Chapoton's Triangles, Lattice Paths and Reciprocity

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Matroids, Ordered Structures, Arrangements in Combinatorics

Manchester, UK

Outline

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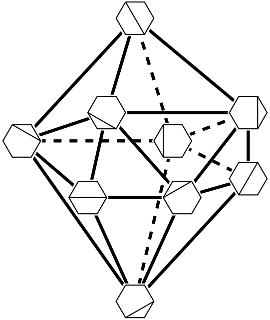
The *F*-Triangle

- **cluster complex**: faces are pairwise noncrossing diagonals of regular $(n+3)$-gon $\rightsquigarrow \mathsf{Clus}(n)$



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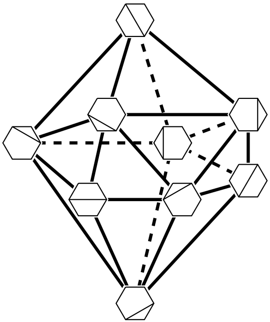

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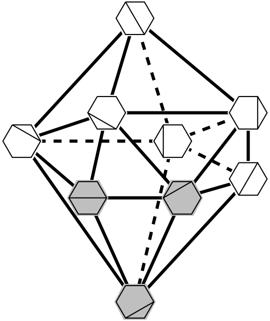

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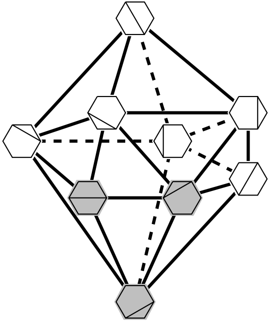

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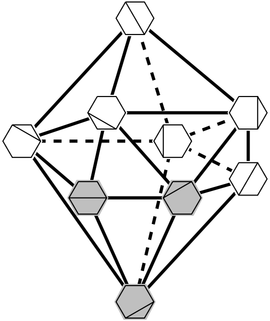
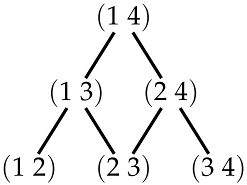


$$\tilde{\mathcal{F}}_3(x,y) = 5x^3+5x^2y+3xy^2+y^3 +10x^2+8xy+3y^2+6x+3y+1$$

The H-Triangle

- **root poset**: triangular poset with $\binom{n+1}{2}$ elements
  $\rightsquigarrow \mathsf{Root}(n)$



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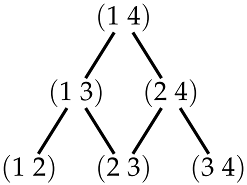

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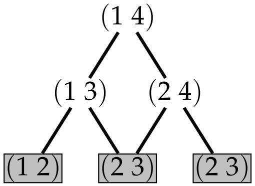

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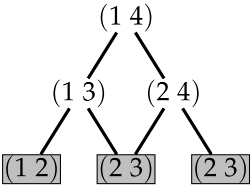

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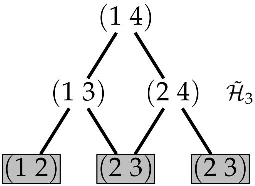

$$\tilde{\mathcal{H}}_3(x,y) = x^3y^3+3x^2y^2+2x^2y+x^2+3xy+3x+1$$

$F = H$

Conjecture (F. Chapoton, 2006)

For $n > 0$,

$$\tilde{\mathcal{F}}_n(x,y) = x^n \tilde{\mathcal{H}}_n\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

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Conjecture (D. Armstrong, 2009)

For $m, n > 0$,

$$\tilde{\mathcal{F}}_n^{(m)}(x,y) = x^n \tilde{\mathcal{H}}_n^{(m)}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

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Theorem (M. Thiel, 2014)

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$\rightsquigarrow$ analytic/inductive proof

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- **Dyck path**: northeast path from $(0,0)$ to (n,n) weakly above the diagonal $\rightsquigarrow \mathsf{Dyck}(n)$

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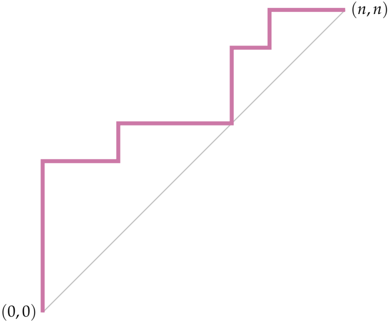

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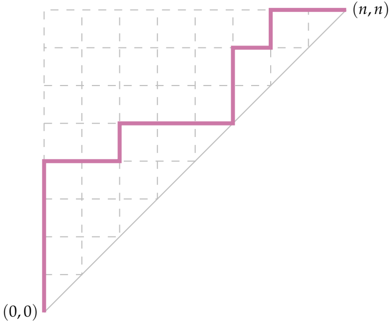

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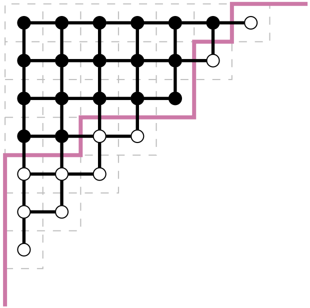

