EL-Shellability of the *m*-Tamari Lattices

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Origin

- the m-Tamari lattices were introduced by Bergeron and Préville-Ratelle in order to express the Frobenius characteristics of the space of higher diagonal harmonics
- Bousquet-Mélou, Fusy and Préville-Ratelle proved the lattice property and a formula for the number of intervals
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- combinatorial realization via m-Dyck paths

- we are interested in topological properties of $\mathcal{T}_n^{(m)}$, which can be determined with the help of EL-shellability

Outline

1. Preliminaries
   - m-Tamari Lattices
   - EL-Shellability of Posets
2. EL-Shellability of $\mathcal{T}_n^{(m)}$
   - A natural Edge-Labeling
   - Constructing Rising Chains
3. Topological Properties of $\mathcal{T}_n^{(m)}$
   - Falling Maximal Chains

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m-Dyck Paths

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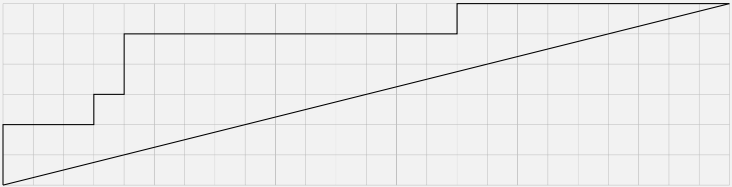
- a 4-Dyck path of height 6



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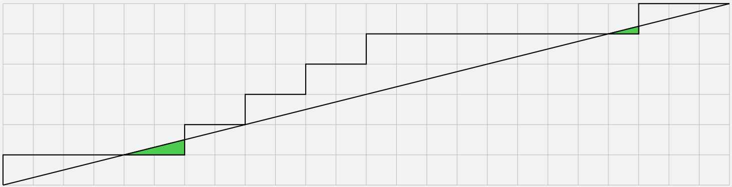

m-Dyck Paths

- $\mathcal{D}_n^{(m)}$: set of m-Dyck paths of height n
- associate an integer sequence $\alpha_p = (a_1, a_2, \ldots, a_n)$ to $p \in \mathcal{D}_n^{(m)}$ that satisfies

$$a_1 \leq a_2 \leq \cdots \leq a_n, \quad \text{and}$$
$$a_i \leq m(i-1), \quad 1 \leq i \leq n$$

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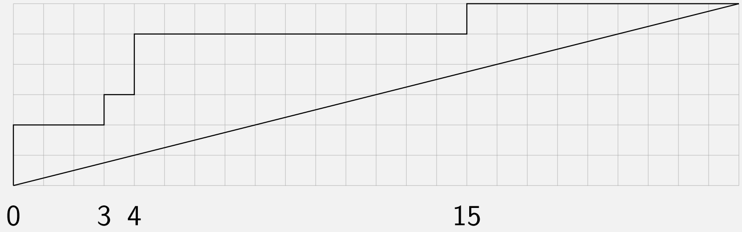

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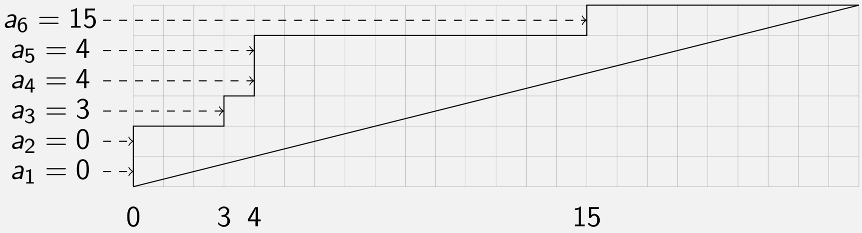

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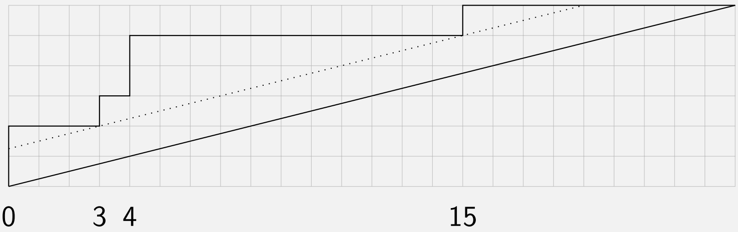
$a_6 = 15$
$a_5 = 4$
$a_4 = 4$
$a_3 = 3$
$a_2 = 0$
$a_1 = 0$



