Connectivity Properties of Factorization Posets

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(joint work with Vivien Ripoll)

Outline

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Generated Groups

- G .. group; $A \subseteq G$ .. generating set; ℓ_A .. word length

$$G = \big\langle r, s, t \mid r^2 = s^3 = t^3 = \mathbb{1}, t = rs \big\rangle_{\text{grp}}$$

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Solution

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$$\begin{array}{cccccc} \mathbb{1} & r & s & t & sr & ss \\ st & tr & ts & tt & sts & sst \end{array}$$

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- *A* **suffix-order**: $u \leq_{\mathsf{suf}} v$ if and only if $\ell_A(vu^{-1}) + \ell_A(u) = \ell_A(v)$

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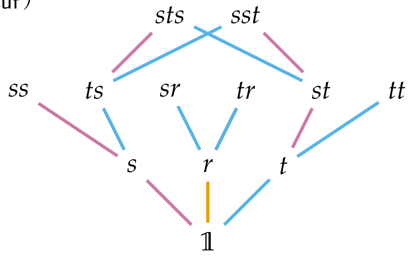
$(G, \leq_{\mathsf{suf}})$



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$$\ell_A(u) + \ell_A(u^{-1}v) = \ell_A(v)$$

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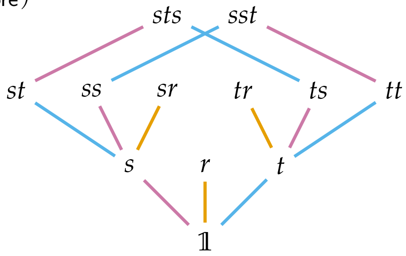

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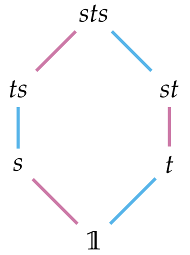

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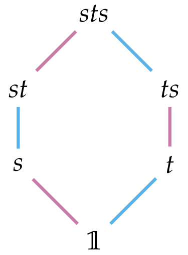

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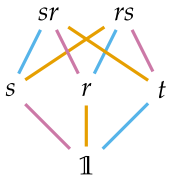

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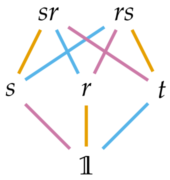

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Factorization Posets

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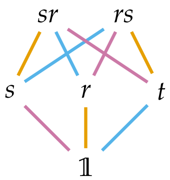

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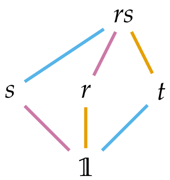

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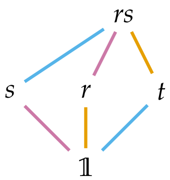

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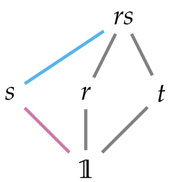

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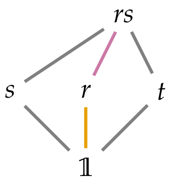

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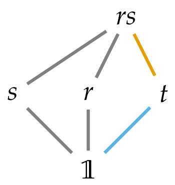

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Lemma (A. Björner, 1984)

For $u \leq_{\mathsf{pre}} v \leq_{\mathsf{pre}} g$, the interval $[u, v]$ in $\mathbf{P}_A(g)$ is isomorphic to $\mathbf{P}_A(u^{-1}v)$.

Outline

Hurwitz Action

- $[k] \overset{\text{def}}{=} \{1,2,\ldots,k\}$ for $k > 0$
- **braid group**:

$$\mathfrak{B}_k = \bigl\langle \sigma_1,\ldots,\sigma_{k-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \in [k-2], \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \bigr\rangle_{\text{grp}}$$

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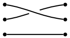
$\sigma_1 \quad = \quad$ 

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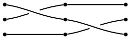
$\sigma_1\sigma_2 \quad =$ 

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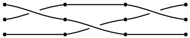
$\sigma_1\sigma_2\sigma_1 \quad =$ 

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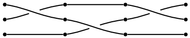
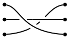
$\sigma_1\sigma_2\sigma_1 \quad = \quad$  $\quad = \quad$ 

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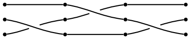
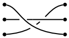
$\sigma_2\sigma_1\sigma_2 \;=$  $=$ 

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Observation (Folklore)

If A is closed under G-conjugation, then the Hurwitz action extends to a group action of $\mathfrak{B}_k$ on $A^{(k)}$.

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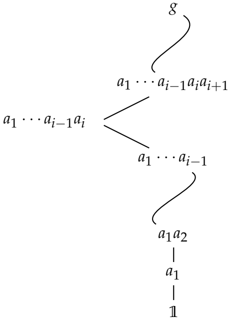

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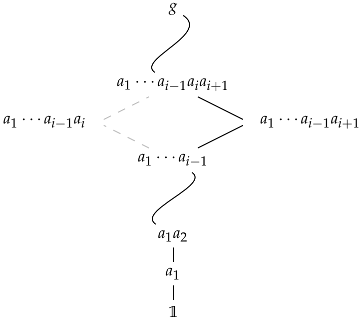

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- origin: Hurwitz' enumeration of branched coverings of a Riemann surface (1891)

$\rightsquigarrow$ $G = \mathfrak{S}_n, A = \bigl\{(i\;j) \mid 1 \leq i < j \leq n\bigr\}, g = (1\;2\;\ldots\;n)$