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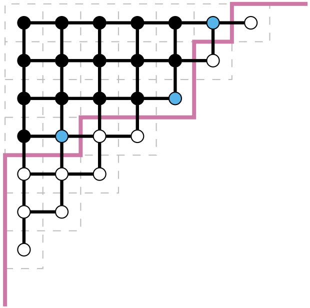

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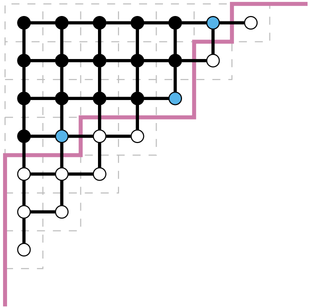
- **Dyck path**: northeast path from $(0,0)$ to (n,n) weakly above the diagonal $\rightsquigarrow \mathsf{Dyck}(n)$
- $A \in \mathsf{Anti}(n)$ corresponds to **valleys** of $\mu \in \mathsf{Dyck}(n)$
- minimal elements contained in A correspond to **returns**



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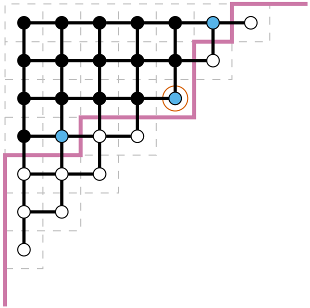

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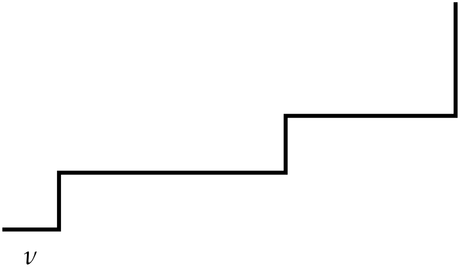
Corollary

For $n \geq 1$,

$$\tilde{\mathcal{H}}_n(x,y) = \sum_{\mu \in \mathsf{Dyck}(n)} x^{\mathsf{val}(\mu)} y^{\mathsf{ret}(\mu)}.$$

Dyck Paths

- let ν be any northeast path from $(0,0)$ to (m,n)



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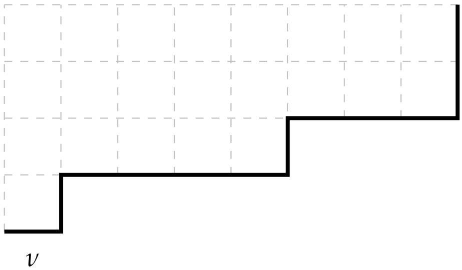

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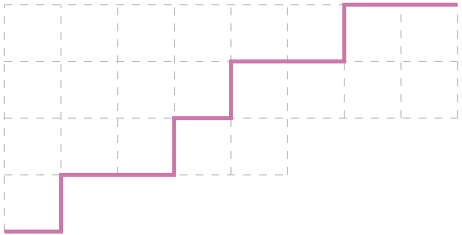

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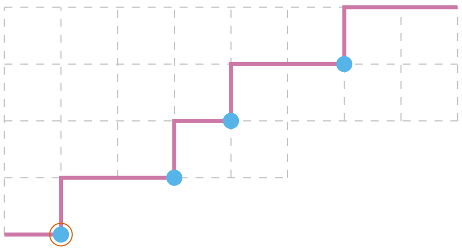

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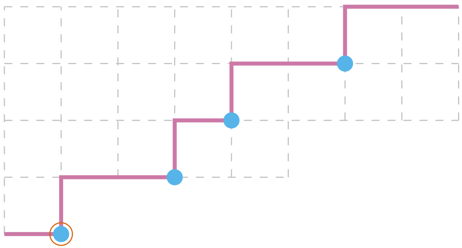

A new F-Triangle

- *define* an F-triangle through Chapoton's transformation
- **degree**: $\mathsf{deg}(\nu) \overset{\mathsf{def}}{=} \max\{\mathsf{val}(\mu) : \mu \in \mathsf{Dyck}(\nu)\}$

$$\mathcal{F}_\nu(x,y) \overset{\mathsf{def}}{=} x^{\mathsf{deg}(\nu)}\mathcal{H}_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

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$\rightsquigarrow$ summing subsets of non-return valleys/returns of μ

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The ν-Associahedron

- **ν-associahedron**:

$$\mathsf{Asso}(\nu) \overset{\mathsf{def}}{=} \{(\mu, V) : \mu \in \mathsf{Dyck}(\nu), V \text{ subset of valleys of } \mu\}$$

- $\dim \mathsf{Asso}(\nu) = \deg(\nu), \dim(\mu, V) = |V|$
- partition $V = V_1 \uplus V_2$ into non-returns and returns

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2019)

For any northeast path ν, $\mathsf{Asso}(\nu)$ is a polytopal complex.

