Counting Proper Mergings

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Outline

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1. Motivation
2. Characterization
3. Enumeration
   - Proper Mergings of Two Chains
   - Proper Mergings of Two Antichains
   - Proper Mergings of an Antichain and a Chain

Motivation

- let $(P,\leq_P)$ be a poset
- consider the elements of P as tasks
- for $p,p'\in P$, consider $p<_P p'$ as saying that the execution of p has to be finished before the execution of p' can begin
- thus, $(P,\leq_P)$ can be seen as a schedule, or an execution plan, and $\leq_P$ can be seen as a set of restrictions

- let $(Q,\leq_Q)$ be another poset
- How many different schedules exist such that
  - $(P,\leq_P)$ and $(Q,\leq_Q)$ are executed "in parallel",
  - no restrictions of $(P,\leq_P)$ and $(Q,\leq_Q)$ are violated or added
  - no two tasks are executed at the same time?
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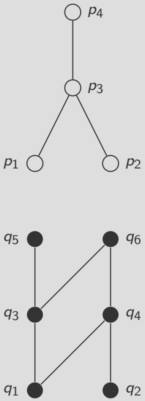
  - no restrictions of $(P, \leq_P)$ and $(Q, \leq_Q)$ are violated or added,
  - no two tasks are executed at the same time?
- we call such a schedule a proper merging of $(P, \leq_P)$ and $(Q, \leq_Q)$

Example



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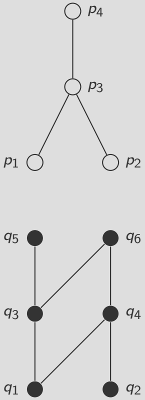

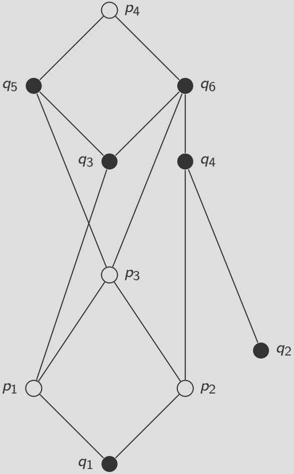

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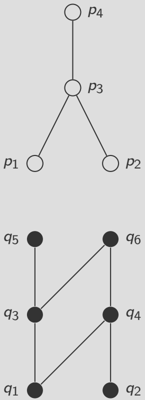

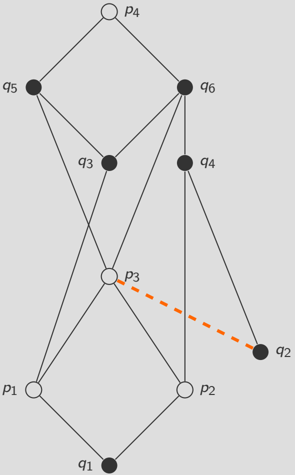

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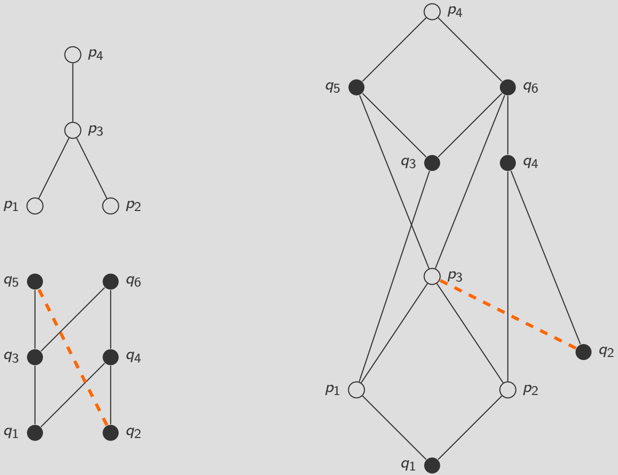

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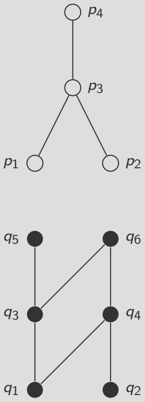

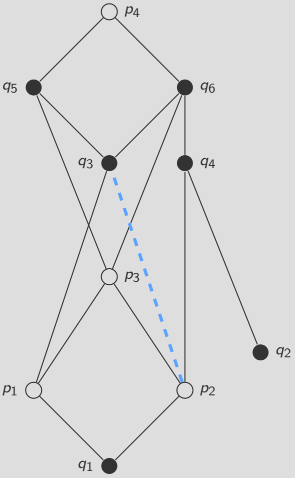

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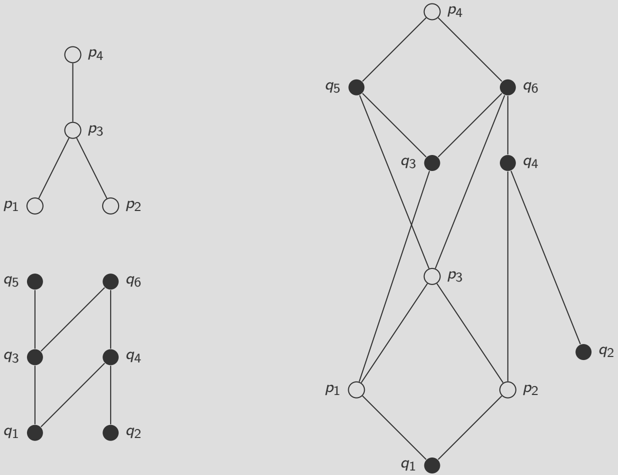