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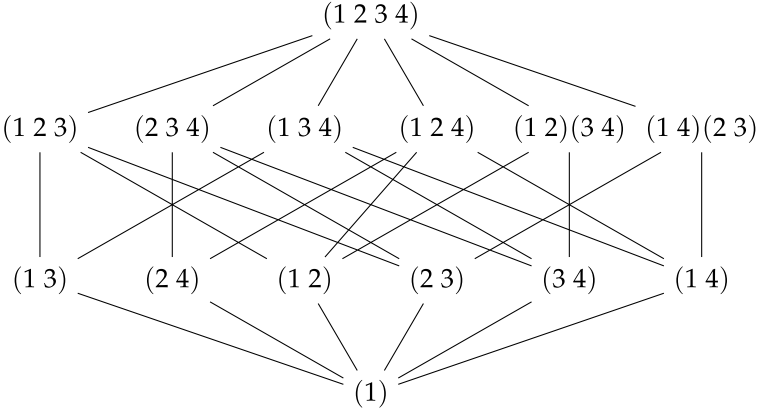

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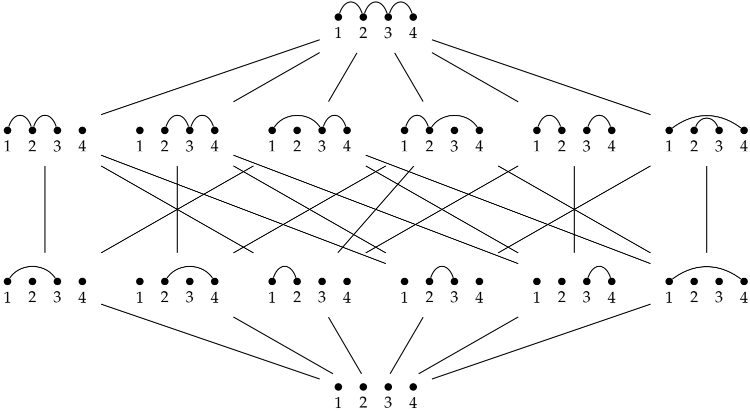

Motivation

- W .. well-generated complex reflection group
- T .. set of *all* reflections of W
- c .. Coxeter element of W

Theorem (P. Deligne, 1974; D. Bessis & R. Corran, 2006; D. Bessis, 2006 (2015))

For any well-generated complex reflection group W and any Coxeter element $c \in W$, the braid group $\mathfrak{B}_{\ell_T(c)}$ acts transitively on $\mathsf{Red}_T(c)$.

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$\rightsquigarrow$ **c-noncrossing W-partitions**: elements of $\mathbf{P}_T(c)$

$\rightsquigarrow$ $\mathbf{Nonc}(W,c)$

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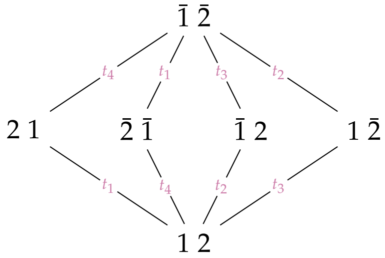

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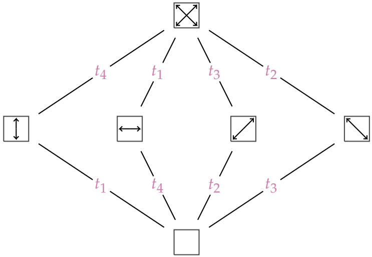

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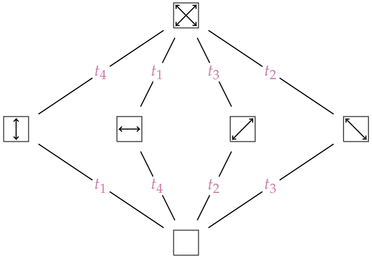

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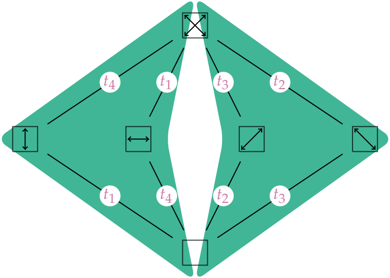

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Limitations

- **quasi-Coxeter element**: exists $(a_1, a_2, \ldots, a_n) \in \mathsf{Red}_T(g)$ such that $W = \langle a_1, a_2, \ldots, a_n \rangle$

Theorem (B. Baumeister, T. Gobet, K. Roberts & P. Wegener, 2017)

Let W be a finite real reflection group. Then $\mathfrak{B}_{\ell_T(g)}$ acts transitively on $\mathsf{Red}_T(g)$ if and only if there exists a parabolic subgroup W' of W for which g is a quasi-Coxeter element.

Limitations

- **quasi-Coxeter element**: exists $(a_1, a_2, \ldots, a_n) \in \mathsf{Red}_T(g)$ such that $W = \langle a_1, a_2, \ldots, a_n \rangle$

Theorem (B. Baumeister, T. Gobet, K. Roberts & P. Wegener, 2017)

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$\rightsquigarrow$ extension to complex reflection groups by J. Lewis and J. Wang (2021)

Other Results

- conditions for Hurwitz-equivalence

  T. Ben-Itzhak & M. Teicher (2003); X. Hou (2008); C. Sia (2009); E. Berger (2011); J. Lewis (2020)

- computation of braid monodromy

  E. Brieskorn (1988); A. Libgober (1999); V. Kulikov & M. Teicher (2000)

- Hurwitz action in finitely-generated real reflection groups

  B. Baumeister, M. Dyer, C. Stump & P. Wegener (2014); P. Wegener (2020)

- subgroups of the symmetric group generated by k-cycles

   & P. Nadeau (2019); , P. Nadeau & N. Williams (2020)