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$$\mathcal{F}_\nu(x,y) = \sum_{\mu\in\mathsf{Dyck}(\nu)} x^{\deg(\nu)-\mathsf{val}(\mu)}(x+1)^{\mathsf{val}(\mu)-\mathsf{ret}(\mu)}(y+1)^{\mathsf{ret}(\mu)}$$

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$$\mathcal{F}_\nu(x,y) = \sum_{\mu \in \mathsf{Dyck}(\nu)} x^{\deg(\nu) - \mathsf{val}(\mu)} (x+1)^{\mathsf{val}(\mu) - \mathsf{ret}(\mu)} (y+1)^{\mathsf{ret}(\mu)}$$

$$= \sum_{\substack{(\mu,V) \in \mathsf{Asso}(\nu) \\ V = V_1 \uplus V_2}} x^{\deg(\nu) - \mathsf{val}(\mu)} x^{\mathsf{val}(\mu) - \mathsf{ret}(\mu) - |V_1|} y^{\mathsf{ret}(\mu) - |V_2|}$$

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$$= \sum_{\substack{(\mu,V) \in \mathsf{Asso}(\nu) \\ V = V_1 \uplus V_2}} x^{\deg(\nu) - \mathsf{ret}(\mu) + |V_2| - (|V_1| + |V_2|)} y^{\mathsf{ret}(\mu) - |V_2|}$$

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- partition $V = V_1 \uplus V_2$ into non-returns and returns

$$\mathcal{F}_\nu(x,y) = \sum_{\mu\in\mathsf{Dyck}(\nu)} x^{\deg(\nu)-\mathsf{val}(\mu)}(x+1)^{\mathsf{val}(\mu)-\mathsf{ret}(\mu)}(y+1)^{\mathsf{ret}(\mu)}$$

$$= \sum_{\substack{(\mu,V)\in\mathsf{Asso}(\nu)\\ V=V_1\uplus V_2}} x^{\mathsf{codim}(\mu,V)-(\mathsf{ret}(\mu)-\lvert V_2\rvert)}y^{\mathsf{ret}(\mu)-\lvert V_2\rvert}$$

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$$\begin{aligned}\mathcal{F}_\nu(x,y) &= \sum_{\mu\in\mathsf{Dyck}(\nu)} x^{\deg(\nu)-\mathsf{val}(\mu)}(x+1)^{\mathsf{val}(\mu)-\mathsf{ret}(\mu)}(y+1)^{\mathsf{ret}(\mu)}\\ &= \sum_{\substack{(\mu,V)\in\mathsf{Asso}(\nu)\\ V=V_1\uplus V_2}} x^{\mathsf{codim}(\mu,V)-(\mathsf{ret}(\mu)-\lvert V_2\rvert)} y^{\mathsf{ret}(\mu)-\lvert V_2\rvert}\\ &= \sum_{(\mu,V)\in\mathsf{Asso}(\nu)} x^{\mathsf{corel}(\mu,V)} y^{\mathsf{rel}(\mu,V)}\end{aligned}$$

$F = H$ for ν-Dyck Paths

- $\mathcal{F}_\nu(x,y) \overset{\mathsf{def}}{=} \sum_{(\mu,V)\in\mathsf{Asso}(\nu)} x^{\mathsf{corel}(\mu,V)} y^{\mathsf{rel}(\mu,V)}$
- $\mathcal{H}_\nu(x,y) \overset{\mathsf{def}}{=} \sum_{\mu\in\mathsf{Dyck}(\nu)} x^{\mathsf{val}(\mu)} y^{\mathsf{ret}(\mu)}$

Theorem (C. Ceballos &, 2021)

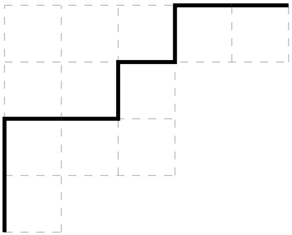
For any northeast path ν, we have

$$\mathcal{F}_\nu(x,y) = x^{\mathsf{deg}(\nu)} \mathcal{H}_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

Outline

The ν-Tamari Lattice

- **rotating** ν-Dyck paths by valleys



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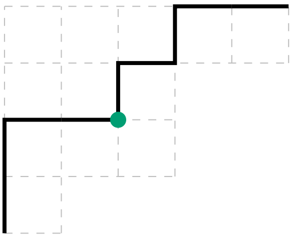

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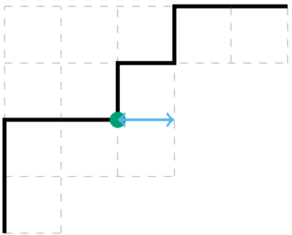

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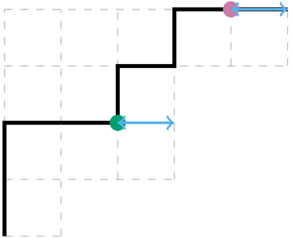

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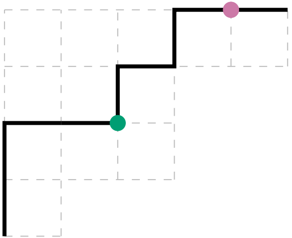