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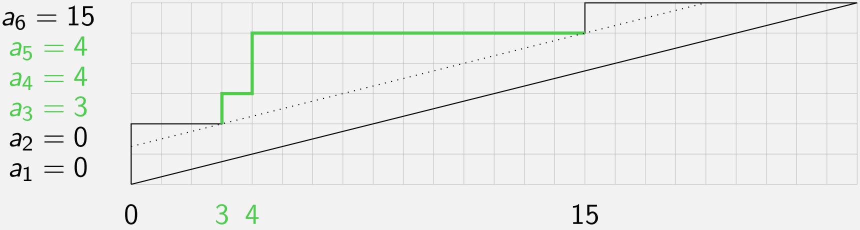

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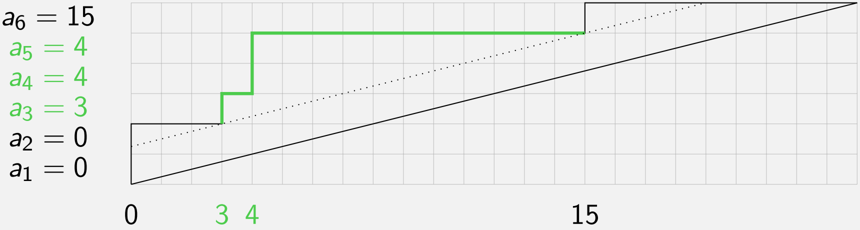

- **primitive subsequence at position** i: unique subsequence $(a_i, a_{i+1}, \ldots, a_k)$ of α_p that satisfies

$$a_j - a_i < m(j-i), \quad i < j \leq k, \text{and}$$
$$\text{either} \quad k = n \quad \text{or} \quad a_{k+1} - a_i \geq m(k+1-i)$$

A Covering Relation on $\mathcal{D}_n^{(m)}$

- let $p \in \mathcal{D}_n^{(m)}$, let u be an upstep of p that is preceded by a rightstep r
- say u is the i-th upstep of p, and let p_i be the m-Dyck subpath of p at position i
- define $p \lessdot q$ if and only if q is obtained from p by exchanging r and p_i

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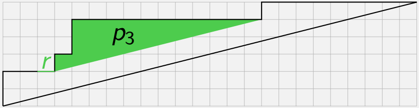
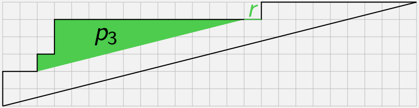


$(0, 0, 3, 4, 4, 15)$

$\lessdot$



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The m-Tamari Lattice

- let $\leq$ denote the transitive and reflexive closure of $\lessdot$
- *m*-**Tamari lattice**: $\mathcal{T}_n^{(m)} = \left(\mathcal{D}_n^{(m)}, \leq\right)$

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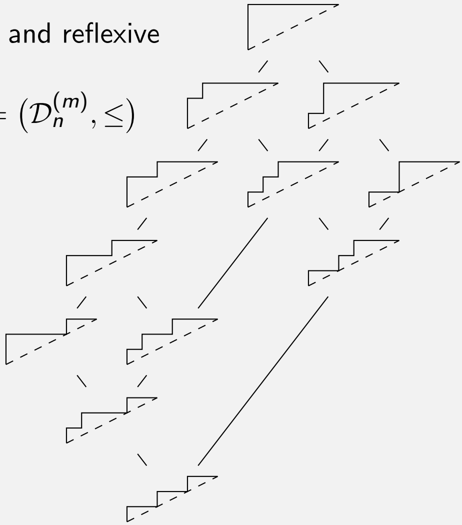
- let $\leq$ denote the transitive and reflexive closure of $\lessdot$
- m-**Tamari lattice**: $\mathcal{T}_n^{(m)} = \left(\mathcal{D}_n^{(m)}, \leq\right)$
- this is $\mathcal{T}_3^{(2)}$



The Main Question

Theorem (Björner & Wachs, 1997)

There exists an EL-labeling for $\mathcal{T}_n^{(1)}$ such that each interval has at most one falling chain.