Outline

Hurwitz Graph

- **Hurwitz graph**: $\mathscr{H}(g) \overset{\mathsf{def}}{=} \bigl(\mathsf{Red}_A(g), \mathscr{HE}(g)\bigr)$, where

$$\mathscr{HE}(g) \overset{\mathsf{def}}{=} \bigl\{(\mathbf{g},\mathbf{g}') \mid \mathbf{g}' = \sigma_i^{\pm 1}\cdot\mathbf{g} \text{ for some } i \in [\ell_A(g)-1]\bigr\}$$

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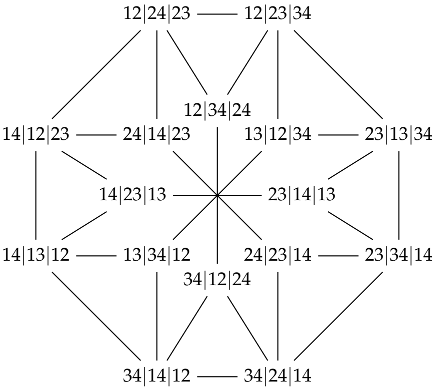
$G = \mathfrak{S}_4$
$g = (1\;2\;3\;4)$



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- **Hurwitz-connected**: $\mathcal{H}(g)$ is connected

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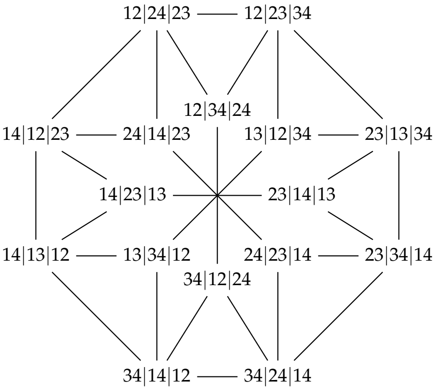

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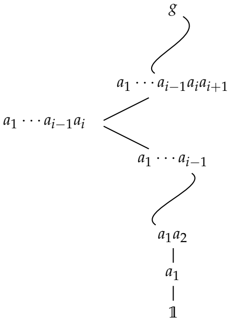

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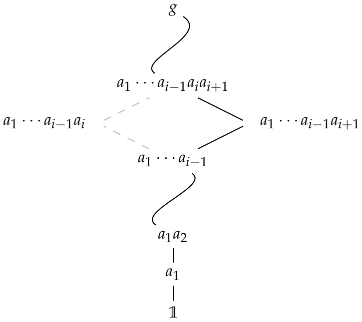

Chain Graph

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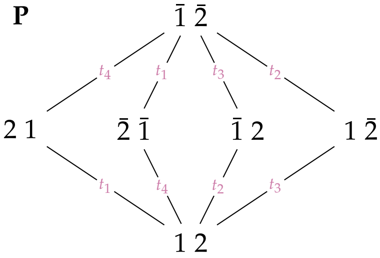
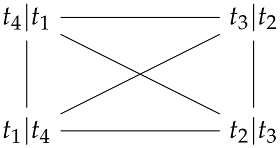


$\mathscr{C}(\mathbf{P})$



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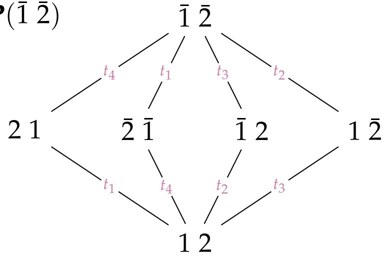

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Chain Graph

- $\mathbf{P} = (P, \leq)$ .. (finite) graded poset with bounds $\hat{0}$ and $\hat{1}$ and rank k
- **chain connected**: $\mathscr{C}(\mathbf{P})$ is connected

Proposition (& V. Ripoll, 2020)

If $\mathbf{P}_A(g)$ is Hurwitz-connected, then it is chain connected.

Shellability

- $\mathbf{P} = (P, \leq)$ .. (finite) graded poset with bounds $\hat{0}$ and $\hat{1}$
- **shelling**: total order $\prec$ on $\mathscr{M}(\mathbf{P})$ such that whenever $M \prec M'$, then there is $N \prec M'$ and $x \in M'$ such that $M \cap M' \subseteq N \cap M' = M' \setminus \{x\}$
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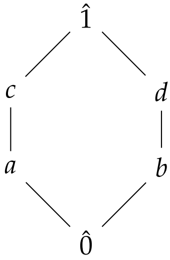

$$C_1 = \{\hat{0}, a, c, \hat{1}\}$$
$$C_2 = \{\hat{0}, b, d, \hat{1}\}$$

$\rightsquigarrow C_1 \cap C_2 = C_2 \setminus \{b, d\}$

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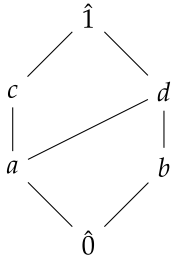

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$$C_1 \cap C_3 \subseteq C_2 \cap C_3$$

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Proposition (🌬 & V. Ripoll, 2020)

Every shellable poset is chain connected.