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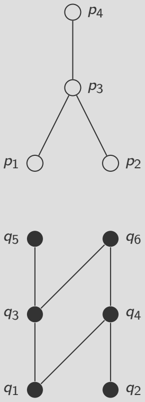

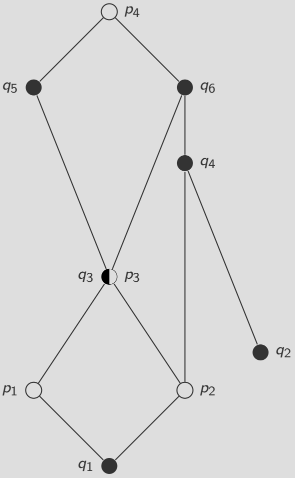

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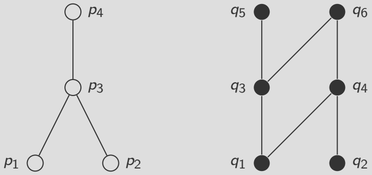
Characterization

- let G, G', M, M' be sets, and let $I \subseteq G \times M, I' \subseteq G' \times M'$ be two binary relations
- row of I: the set $g^I = \{m \in M \mid (g,m) \in I\}$ for $g \in G$
- column of I: the set $m^I = \{g \in G \mid (g,m) \in I\}$ for $m \in M$
- intent of I: an intersection over a subset of the rows of I
- extent of I: an intersection over a subset of the columns of I
- bond between I and I': a binary relation $R \subseteq G \times M'$ such that for all $g \in G$, the row g^R is an intent of I', and for all $m \in M'$, the column m^R is an extent of I

Example

	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	×		×	×						
p_2		×	×	×						
p_3			×	×						
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×



Example

	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	$\times$		$\times$	$\times$			$\times$		$\times$	$\times$
p_2		$\times$	$\times$	$\times$				$\times$		$\times$
p_3			$\times$	$\times$						$\times$
p_4				$\times$						
q_1					$\times$		$\times$	$\times$	$\times$	$\times$
q_2						$\times$		$\times$		$\times$
q_3							$\times$		$\times$	$\times$
q_4								$\times$		$\times$
q_5									$\times$	
q_6										$\times$

Example

	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	×		×	×			×		×	×
p_2		×	×	×				×		×
p_3			×	×			×			×
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×

Example

	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	×		×	×			×		×	×
p_2		×	×	×			×	×	×	×
p_3			×	×			×		×	×
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×

Characterization

- let $(P,\leq_P)$ and $(Q,\leq_Q)$ be disjoint posets, and let $R\subseteq P\times Q$, and $T\subseteq Q\times P$
- for $p,q\in P\cup Q$, define $p\leftarrow_{R,T} q$ if and only if

$$p\leq_P q \text{ or } p\leq_Q q \text{ or } (p,q)\in R \text{ or } (p,q)\in T$$

- merging of $(P,\leq_P)$ and $(Q,\leq_Q)$: a pair (R,T) such that $(P\cup Q,\leftarrow_{R,T})$ is a quasi-ordered set
- proper merging of $(P,\leq_P)$ and $(Q,\leq_Q)$: a merging (R,T) such that $R\cap T^{-1}=\emptyset$

Characterization

Proposition (Meschke, 2011)

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be disjoint posets, and let $R \subseteq P \times Q$ and $T \subseteq Q \times P$. The relation $\leftarrow_{R,T}$ is reflexive and transitive if and only if all of the following are satisfied:

1. *R is a bond between $\ngeq_P$ and $\ngeq_Q$,*
2. *T is a bond between $\ngeq_Q$ and $\ngeq_P$,*
3. *$R \circ T$ is contained in $\leq_P$,*
4. *$T \circ R$ is contained in $\leq_Q$.*

Moreover, $\leftarrow_{R,T}$ is antisymmetric if and only if $R \cap T^{-1} = \emptyset$.