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- Can this result be generalized to $\mathcal{T}_n^{(m)}$ for $m \geq 1$?

Basics on Posets

- **bounded poset**: a poset that has a unique minimal and a unique maximal element
- let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a bounded poset
- $\overline{\mathbb{P}}$ is the poset that arises from $\mathbb{P}$ by removing the maximal and minimal element (the so-called **proper part of** $\mathbb{P}$)
- **chain**: linearly ordered subset c of P
  notation: $c : p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s$
- **maximal chain in** $[p, q]$: there is no $p' \in [p, q]$ and no $0 \leq i < s$ such that
  $p = p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_i <_{\mathbb{P}} p' <_{\mathbb{P}} p_{i+1} <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s = q$
  is a chain

Edge-Labelings

- **cover relation** $p \lessdot_{\mathbb{P}} q$: $p <_{\mathbb{P}} q$ and there is no $p' \in P$ with $p <_{\mathbb{P}} p' <_{\mathbb{P}} q$
- $\mathcal{E}(\mathbb{P}) = \big\{(p, q) \mid p \lessdot_{\mathbb{P}} q\big\}$ is the set of covering relations on $\mathbb{P}$
- **edge-labeling** λ: map $\lambda : \mathcal{E}(\mathbb{P}) \to \Lambda$, for some poset Λ
- $\lambda(c) = \big(\lambda(p_0, p_1), \lambda(p_1, p_2), \ldots, \lambda(p_{s-1}, p_s)\big)$ is the label-sequence of c
- **rising chain**: a chain c such that $\lambda(c)$ is strictly increasing
- **ER-labeling**: an edge-labeling such that for every interval of $\mathbb{P}$ there is exactly one rising maximal chain
- **EL-labeling**: an ER-labeling such that the rising chain in every interval is lexicographically first among all maximal chains

EL-Shellability

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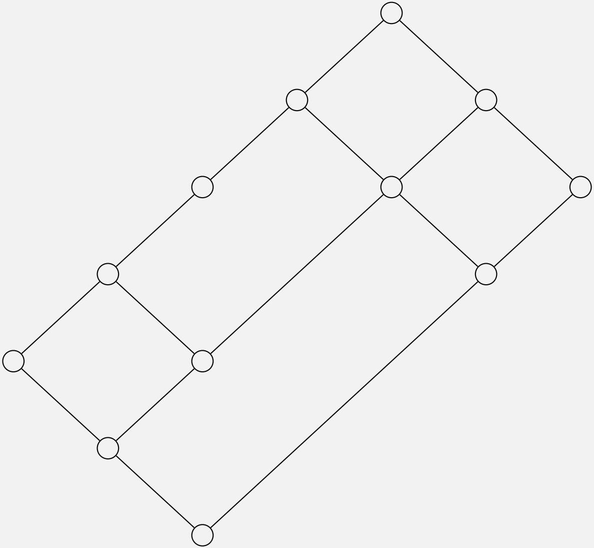
- the order complex $\Delta(\overline{\mathbb{P}})$ of an EL-shellable poset $\mathbb{P}$ is shellable and hence Cohen-Macaulay
- the geometric realization of $\Delta(\overline{\mathbb{P}})$ is homotopy equivalent to a wedge of spheres
- the i-th Betti number of $\Delta(\overline{\mathbb{P}})$ is given by the number of falling maximal chains of length $i+2$
- hence, the Euler characteristic $\chi\big(\Delta(\overline{\mathbb{P}})\big)$ can be computed from the labeling
- if $0_{\mathbb{P}}$ is the unique minimal element and $1_{\mathbb{P}}$ the unique maximal element of $\mathbb{P}$, we have $\chi\big(\Delta(\overline{\mathbb{P}})\big) = \mu(0_{\mathbb{P}}, 1_{\mathbb{P}})$