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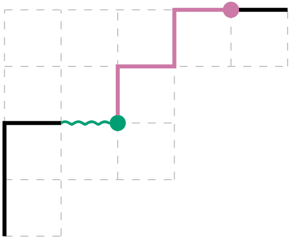

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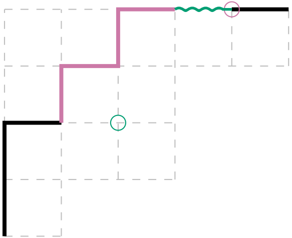
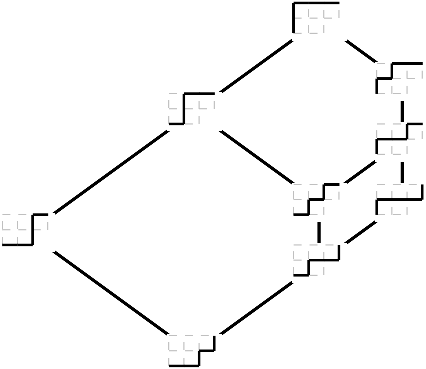


The ν-Tamari Lattice

- **ν-Tamari lattice**: rotation order on $\mathsf{Dyck}(\nu)$ $\rightsquigarrow$ $\mathsf{Tam}(\nu)$



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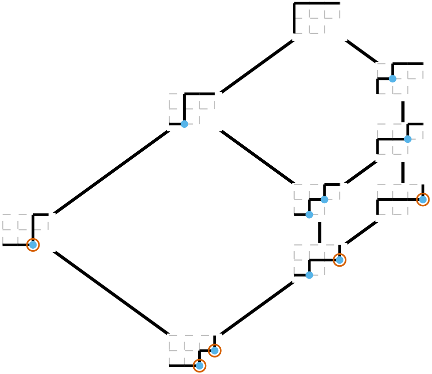

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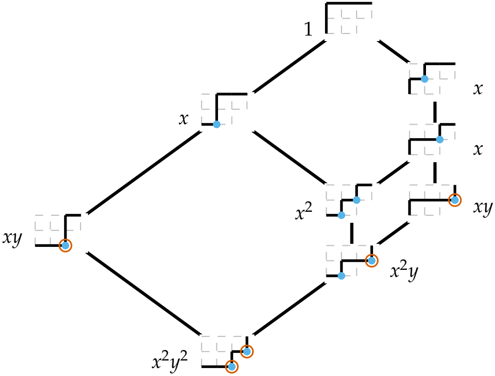

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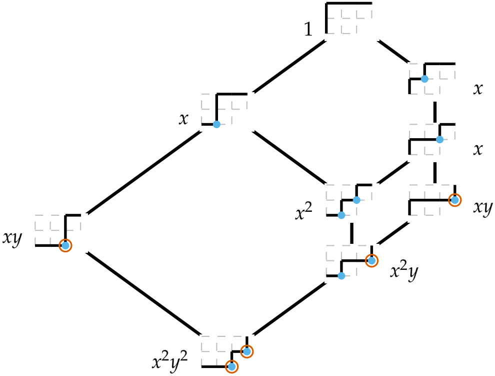

$$\mathcal{H}_{EENEN}(x,y) = x^2y^2 + x^2y + x^2 + 2xy + 3x + 1$$

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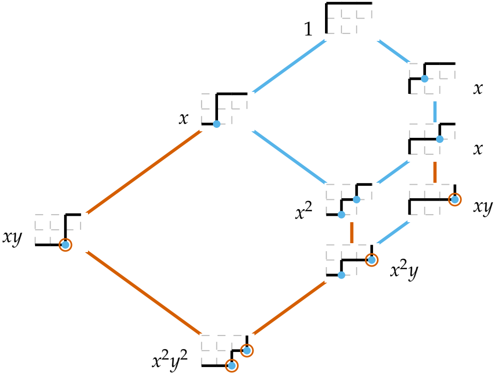

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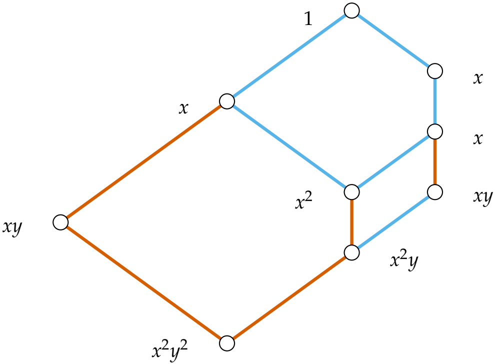

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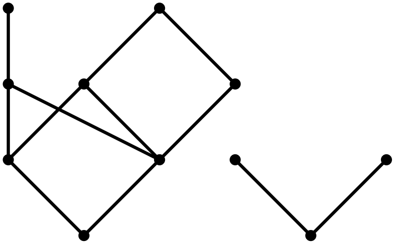
$F = H$ for Posets