Lexicographic Shellability

- $\mathbf{P} = (P, \leq)$ .. (finite) graded poset with bounds $\hat{0}$ and $\hat{1}$

- **EL-labeling**: edge-labeling such that for each interval the lexicographically smallest chain is uniquely rising
- **EL-shellable**: poset that admits an EL-labeling

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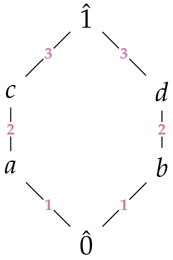

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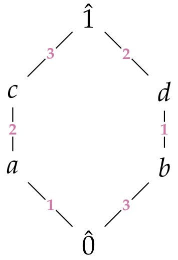

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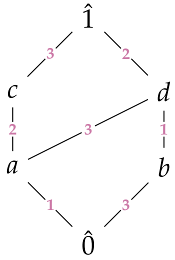

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Proposition (A. Björner, 1980)

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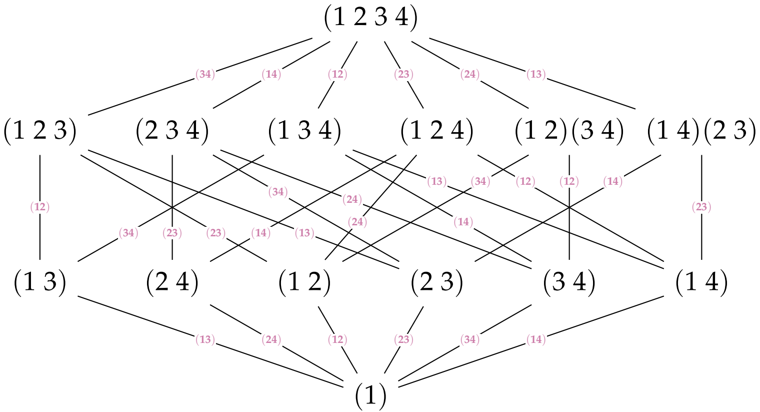
Observation ( )

Any factorization poset $\mathbf{P}_A(g)$ *admits a canonical edge labeling given by* $\lambda_g(u,v) \stackrel{\mathsf{def}}{=} u^{-1}v \in A$.

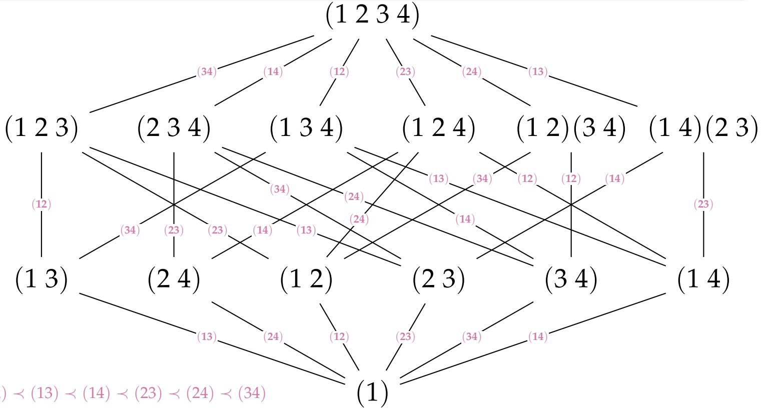
Lexicographic Shellability



Lexicographic Shellability

Proposition (A. Björner & P. Edelman, 1980)

For $n > 0$, the lexicographic order on transpositions makes λ_c an EL-labeling of $\mathbf{Nonc}(\mathfrak{S}_n, c)$, where $c = (1\ 2\ \ldots\ n)$.



$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$

Compatible Generator Orders

- $A_g \overset{\mathsf{def}}{=} \{a \in A \mid a \leq_{\mathsf{pre}} g\}$ $\rightsquigarrow \mathsf{Red}_A(g) = \mathsf{Red}_{A_g}(g)$
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Definition (& V. Ripoll, 2020)

A total order of A_g is **g-compatible** if every $h \leq_{\mathsf{pre}} g$ with $\ell_A(h) = 2$ has a unique $\prec$-rising factorization.

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$\rightsquigarrow$ compatibility is a "local" version of EL-shellability

Lexicographic Shellability

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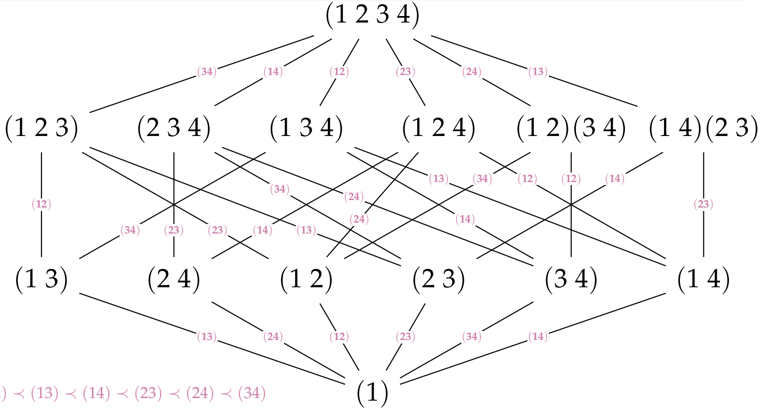

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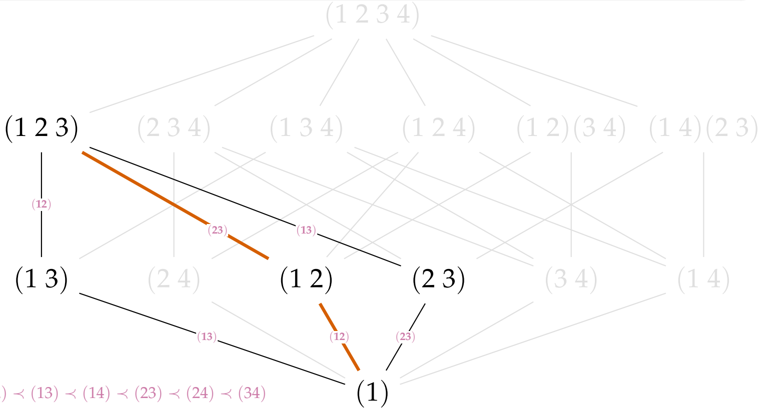
Corollary (A. Björner & P. Edelman, 1980)

For $n > 0$, the lexicographic order on transpositions of $\mathfrak{S}_n$ is c-compatible, where $c = (1\,2\,\ldots\,n)$.



