T-decomposition $\rightsquigarrow \ell_T$
- absolute order: $u \leq_T v$ if and only if
$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

Noncrossing Partition Lattices

- W .. well-generated reflection group

Definition (Brady, 2001; Brady & Watt, 2002; Bessis, 2003; Bessis, 2007)

The lattice of noncrossing partitions of W is defined to be the interval $[e, c]_T$ between the identity e and some Coxeter element c of W in absolute order. $\rightsquigarrow \mathcal{NC}_W(c)$

Noncrossing Partition Lattices

- combinatorial definitions
  - type A: Kreweras, 1971
  - type B: Reiner, 1997
  - type D: Athanasiadis & Reiner, 2004
  - type $G(d,d,n)$: Bessis & Corran, 2006
  - type $G(d,1,n)$: essentially type B

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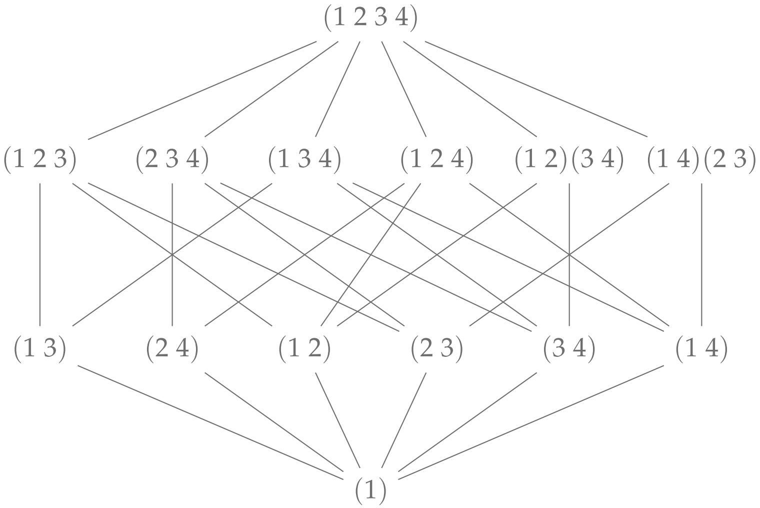
Example: $\mathcal{NC}_{\mathfrak{S}_4}$



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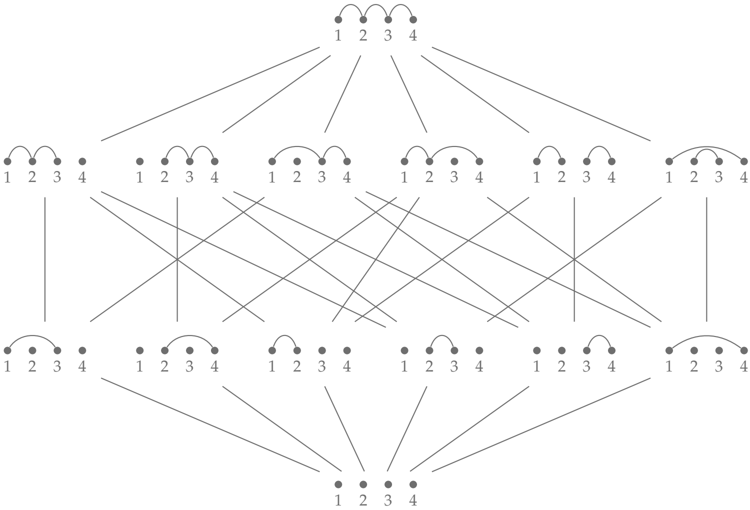

Properties

Theorem (Reiner, Ripoll & Stump, 2014)

For any well-generated reflection group W, and any two Coxeter elements $c, c' \in W$, we have $\mathcal{NC}_W(c) \cong \mathcal{NC}_W(c')$.

Properties

Theorem (Kreweras, 1971; Reiner, 1997; Brady, 2001; Brady & Watt, 2002; Bessis, 2003; Athanasiadis & Reiner, 2004; Bessis & Corran, 2006; Bessis, 2007; Brady & Watt, 2008)

$\mathcal{NC}_W$ is indeed a lattice for any well-generated reflection group W.

- uniform proof only for Coxeter groups

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EL-Shellability

Theorem (Björner & Edelman, 1980; Reiner, 1997)

If $W = A_n$ or $W = B_n$, then $\mathcal{NC}_W$ is EL-shellable for any $n > 0$.