- $\mathbf{P} = (P, \leq); p \in P$
- $\mathsf{Succ}(p) \overset{\mathsf{def}}{=} \{p' \in P : p \lessdot p'\}$
- $\mathsf{out}(p) \overset{\mathsf{def}}{=} \left|\mathsf{Succ}(p)\right|$



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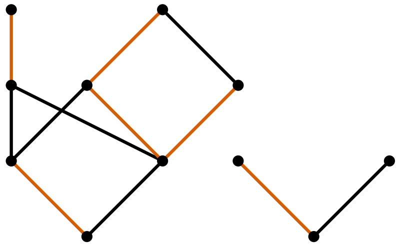

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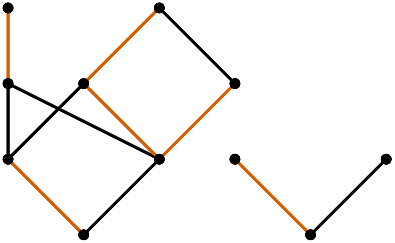
- $\mathsf{mrk}(p) \overset{\mathsf{def}}{=} \big|\{p' \in \mathsf{Succ}(p) : \lambda(p, p') = 1\}\big|$



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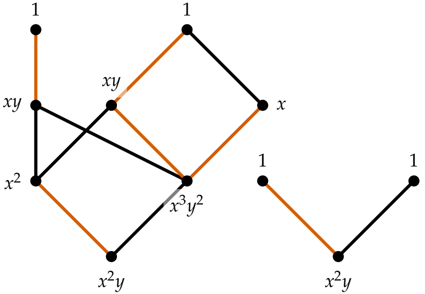

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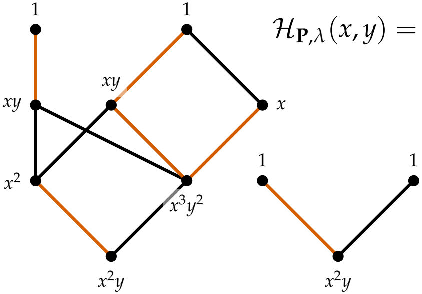

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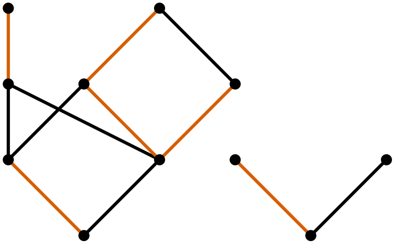
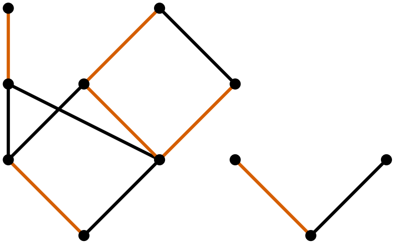


$F = H$ for Posets

- $\mathbf{P} = (P, \leq); p \in P\,; S \subseteq \mathsf{Succ}(p)$
- $\mathsf{deg}(\mathbf{P}) \overset{\mathsf{def}}{=} \mathsf{max}\{\mathsf{out}(p) \colon p \in P\}$
- $\mathsf{rem}(p, S) \overset{\mathsf{def}}{=} \mathsf{mrk}(p) - \big|\{s \in S \colon \lambda(p,s) = 1\}\big|$
- $\mathsf{corem}(p, S) \overset{\mathsf{def}}{=} \mathsf{deg}(\mathbf{P}) - |S| - \mathsf{rem}(p, S)$



$F = H$ for Posets

- $\mathbf{P} = (P, \leq)$; $p \in P$; $S \subseteq \mathsf{Succ}(p)$
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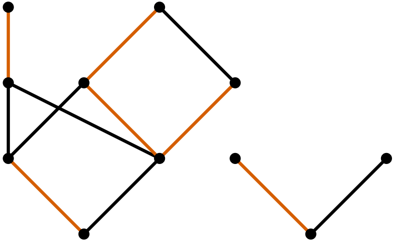

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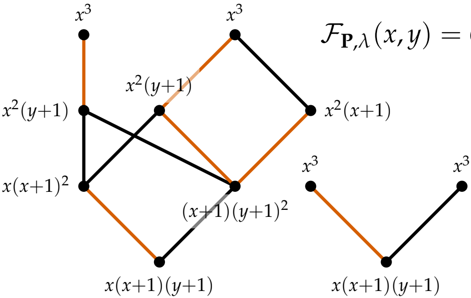

$$\mathcal{F}_{\mathbf{P},\lambda}(x,y) = 6x^3+4x^2y+xy^2+7x^2 +4xy+y^2+4x+2y+1$$

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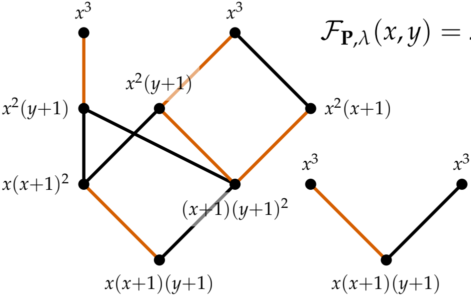

$$\mathcal{F}_{\mathbf{P},\lambda}(x,y) = x^{\deg(\mathbf{P})} \mathcal{H}_{\mathbf{P},\lambda}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right)$$