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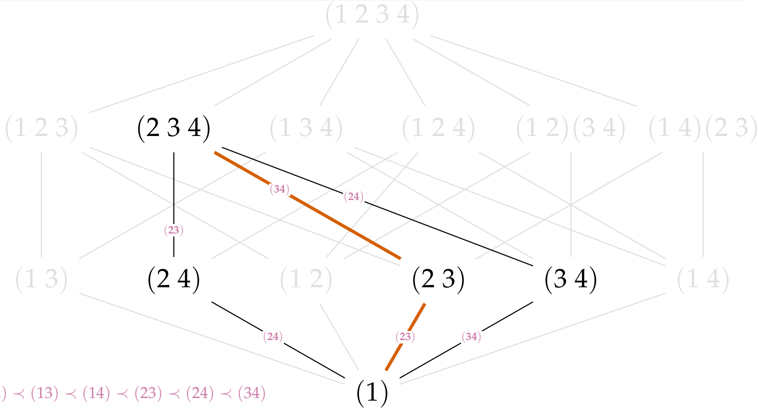

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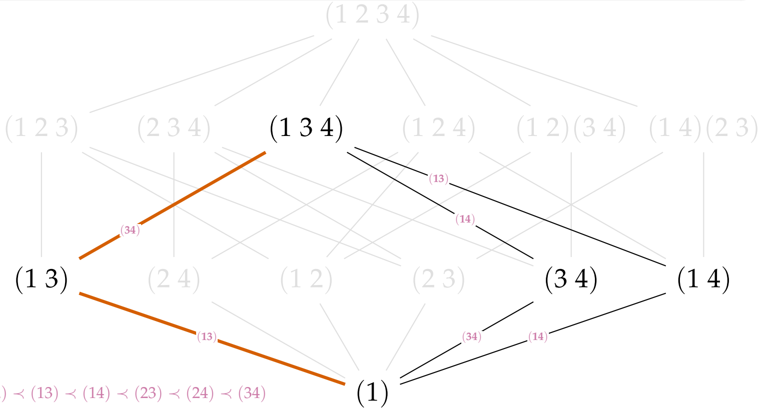

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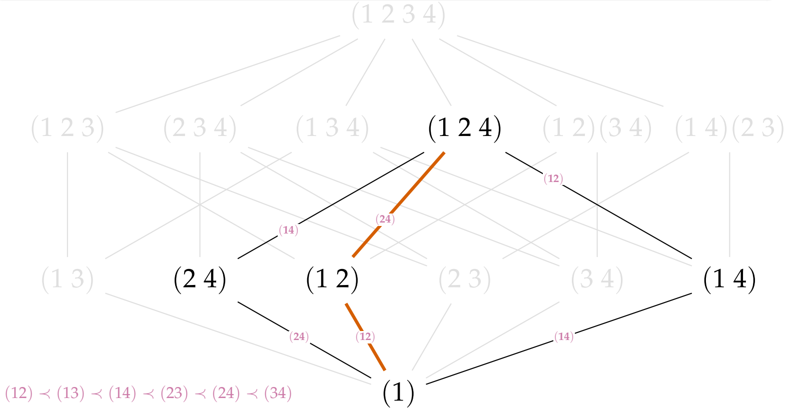

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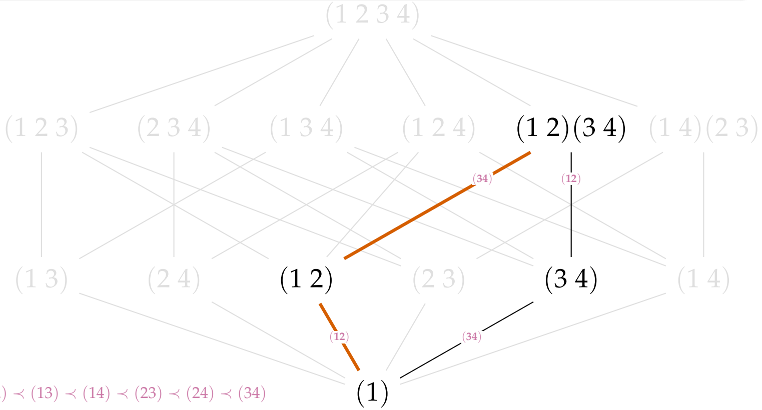

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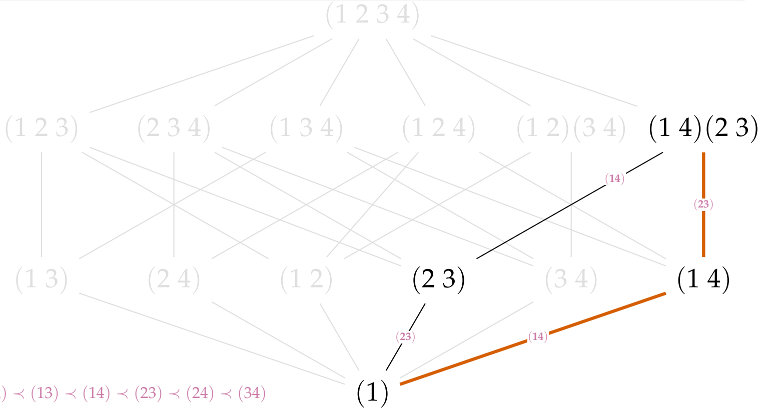