Möbius Function and Falling Chains



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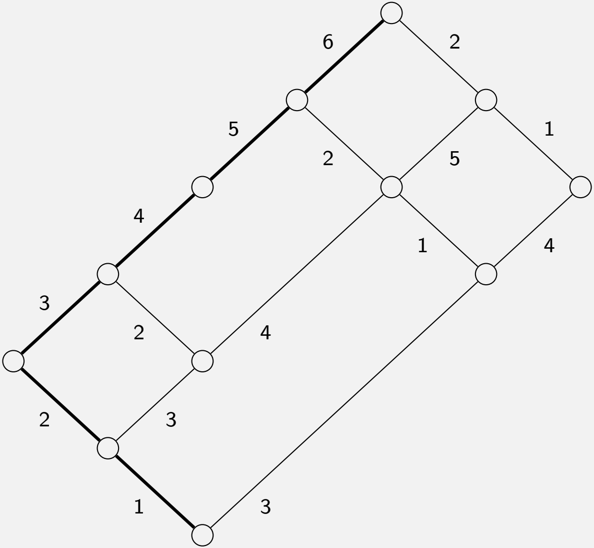

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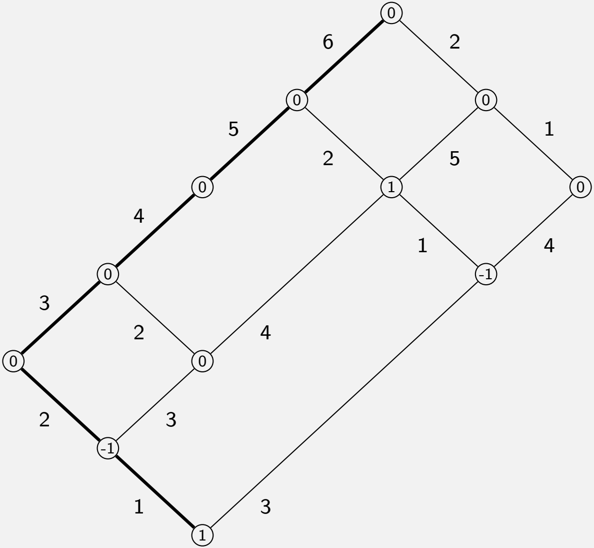

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An Edge-Labeling

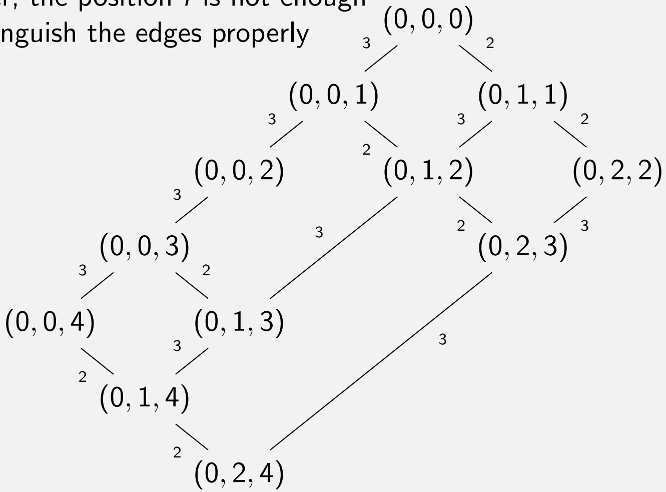
- an edge (p, p') in $\mathcal{T}_n^{(m)}$ is determined by the two sequences α_p and $\alpha_{p'}$, which satisfy

$$\alpha_p = \alpha_{p'} + (\underbrace{0, 0, \ldots, 0}_{i-1}, \underbrace{1, 1, \ldots, 1}_{k-i+1}, \underbrace{0, 0, \ldots, 0}_{n-k})$$

- the value k is uniquely determined by i
- given $\alpha_p = (a_1, a_2, \ldots, a_n)$ and i, we can uniquely determine $\alpha_{p'}$, and hence the covering pair (p, p')

An Edge-Labeling

- however, the position i is not enough to distinguish the edges properly



An Edge-Labeling

- to overcome this, we also take the value a_i into account and consider the edge-labeling

$$\begin{aligned}\lambda : \mathcal{E}\big(\mathcal{T}_n^{(m)}\big) &\to \mathbb{N}\times\mathbb{N}\\ (p,p') &\mapsto (i,a_i),\end{aligned}$$