- restrict labeling that comes from the semimodularity of the partition lattice

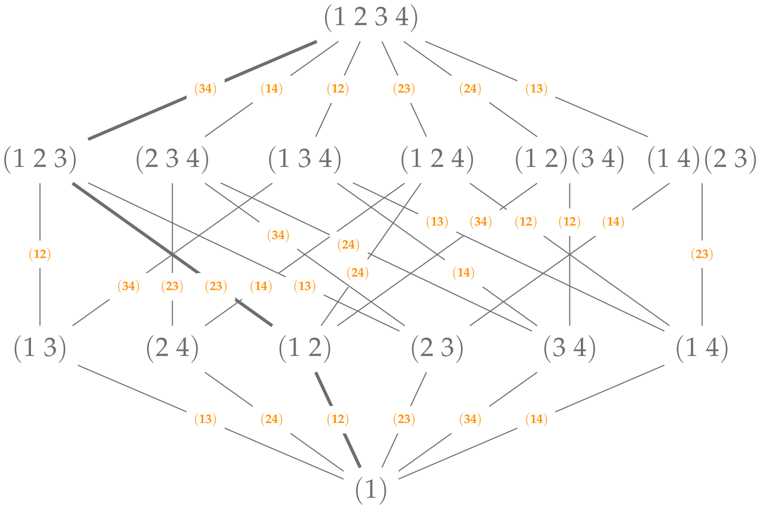
EL-Shellability

Theorem (Athanasiadis, Brady & Watt, 2007)

If W is a Coxeter group, then $\mathcal{NC}_W$ is EL-shellable.

- label (u,v) by $u^{-1}v \in T$
- use compatible reflection order
- this is uniform!

Example: $\mathcal{NC}_{\mathfrak{S}_4}$



$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$

Generalizations?

- what about well-generated reflection groups that are no Coxeter groups?
  - can we generalize the proof of Athanasiadis, Brady and Watt?

Compatible Reflection Orders

- W .. Coxeter group
- $c \in W$ .. Coxeter element
- reflection order: either $t_\alpha \prec t_{a\alpha+b\beta} \prec t_\beta$ or $t_\beta \prec t_{a\alpha+b\beta} \prec t_\alpha$

Definition (Athanasiadis, Brady & Watt, 2007)

A reflection order is c-compatible if for any rank-2 subgroup of W, whose simple reflections are s, t, we have $s \prec t$ whenever $st \leq_T c$.

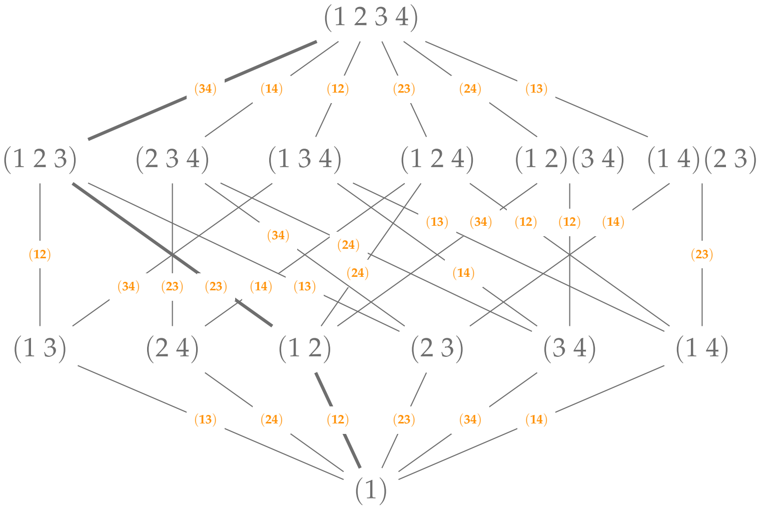
Compatible Reflection Orders

- W .. well-generated reflection group
- $c \in W$ .. Coxeter element
- T_c .. reflections below c

Definition (, 2015)

A total order of T_c is *c-compatible* if for any $w \leq_T c$ with $\ell_T(w) = 2$, there exists a unique rising reduced T-decomposition of w.

Example: $\mathcal{NC}_{\mathfrak{S}_4}$



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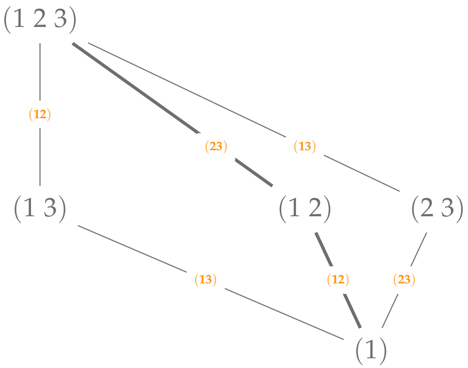

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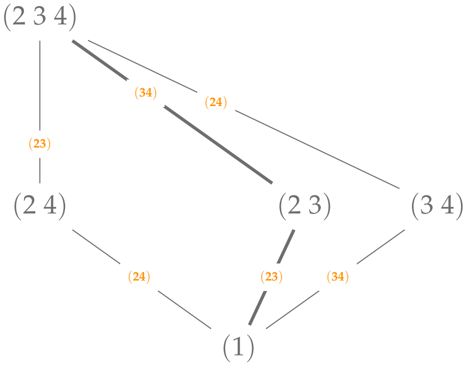