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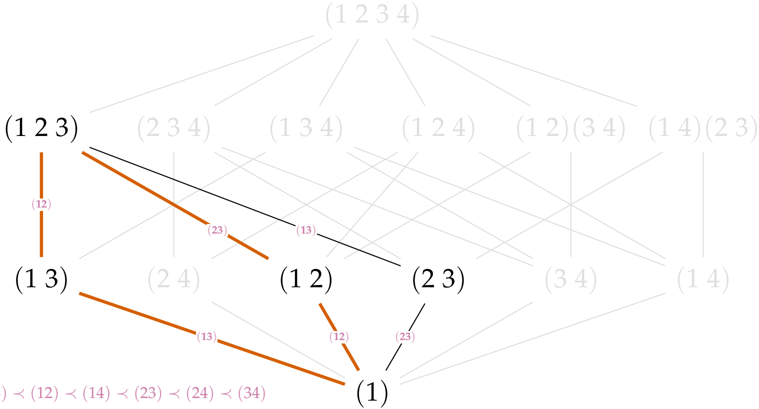

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Theorem (C. Athanasiadis, T. Brady & C. Watt, 2007; , 2015)

For every well-generated complex reflection group W and every Coxeter element $c \in W$, the set of all reflections admits a c-compatible order.

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For every well-generated complex reflection group W and every Coxeter element $c \in W$, the set of all reflections admits a c-compatible order.

$\rightsquigarrow$ crucial component in the proof of the EL-shellability

Compatible Generator Orders

Conjecture (& V. Ripoll, 2020)

If $\mathsf{Red}_A(g)$ *is finite, every interval of* $\mathbf{P}_A(g)$ *is chain connected and* A_g *admits a* g*-compatible generator order, then* λ_g *is an EL-labeling.*

Compatible Generator Orders

Lemma (& V. Ripoll, 2020)

Suppose that $\ell_A(g) = 2$. There exists a g-compatible order of A_g if and only if $\mathbf{P}_A(g)$ is Hurwitz-connected.

Compatible Generator Orders

Lemma (& V. Ripoll, 2020)

Suppose that $\ell_A(g) = 2$. There exists a g-compatible order of A_g if and only if $\mathbf{P}_A(g)$ is Hurwitz-connected.

$\rightsquigarrow$ does not extend to $\ell_A(g) > 2$

Compatible Generator Orders

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Corollary (& V. Ripoll, 2020)

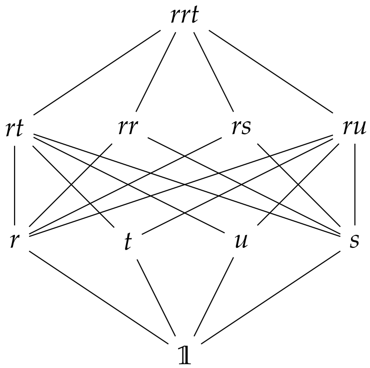
If $\mathsf{Red}_A(g)$ *is finite and* λ_g *is an EL-labeling, then* $\mathbf{P}_A(g)$ *is Hurwitz-connected.*

Limitations

- Hurwitz-transitivity does not necessarily imply rank-2 Hurwitz-transitivity

$$G = \langle r,s,t,u \mid r^2 = s^2, t^2 = u^2, rs = sr, tu = ut, rt = ts = su = ur, st = tr = ru = us\rangle_{\text{grp}}$$

$\mathbf{P}(rrt)$



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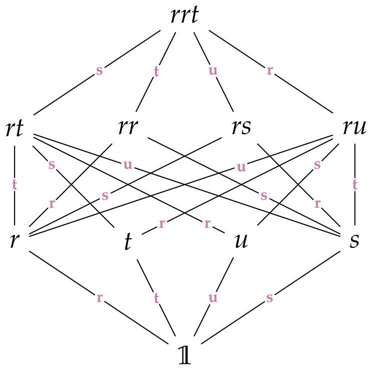

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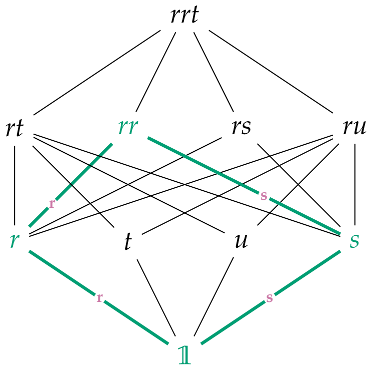

Limitations

- Hurwitz-transitivity does not necessarily imply rank-2 Hurwitz-transitivity

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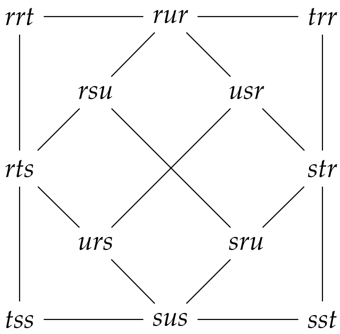

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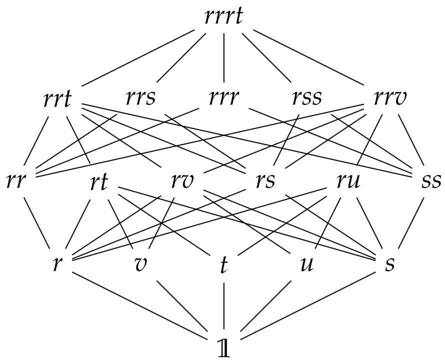

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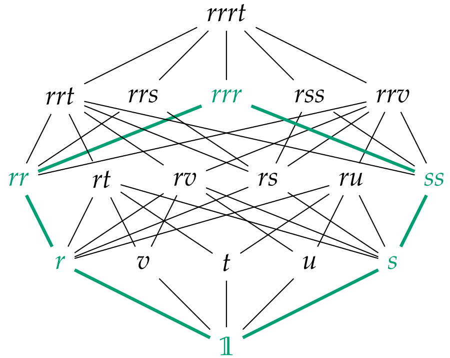