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Theorem (C. Ceballos & , 2021)

For every finite poset $\mathbf{P}$ *and every* 01*-labeling* λ,

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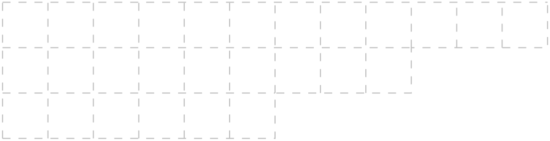
$\rightsquigarrow$ multivariate extension

Back to the Cluster Complex

- fix integers $m, n, t > 0$
- let $\nu_{m,n,t} \overset{\text{def}}{=} E^{mt}N(E^mN)^{n-t-1}$ $\rightsquigarrow$ $\mathsf{Dyck}(m,n,t)$



$m = 3, n = 5, t = 2$

Back to the Cluster Complex

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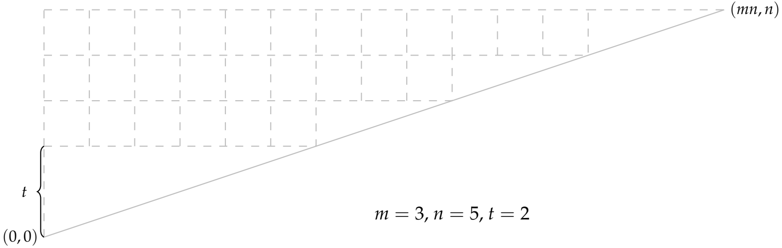
- $t = 1$: m-Dyck paths



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For $m, n > 0$,

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The coefficients are the (reverse) ***Fuß–Narayana numbers****.*

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Back to the Cluster Complex

- $\mathcal{F}_{m,n,1}(x,y)$ is *not* the correct F-triangle
- solution: count only *particular* valleys

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- ***m*-valley**: a valley whose x-coordinate is divisible by m $\rightsquigarrow \mathsf{mval}(\mu)$

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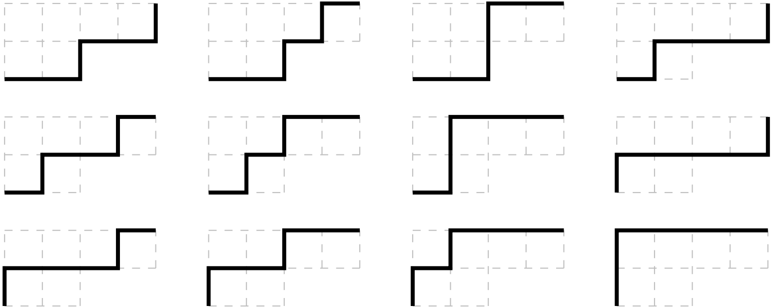
Example: $m = 2, n = 3, t = 1$



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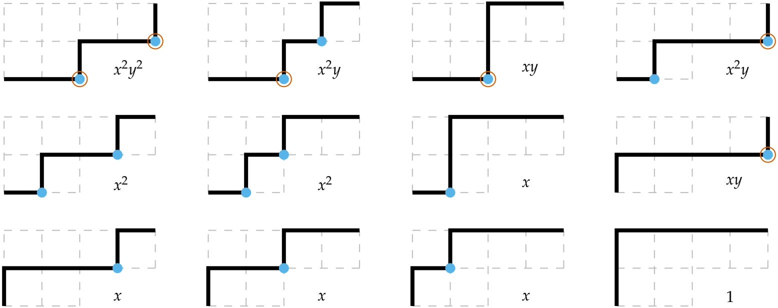

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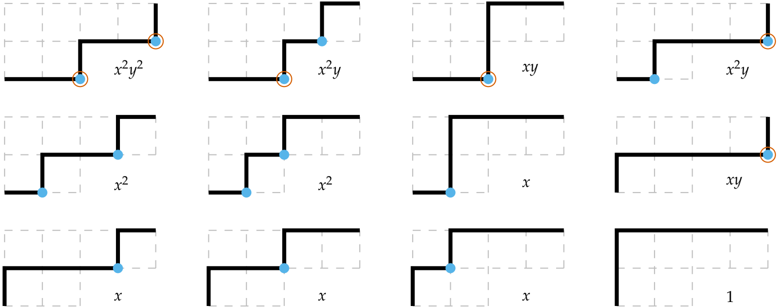

$$\mathcal{H}_{2,3,1}(x,y) = x^2y^2+2x^2y+2x^2+2xy+4x+1$$

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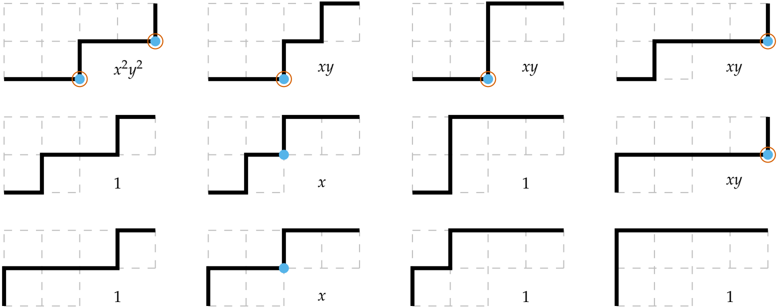