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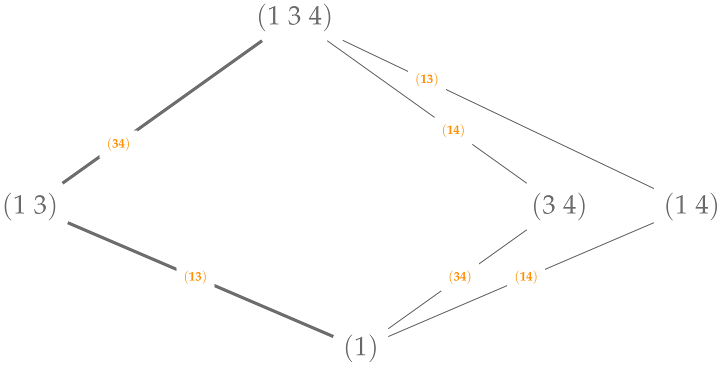

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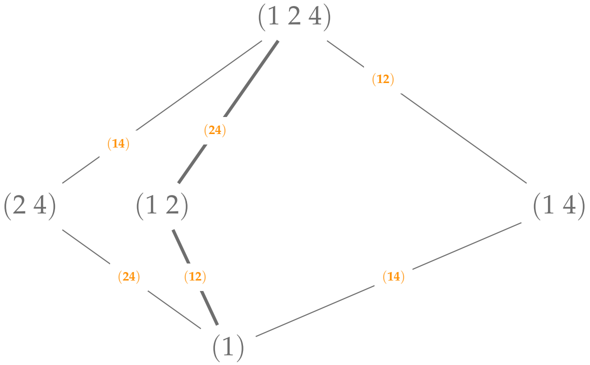

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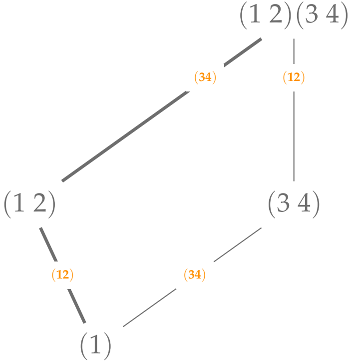

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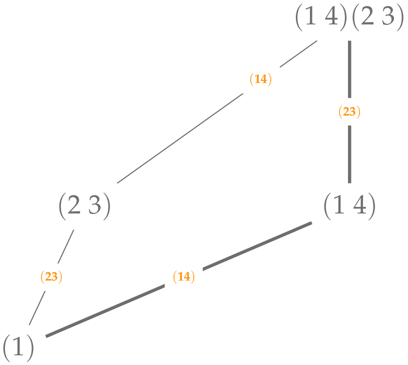

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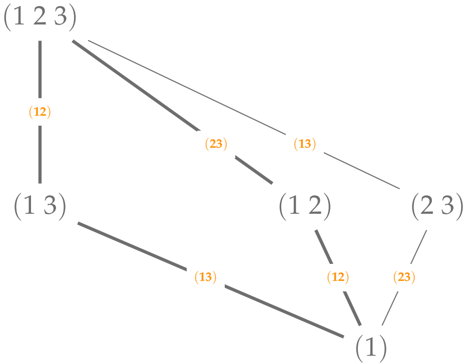

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More EL-Shellability

Theorem (, 2015)

If $W \cong G(d,d,n)$ or if W is one of the exceptional well-generated reflection groups that is not a Coxeter group, then $\mathcal{NC}_W$ is EL-shellable.

- label (u,v) by $u^{-1}v \in T$
- use compatible order for $G(d,d,n)$, and a computer verification for the exceptional types
- not uniform!

Outline

Unite the Tribes

- W .. well-generated reflection group
- $c \in W$ .. Coxeter element
- T_c .. reflections below c

Theorem (, 2014)

Every c-compatible order of T_c is a recursive atom order of $\mathcal{NC}_W(c)$.

- proof by induction on rank of W
- "almost" uniform!

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CL-Shellability

Proposition (, 2014)

c-compatible orders exist in well-generated reflection groups.

- not uniform!

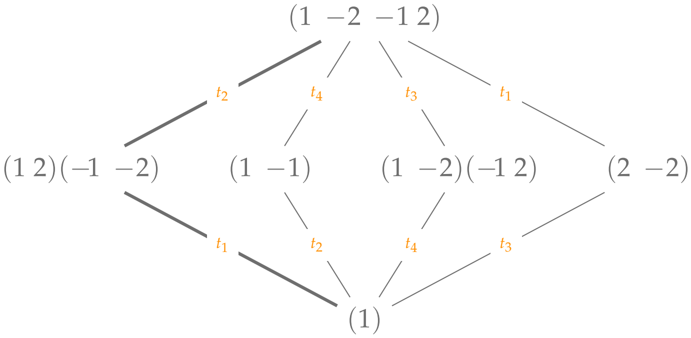
CL-Shellability

Theorem (, 2014)

If W is a well-generated reflection group, then $\mathcal{NC}_W$ is CL-shellable.