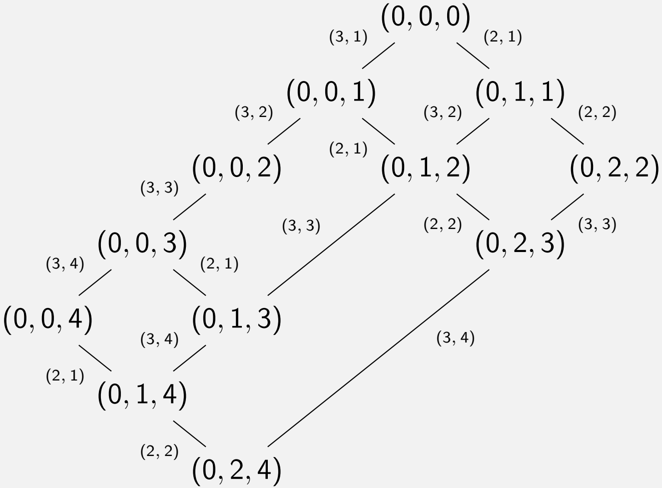
where $\alpha_p = (a_1, a_2, \ldots, a_n)$ and

$$\alpha_p = \alpha_{p'} + (\underbrace{0,0,\ldots,0}_{i-1},1,1,\ldots,1,0,0,\ldots,0)$$

- we consider the following linear order on the set of edge-labels

$$(i,a_i) < (j,a_j) \quad \text{if and only if} \quad i<j \quad \text{or} \quad i=j \text{ and } a_i > a_j$$

An Edge-Labeling



Constructing Rising Chains

- let $\alpha_p = (0,0,3,4,4,15)$ and $\alpha_q = (0,0,1,1,1,13)$

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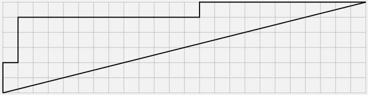

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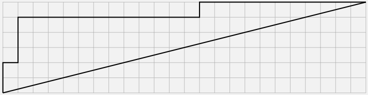

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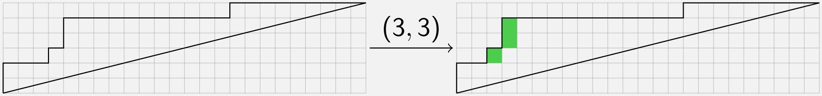
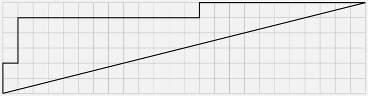


$(0,0,3,4,4,15) \xrightarrow{(3,3)} (0,0,2,3,3,15)$



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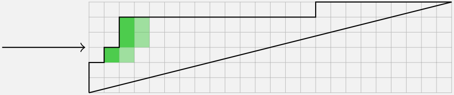

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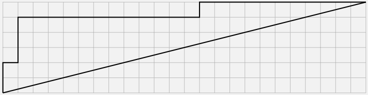

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$\xrightarrow{(3,2)}$

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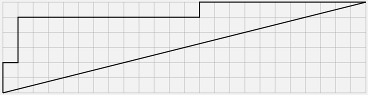

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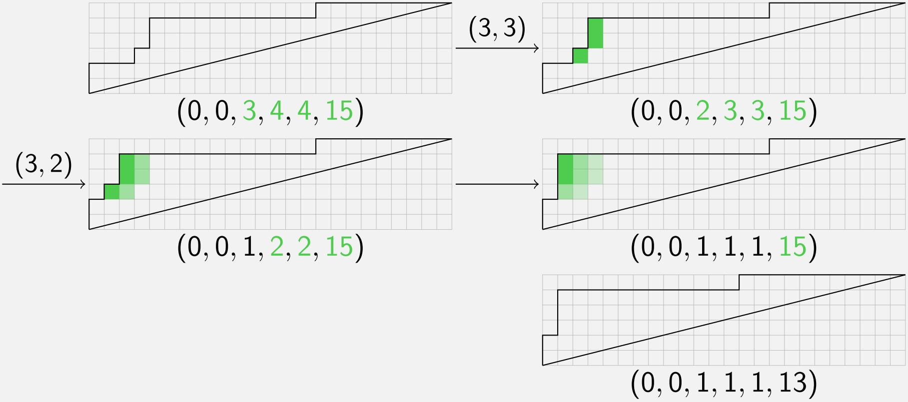