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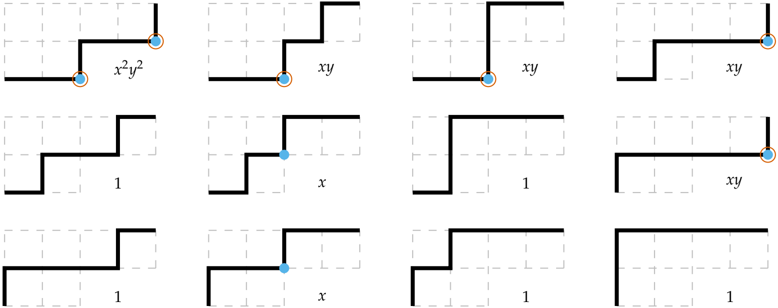

$$\mathcal{H}_{2,3,1}(x,y) = x^2y^2+2x^2y+2x^2+2xy+4x+1$$

$$\mathcal{H}^{(2)}_{2,3,1}(x,y) = x^2y^2+4xy+2x+5$$

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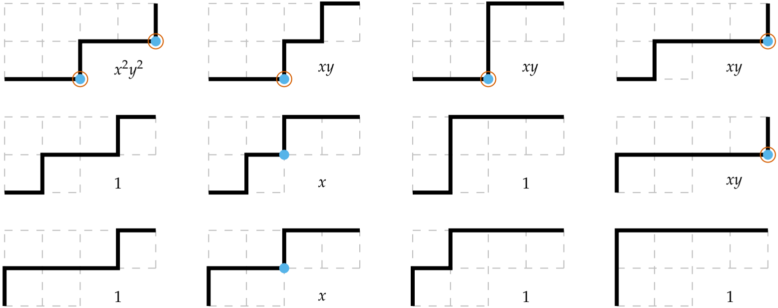


$$\mathcal{H}_{2,3,1}(x,1) = 5x^2+6x+1$$

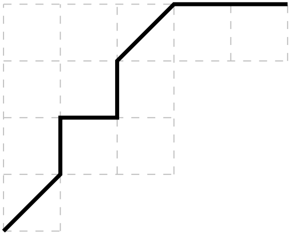
$$\mathcal{H}^{(2)}_{2,3,1}(x,1) = x^2+6x+5$$

Outline

ν-Schröder Paths

- **ν-Schröder path**: lattice path from $(0,0)$ to (m,n) weakly above ν with possible diagonal steps

$\rightsquigarrow \mathsf{Schröder}(\nu)$



ν-Schröder Paths

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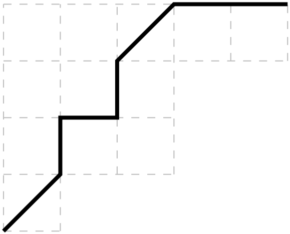
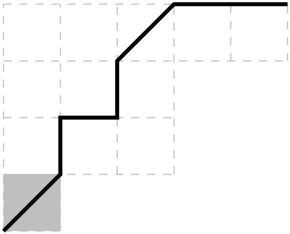
- two statistics dg and cd control diagonals and **cornered diagonals**



ν-Schröder Paths

- **ν-Schröder path**: lattice path from $(0,0)$ to (m,n) weakly above ν with possible diagonal steps
  $\rightsquigarrow \mathsf{Schröder}(\nu)$
- two statistics dg and cd control diagonals and **cornered diagonals**



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Theorem (M. von Bell & M. Yip, 2020)

For any northeast path ν, the sets $\mathsf{Asso}(\nu)$ and $\mathsf{Schröder}(\nu)$ are in bijection.

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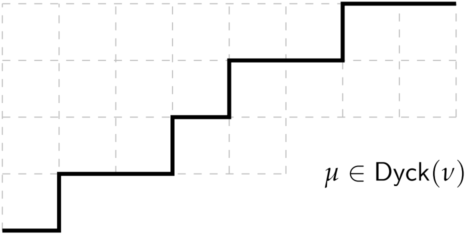

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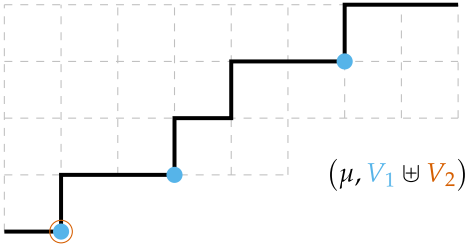

$(\mu, V_1 \uplus V_2) \in \mathsf{Asso}(\nu)$

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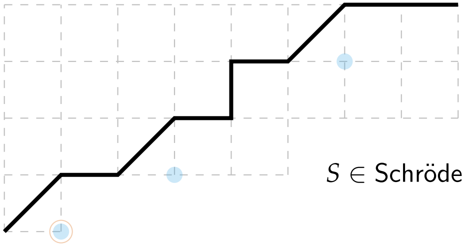