- "almost" uniform!

Example: B_2



$$t_1 \prec t_3 \prec t_4 \prec t_2$$

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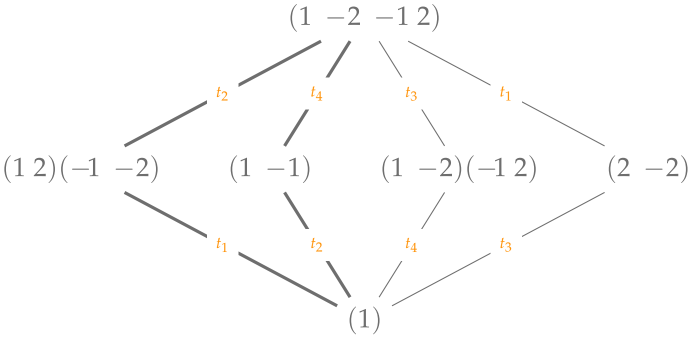

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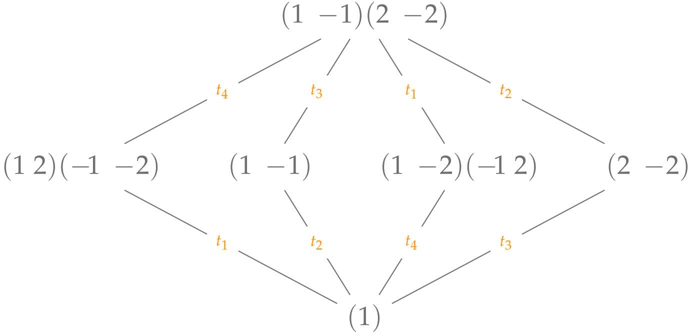

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Hurwitz Action

- braid group: $\mathfrak{B}_k = \langle \sigma_1, \ldots, \sigma_{k-1} \mid (\sigma_i\sigma_{i+1})^3 = (\sigma_i\sigma_j)^2 = e, \text{ for } |j-i| > 1 \rangle$
- minimal T-decompositions of $w \in W$ $\rightsquigarrow \mathrm{Red}_T(w)$
- $\mathfrak{B}_{\ell_T(w)}$ acts on $\mathrm{Red}_T(w)$ by
  $\sigma_i \cdot (w_1 \cdots w_k) = w_1 \cdots w_{i-1}(w_i w_{i+1} w_i^{-1}) w_i w_{i+2} \cdots w_k$
  $\rightsquigarrow$ Hurwitz move

Hurwitz Action

Theorem (Deligne, 1974; Bessis & Corran, 2006; Bessis, 2007)

For any well-generated reflection group W, and any Coxeter element $c \in W$, the group $\mathfrak{B}_{\ell_T(w)}$ acts transitively on $\mathrm{Red}_T(w)$, whenever $w \leq_T c$.

- uniform proof only for Coxeter groups

A Connection

- W .. well-generated reflection group
- $c \in W$ .. Coxeter element

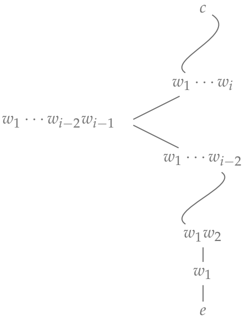
Theorem (, 2015)

If $\mathfrak{B}_2$ acts transitively on $Red_T(w)$ for every $w \leq_T c$ with $\ell_T(w) = 2$, and $\mathcal{NC}_W(c)$ is lexicographically shellable, then $\mathfrak{B}_n$ acts transitively on $Red_T(c)$.

Sketch of Proof

- minimal T-decompositions of c correspond to maximal chains in $\mathcal{NC}_W(c)$
- Hurwitz moves correspond to "taking detours"

$$c = w_1 w_2 \cdots w_{i-2} w_{i-1} w_i w_{i+1} \cdots w_n$$



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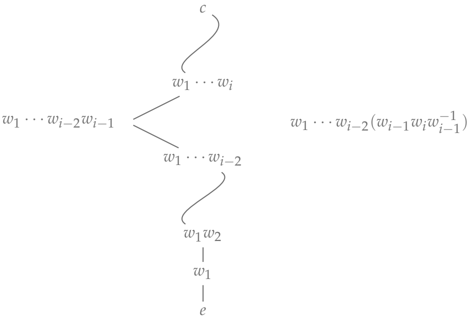