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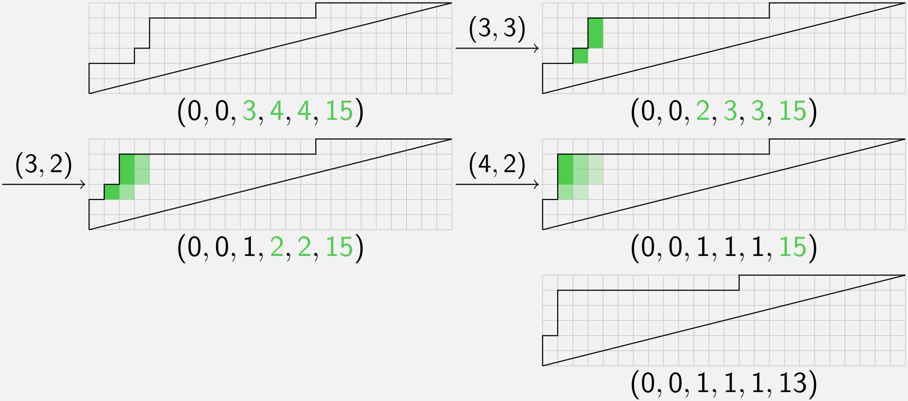

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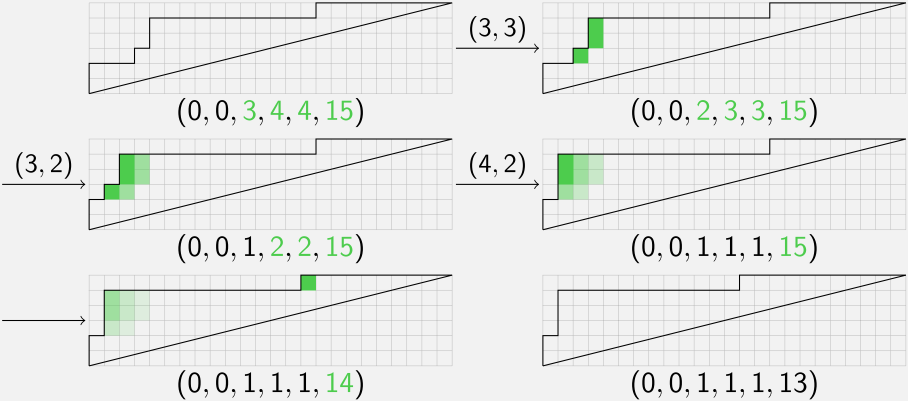

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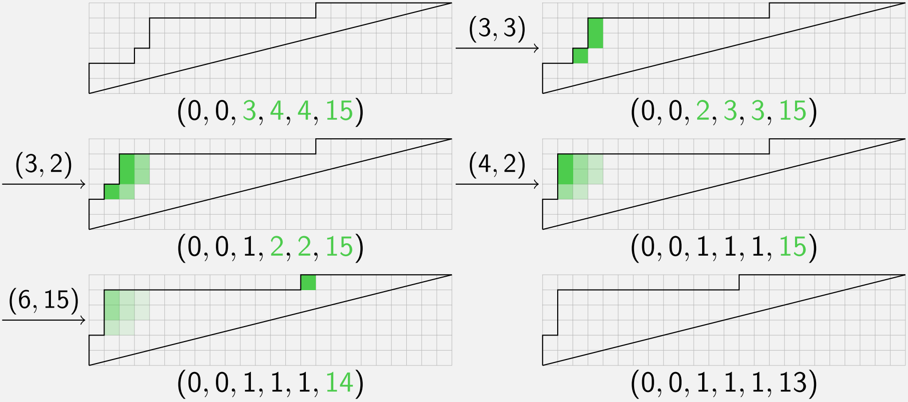