$S \in \mathsf{Schröder}(\nu)$

ν-Schröder Paths

- enumerate ν-Schröder paths with respect to diagonals and cornered diagonals:

$$\mathcal{S}_\nu(x,y) \stackrel{\mathsf{def}}{=} \sum_{S \in \mathsf{Schröder}(\nu)} x^{\mathsf{dg}(S)} y^{\mathsf{cd}(S)}$$

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For $m,n,t > 0$, we have

$$\mathcal{P}^{(m)}_{m,n,t}(x,y) = \sum_{a=0}^{n-t} \sum_{b=0}^{a} \left(\frac{a+t}{n} - \frac{a-b}{n-b-1} \right) \times \binom{n-b-1}{a-b} \binom{mn+n-t-a-1}{n-t-a} x^a y^b.$$

Once again the Cluster Complex

- $\mathcal{H}^{(m)}_{m,n,t}(x,y) = \sum_{\mu \in \mathsf{Dyck}(m,n,t)} x^{\mathsf{mval}(\mu)} y^{\mathsf{ret}(\mu)}$
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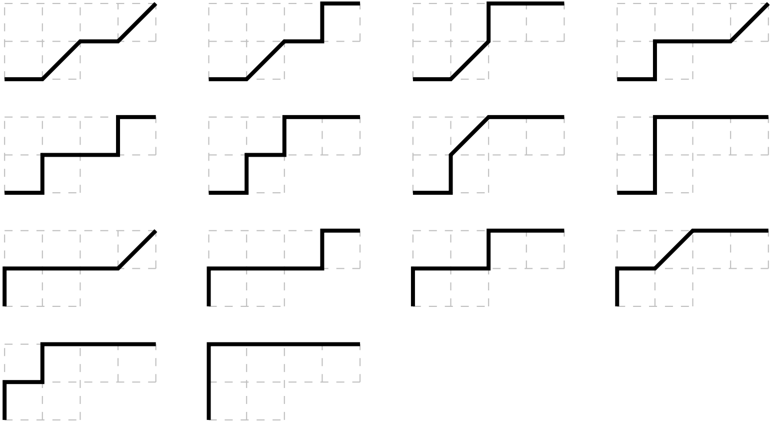
Corollary (& E. Tzanaki, 2022)

For $m, n > 0$, we have

$$\mathcal{P}^{(m)}_{m,n,1}(x,y) = (-1)^{n-1} \mathcal{H}^{(m)}_{-m,n,1}(-x,y).$$

Example: $m = 2, n = 3, t = 1$



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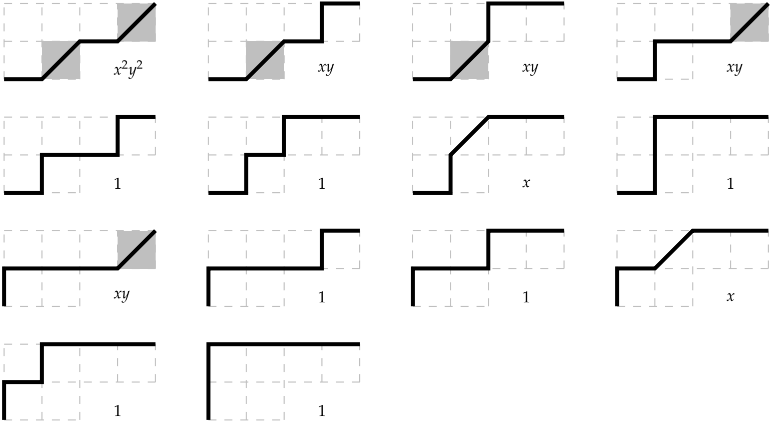

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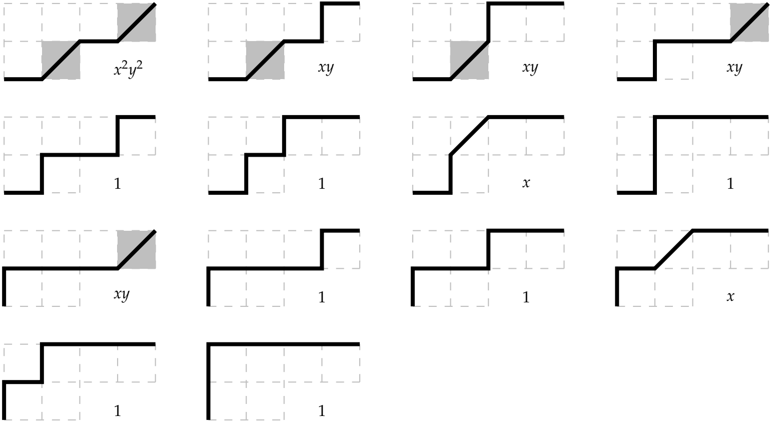

$$\mathcal{P}^{(2)}_{2,3,1}(x,y) = x^2y^2+4xy+2x+7$$

Thank You.

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