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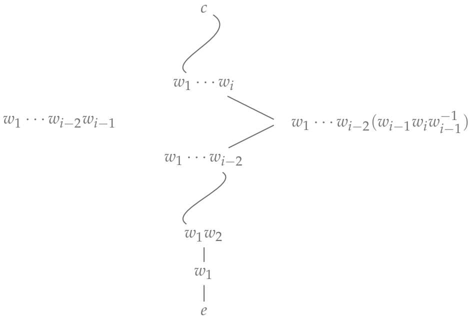
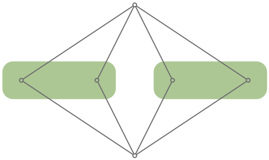


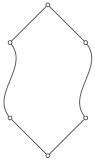
Sketch of Proof

- non-transitivity can be caused by two scenarios

rank-2 violation



large "gaps"



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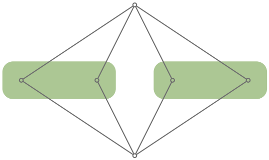

Contradicts rank-2 assumption!

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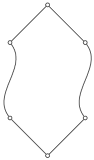

Contradicts shellability assumption!

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If there exists a c-compatible order of T_c, then $\mathfrak{B}_n$ acts transitively on $Red_T(c)$.

Outline

Conclusions

- c-compatible reflection orders are good!
- for Coxeter groups, they allow uniform solutions for:
  - lattice property of $\mathcal{NC}_W$
  - lexicographic shellability of $\mathcal{NC}_W$
  - bases of homology of $\Delta\big(\overline{\mathcal{NC}_W}\big)$

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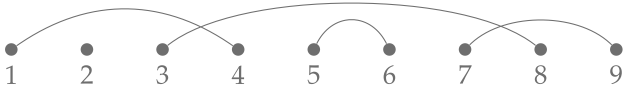
Problem

Give a uniform description of a c-compatible order of the reflections below c for all well-generated reflection groups, and some (uniform) choice of Coxeter element c.

Merci Beaucoup.

Noncrossing Partitions

- noncrossing (set) partition $\rightsquigarrow NC_n$



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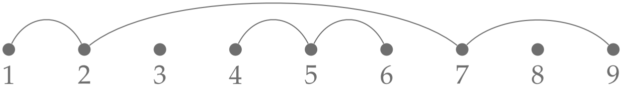
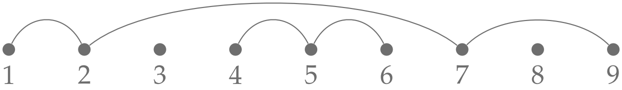


Noncrossing Partitions

- **refinement order**: join blocks $\rightsquigarrow \leq_{\mathrm{ref}}$



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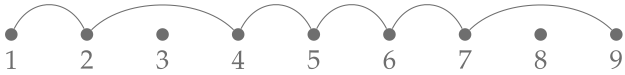
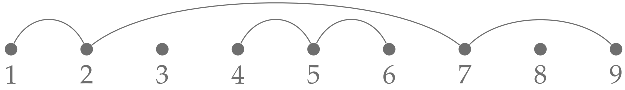


A Map to Permutations

- group blocks into cycles



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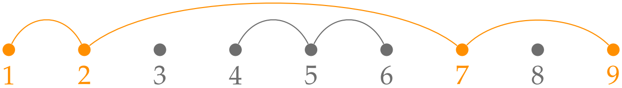

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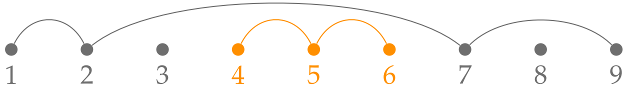

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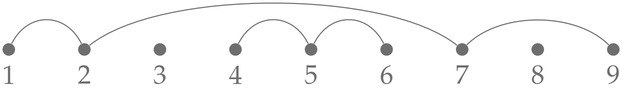

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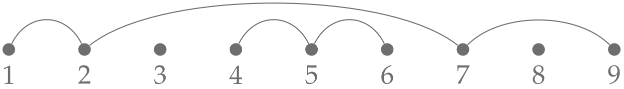

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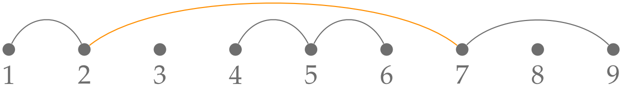

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A Map to Permutations

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$$(1245679) = (1279)(456) \cdot (26)$$

A Symmetric Group Object

- T .. transpositions in $\mathfrak{S}_n$
- e .. identity permutation
- c .. some long cycle

Theorem (Biane, 1997)

For $n > 0$, the poset $\mathcal{NC}_n = (NC_n, \leq_{ref})$ is isomorphic to the interval $[e, c]$ in the Cayley graph of $(\mathfrak{S}_n, T)$.