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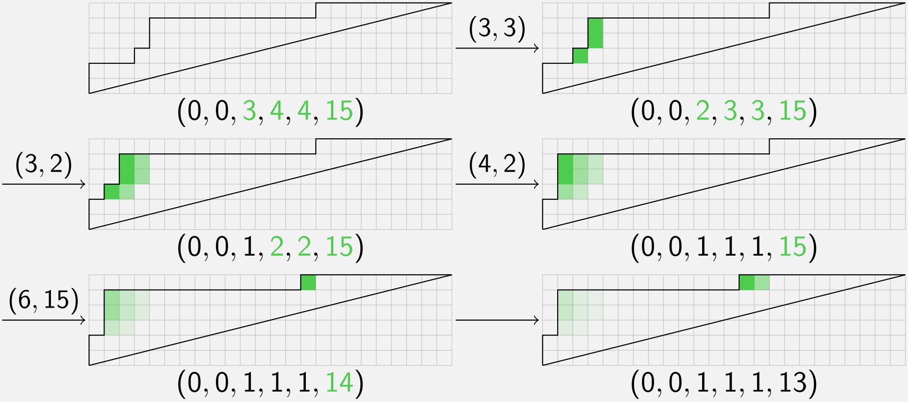

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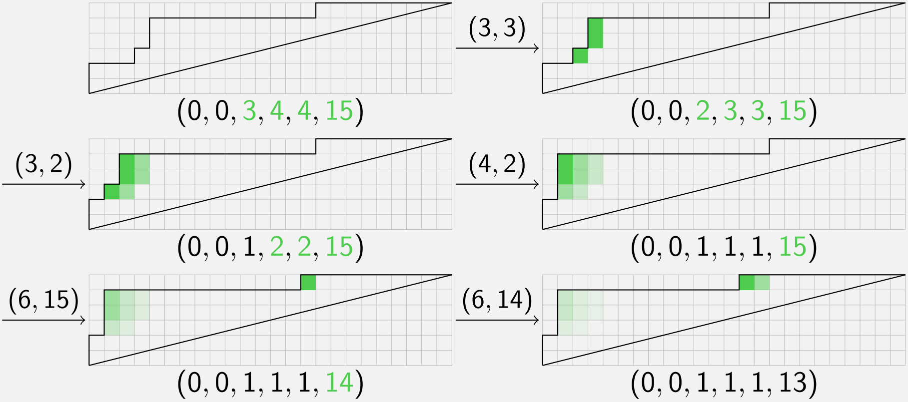

EL-Shellability of $\mathcal{T}_n^{(m)}$

Theorem

For every $m, n \in \mathbb{N}$, the edge-labeling λ is an EL-labeling for $\mathcal{T}_n^{(m)}$.

Outline

Topological Consequences

Proposition (Björner & Wachs, 1996)

Let $\mathbb{P}$ be a bounded poset and $[p, q]$ an interval in $\mathbb{P}$. If $\mathbb{P}$ is EL-shellable, then

$$\begin{aligned}\mu(p,q) &= \textit{number of even length falling maximal chains in } [p,q]\\ &\quad - \textit{number of odd length falling maximal chains in } [p,q]\end{aligned}$$

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Let $\mathbb{P}$ be an EL-shellable poset. Then, the order complex $\Delta(\overline{\mathbb{P}})$ of $\overline{\mathbb{P}}$ has the homotopy type of a wedge of spheres, and the dimension of the i-th homology group of $\Delta(\overline{\mathbb{P}})$ is given by the number of falling maximal chains of length $i+2$.

Falling Maximal Chains

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- let $\alpha_p = (a_1, a_2, \ldots, a_n), \alpha_q = (b_1, b_2, \ldots, b_n)$ and let $D = \{j \mid a_j \neq b_j \text{ and } a_j \geq a_{j-1} + m\} = \{j_1, j_2, \ldots, j_s\}$
- if $\alpha_{p^{(0)}} \lessdot \alpha_{p^{(1)}} \lessdot \cdots \lessdot \alpha_{p^{(s)}}$ is a falling maximal chain in $[p,q]$, it must have the label sequence

$$(j_s, a_{j_s}^{(0)}), (j_{s-1}, a_{j_{s-1}}^{(1)}), \ldots, (j_1, a_{j_1}^{(s-1)})$$

- this follows, since each of the values $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ must be decreased along a maximal chain in $[p,q]$ at least once

Conclusions

Corollary

Let $p \leq q$ in $\mathcal{T}_n^{(m)}$. Then, $\mu(p,q) \in \{-1,0,1\}$.

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Corollary

Each open interval in $\mathcal{T}_n^{(m)}$ has the homotopy type of either a sphere or a point.

Thank you!