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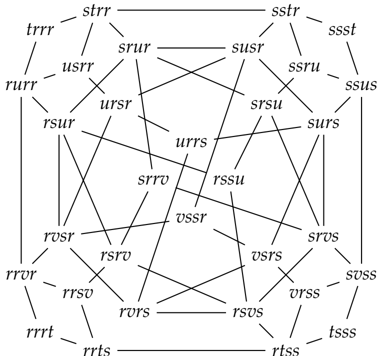
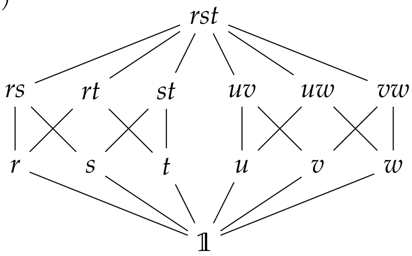


Limitations

- existence of a compatible order does not necessarily imply shellability

$$G = \langle r, s, t, u, v, w \mid \text{commutations}, rst = uvw \rangle_{\text{grp}}$$

$\mathbf{P}(rst)$



Well-Covered Factorization Posets

- fix a total order $\prec$ of A_g
- **well covered**: if $b \in A_g$ is not $\prec$-minimal, there exists some $a \prec b$ such that a and b have a common upper cover in $\mathbf{P}_A(g)$

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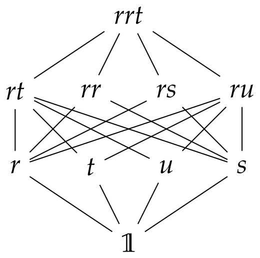

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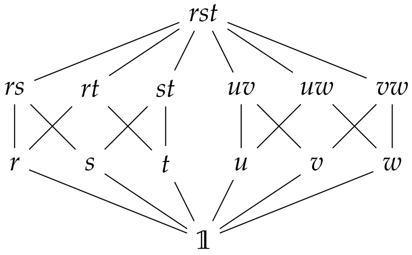

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Theorem (& V. Ripoll, 2020)

Let $\prec$ be a total order of A_g. Then, λ_g is an EL-labeling of $\mathbf{P}_A(g)$ with respect to $\prec$ if and only if $\prec$ is g-compatible and $\mathbf{P}_A(g)$ is totally well covered with respect to $\prec$.

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$\rightsquigarrow$ modeled after *recursive atom orders* of A. Björner and M. Wachs (1983)

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Theorem (& V. Ripoll, 2020)

If $\mathsf{Red}_A(g)$ is finite and $\mathbf{P}_A(g)$ admits a g-compatible order $\prec$ of A_g and $\mathbf{P}_A(g)$ is totally well covered with respect to $\prec$, then $\mathbf{P}_A(g)$ is chain connected, Hurwitz-connected and shellable.

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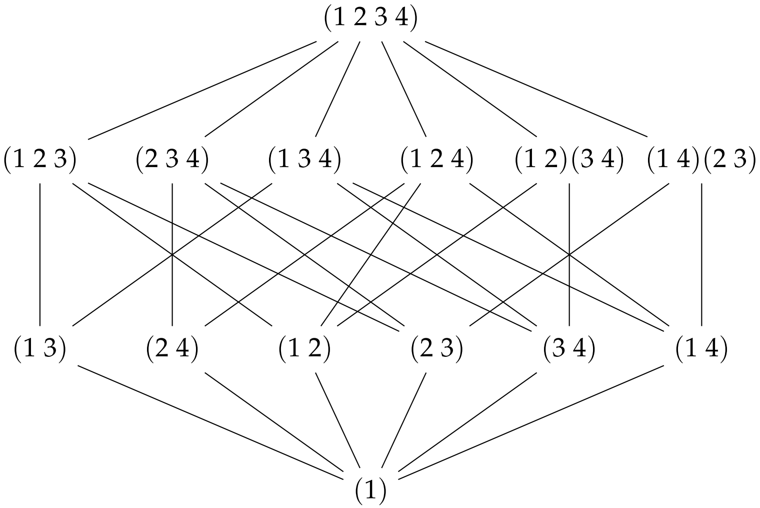
Conjecture (& V. Ripoll, 2020)

If every interval of $\mathbf{P}_A(g)$ is chain connected and there exists a g-compatible order $\prec$ of A_g, then $\mathbf{P}_A(g)$ is totally well covered with respect to $\prec$.

Outline

The Cycle Graph of $\mathbf{P}_A(g)$



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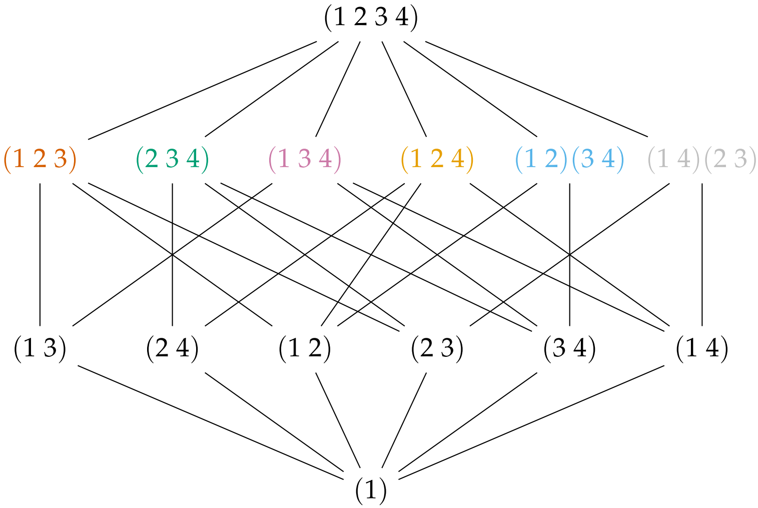