- use this connection as a starting point to generalize $\mathcal{NC}_n$ to all well-generated reflection groups

A More General Setting

- generated group: group G with a distinguished generating set A $\rightsquigarrow (G, A)$
- define a word length $\rightsquigarrow \ell_A$
- partial orders:

$$u \leq v \quad \text{if and only if} \quad \ell_A(v) = \ell_A(u) + \ell_A(u^{-1}v)$$

- A closed under conjugation $\rightsquigarrow$ well-defined Hurwitz action

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More General Results

- (G, A) .. generated group, A closed under conjugation
- $x \in G$ .. some element
- A_x .. generators below x

Theorem (, 2015)

Suppose that $\mathfrak{B}_2$ acts transitively on $\mathrm{Red}_T(A)g$, whenever $\ell_A(g) = 2$. If $[e, x]_A$ is lexicographically shellable, then $\mathfrak{B}_{\ell_A(x)}$ acts transitively on $\mathrm{Red}_T(A)x$.

More General Results

- (G, A) .. generated group, A closed under conjugation
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Proposition (, 2015)

If there exists a x-compatible order of A_x, then $\mathfrak{B}_2$ acts transitively on $\mathrm{Red}_T(A)g$, whenever $\ell_A(g) = 2$.

More General Results

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Theorem (, 2015)

Every x-compatible order of A_x is a recursive atom order of $[e,x]_A$.

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If there exists a x-compatible order of A_x, then $\mathfrak{B}_{\ell_A(x)}$ acts transitively on $Red_T(A)x$.

More General Questions

- how frequently do generated groups with conjugation closed generating sets appear?
- how frequently do x-compatible orders exist in these groups?

The Braid Group of a Reflection Group

- braid group: fundamental group of complement of reflection hyperplanes $\rightsquigarrow \mathfrak{B}(W)$
- $\mathfrak{B}_n = \mathfrak{B}(A_{n-1})$
- group presentation:

$$\begin{array}{c} W = \langle T \mid t_i^{\varepsilon_i} = e, R \rangle \\ \downarrow \\ \mathfrak{B}(W) = \langle T \mid R \rangle \end{array}$$

- consider $\big(\mathfrak{B}(W), \leq_T\big)$
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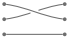
Example: $\mathfrak{B}_3$

$t_1 =$ 

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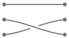
$t_3 =$ 

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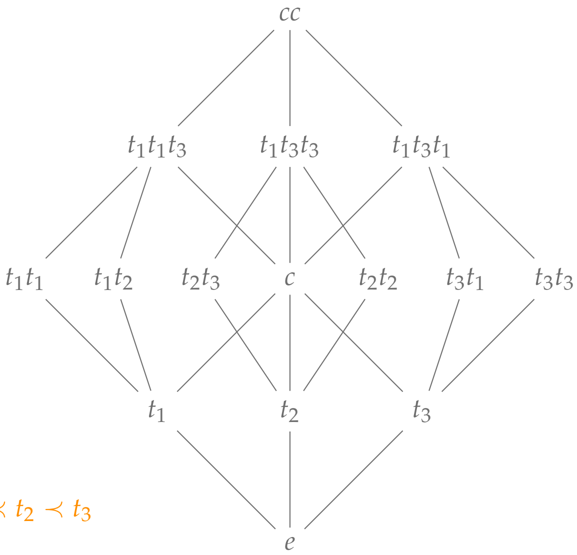

$t_1 \prec t_2 \prec t_3$

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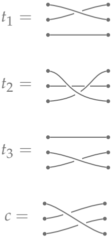

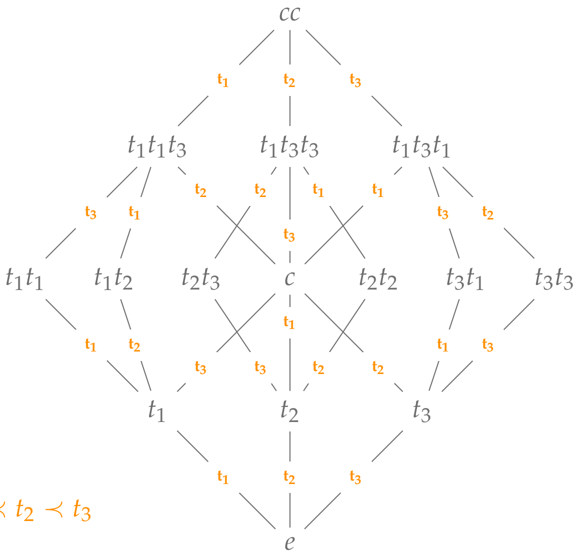

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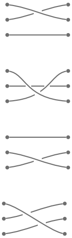

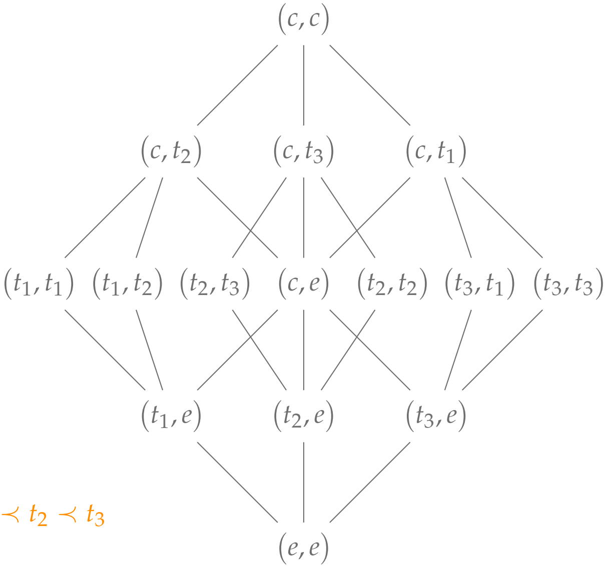

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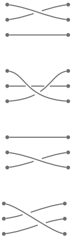

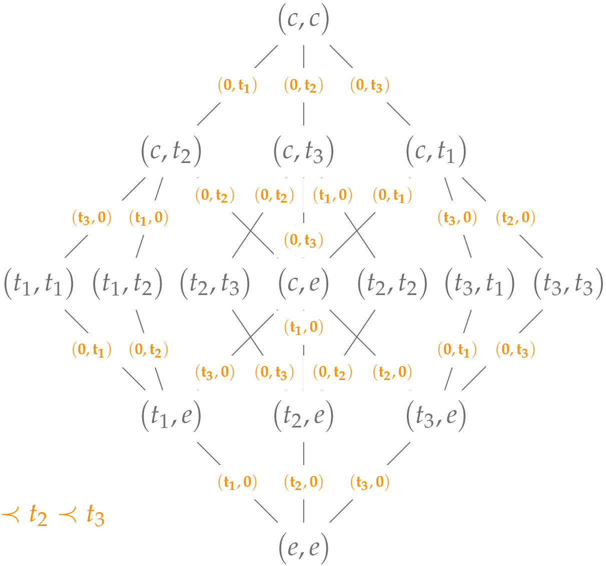

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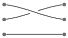
$t_1 =$ 

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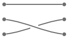
$t_3 =$ 

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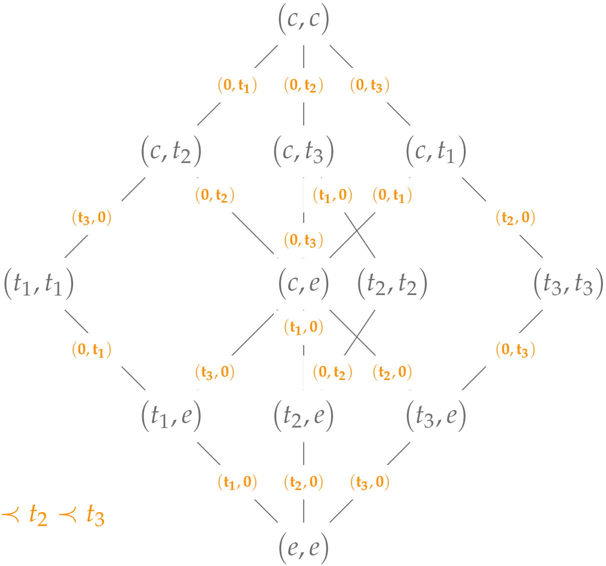

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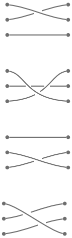

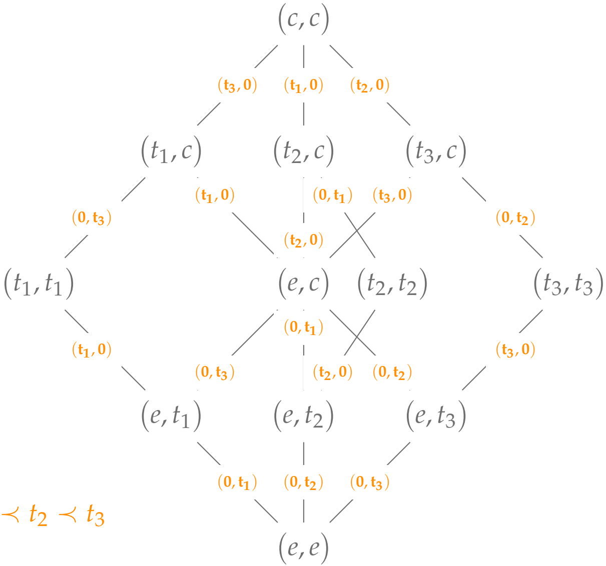

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Multichains of Noncrossing Partitions

- elements in $NC_W^{[m]}$ are m-multichains in $\mathcal{NC}_W$
  - but: different partial order than Armstrong's m-divisible noncrossing partitions!