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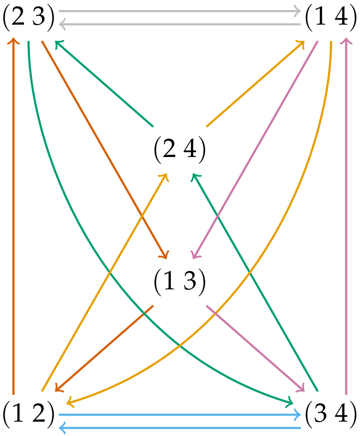

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- **cycle graph**: labeled directed graph $\Gamma_A(g) \overset{\text{def}}{=} (V_g, \vec{E}_g, \sigma_g)$, where:
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Lemma (& V. Ripoll, 2020)

For any $h \in B_g$, the set of edges labeled by h in $\Gamma_A(g)$ is a disjoint union of directed cycles. Each such cycle corresponds to a connected component of $\mathscr{H}(h)$.

The Cycle Graph of $\mathbf{P}_A(g)$

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Proposition (V. Ripoll, 2020)

For any $g \in G$, $\mathsf{df}(g) \geq |B_g|$. Moreover, $\mathsf{df}(g) = |B_g|$ if and only if $\mathbf{P}_A(g)$ admits a g-compatible order of A_g.

The Reduced Cycle Graph of $\mathbf{P}_A(g)$

- fix a g-compatible order $\prec$ of A_g
- **reduced cycle graph**: for every $h \in B_g$ remove the unique $\prec$-rising factorization of h $\rightsquigarrow \Gamma_A^{\prec}(g)$

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Proposition (& V. Ripoll, 2020)

The order $\to$ is total (and therefore equal to $\prec$) if and only if $\Gamma_A^{\prec}(g)$ is connected (as a directed graph).

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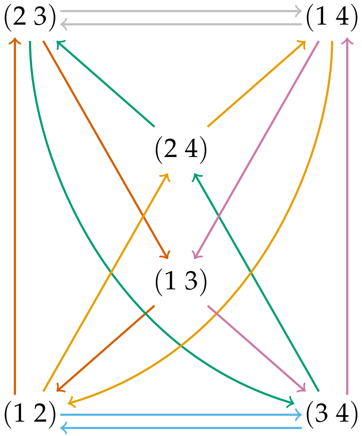
Proposition (& V. Ripoll, 2020)

$\mathbf{P}_A(g)$ is well covered with respect to $\prec$ if and only if $\Gamma_A^{\prec}(g)$ has a unique sink. In particular, if $\to$ is total, then $\mathbf{P}_A(g)$ is well covered.

The Cycle Graph of $\mathbf{P}_A(g)$



$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$

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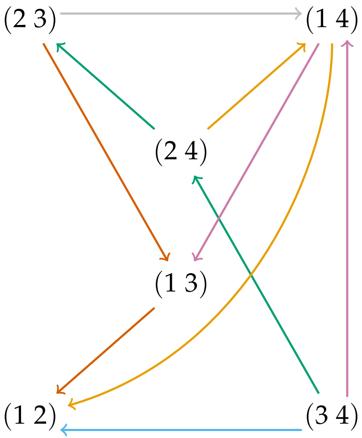

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A Partial Result

Theorem (& V. Ripoll, 2020)

Let $\ell_A(g) \geq 3$ *such that* $\mathsf{Red}_A(g)$ *is finite, and let* $\mathbf{P}_A(g)$ *be a factorization poset in which every interval is chain-connected. Suppose that there is some* $a \in A_g$ *that lies in a unique monochromatic cycle of* $\Gamma_A(g)$ *which is not a loop. If there exists a* g*-compatible order* $\prec$ *of* A_g*, then* $\mathbf{P}_A(g)$ *is totally well covered with respect to* $\prec$*.*

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- idea: characterize the cycle graphs that admit a compatible order
- there are two non-trivial options

Open Problems

- are factorization posets of quasi-Coxeter elements in (well-generated) reflection groups shellable?
- are factorization posets of cycles $(1\;2\;\ldots\;kn{+}1)$ in the subgroup of $\mathfrak{S}_{kn+1}$ generated by all $(k+1)$-cycles shellable?

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- study Hurwitz graphs from a graph-theoretic perspective

Thank You.

Generated Groups

- G .. group; $A \subseteq G$ .. generating set; ℓ_A .. word length

Return

$$G = \left\langle r, s, t \mid r^2 = s^3 = t^3 = \mathbb{1}, t = rs \right\rangle_{\text{grp}}$$

$$\begin{array}{cccccc} \mathbb{1} & r & s & t & sr & ss \\ st & tr & ts & tt & sts & sst \end{array}$$

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$$r = (1\ 2)(3\ 4)$$
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(1)	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(2\ 4\ 3)$	$(1\ 3\ 4)$	$(1\ 3\ 2)$
$(1\ 2\ 4)$	$(1\ 4\ 2)$	$(1\ 4\ 3)$	$(2\ 3\ 4)$	$(1\ 4)(2\ 3)$	$(1\ 3)(2\ 4)$

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$$\mathfrak{A}_4 = \langle r, s, t \mid r^2 = s^3 = t^3 = \mathbb{1}, t = rs \rangle_{\text{grp}}$$

(1)	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(2\,4\,3)$	$(1\,3\,4)$	$(1\,3\,2)$
$(1\,2\,4)$	$(1\,4\,2)$	$(1\,4\,3)$	$(2\,3\,4)$	$(1\,4)(2\,3)$	$(1\,3)(2\,4)$

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