The Poset of m-Multichains

- $\mathcal{P} = (P, \leq)$ .. some poset
- m-multichain: $(x_1, x_2, \ldots, x_m)$ with $x_1 \leq x_2 \leq \cdots \leq x_m$ $\rightsquigarrow P^{[m]}$
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Theorem (, 2014)

Let $\mathcal{P} = (P, \leq)$ be a bounded poset, and let $\mathcal{P}^{[m]}$ denote its poset of m-multichains, ordered componentwise by $\leq$. Then, $\mathcal{P}$ is lexicographically shellable if and only if $\mathcal{P}^{[m]}$ is lexicographically shellable for every $m > 0$.

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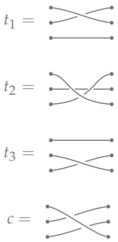

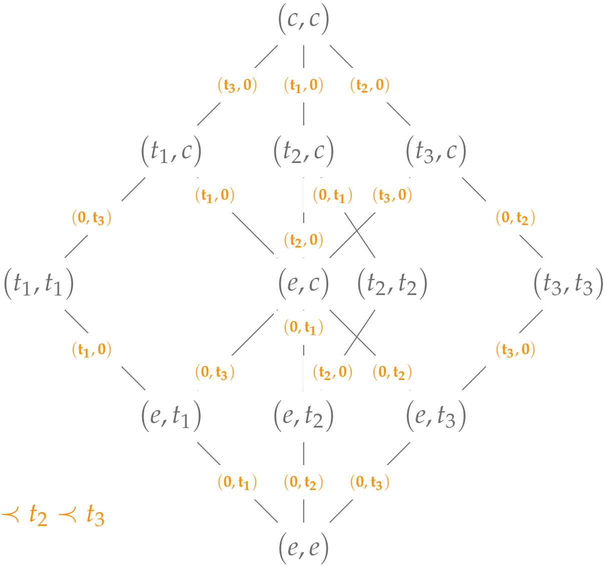

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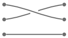
$t_1 =$ 

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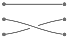
$t_3 =$ 

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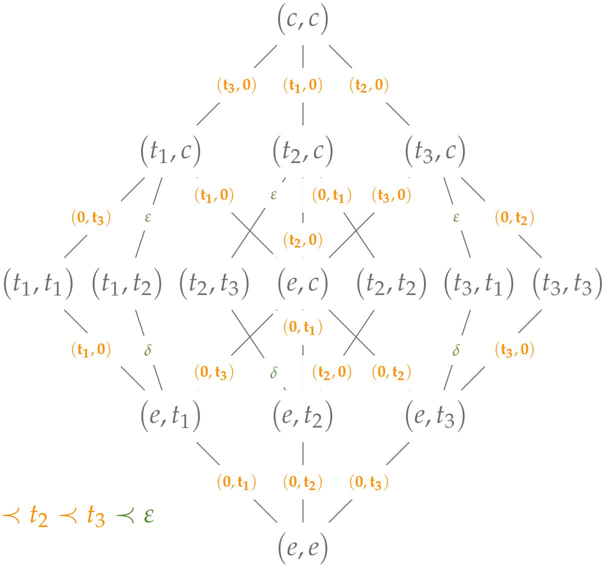
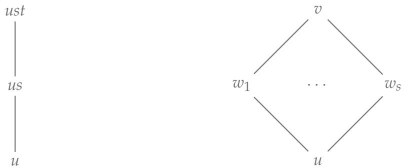


$\delta \prec t_1 \prec t_2 \prec t_3 \prec \varepsilon$

Rank-2 Intervals

- rank-2 intervals of $[e, c^m]_T$ in $\mathfrak{B}(W)$ are of one of the two forms
  - $\rightsquigarrow$ rank-2 transitivity of the Hurwitz action
  - $\rightsquigarrow$ proving lexicographic shellability yields Hurwitz transitivity "for free"!



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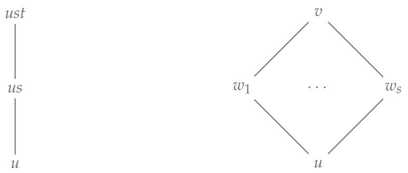

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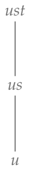

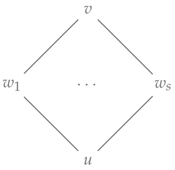

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