- in other words, $(P \cup Q, \leftarrow_{R,T})$ is a poset if and only if (R, T) is a proper merging of $(P, \leq_P)$ and $(Q, \leq_Q)$

Characterization

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Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be disjoint posets, and let $R \subseteq P \times Q$ and $T \subseteq Q \times P$. The relation $\leftarrow_{R,T}$ is reflexive and transitive if and only if all of the following are satisfied:

1. *R is a bond between $\not\geq_P$ and $\not\geq_Q$,*
2. *T is a bond between $\not\geq_Q$ and $\not\geq_P$,*
3. *$R \circ T$ is contained in $\leq_P$,*
4. *$T \circ R$ is contained in $\leq_Q$.*

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- in other words, $(P \cup Q, \leftarrow_{R,T})$ is a poset if and only if (R, T) is a proper merging of $(P, \leq_P)$ and $(Q, \leq_Q)$

A Lattice Structure

- let $\mathfrak{M}_{P,Q}$ denote the set of mergings of $(P,\leq_P)$ and $(Q,\leq_Q)$
- define a partial order via

$$(R,T)\preceq(R',T')\quad\text{if and only if}\quad R\subseteq R'\text{ and }T\supseteq T',$$

A Lattice Structure

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Theorem (Meschke, 2011)

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be disjoint posets. The poset $(\mathfrak{M}_{P,Q}, \preceq)$ is in fact a distributive lattice, where the least element is $(\emptyset, P \times Q)$ and the greatest element is $(P \times Q, \emptyset)$.
Moreover, the poset $(\mathfrak{M}^{\bullet}_{P,Q}, \preceq)$ is a distributive sublattice of the previous.

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Enumeration

- Is it easy to determine the number of (proper) mergings of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$?
- the number of (proper) mergings depends heavily on the structure of $(P, \leq_P)$ and $(Q, \leq_Q)$

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		$\circ$	$\circ\ \circ$	$\circ\ \circ\ \circ$	$\circ\ \circ\ \circ\ \circ$	$\circ\ \circ\ \circ\ \circ\ \circ$
	1	18	230	2676	30386	344748
	1	15	155	1443	12899	113235

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	1	18	142	723	2782	8796
	1	15	105	409	1764	5292

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- the number of (proper) mergings depends heavily on the structure of $(P, \leq_P)$ and $(Q, \leq_Q)$
- we present the enumeration of three special cases:
  1. proper mergings of two chains
  2. proper mergings of two antichains
  3. proper mergings of an antichain and a chain

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Preparation

- let $C = \{c_1, c_2, \ldots, c_n\}$ be a set and define $c_i \leq_{\mathfrak{c}} c_j$ if and only if $i \leq j$
- we notice that $c_i \not\geq_{\mathfrak{c}} c_j$ if and only if $i < j$, or equivalently $c_i <_{\mathfrak{c}} c_j$ for all $i, j \in \{1, 2, \ldots, n\}$
- thus, the extents of $\not\geq_{\mathfrak{c}}$ are of the form $\{c_1, c_2, \ldots, c_k\}$ for some $k \in \{0, 1, \ldots, n\}$ and the intents are of the form $\{c_k, c_{k+1}, \ldots, c_n\}$ for some $k \in \{1, 2, \ldots, n+1\}$

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c_2		$\times$	$\times$	$\times$
c_3			$\times$	$\times$
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$\leq_{\mathfrak{c}}$	c_1	c_2	c_3	c_4
c_1	×	×	×	×
c_2		×	×	×
c_3			×	×
c_4				×

$\not\geq_{\mathfrak{c}}$	c_1	c_2	c_3	c_4
c_1		×	×	×
c_2			×	×
c_3				×
c_4				

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Preparation

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$\leq_{\mathfrak{c}}$	c_1	c_2	c_3	c_4
c_1	×	×	×	×
c_2		×	×	×
c_3			×	×
c_4				×

$\not\geq_{\mathfrak{c}}$	c_1	c_2	c_3	c_4
c_1		×	×	×
c_2			×	×
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c_4				

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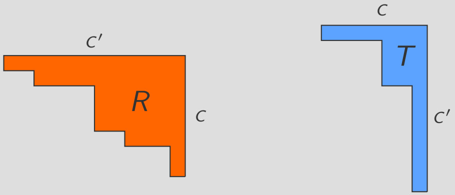
The Idea

- let $C=\{c_1,c_2,\ldots,c_m)$ and $C'=\{c'_1,c'_2,\ldots,c'_n\}$ be sets and let $\mathfrak{c}=(C,\leq_{\mathfrak{c}})$ and $\mathfrak{c}'=(C',\leq_{\mathfrak{c}'})$ be two chains

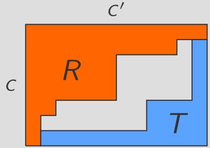
The Idea

- let $C = \{c_1, c_2, \ldots, c_m)$ and $C' = \{c'_1, c'_2, \ldots, c'_n\}$ be sets and let $\mathfrak{c} = (C, \leq_{\mathfrak{c}})$ and $\mathfrak{c}' = (C', \leq_{\mathfrak{c}'})$ be two chains
- if (R, T) is a merging of $\mathfrak{c}$ and $\mathfrak{c}'$, then R and T must be right-justified and top-justified



The Idea

- let $C = \{c_1, c_2, \ldots, c_m)$ and $C' = \{c'_1, c'_2, \ldots, c'_n\}$ be sets and let $\mathfrak{c} = (C, \leq_{\mathfrak{c}})$ and $\mathfrak{c}' = (C', \leq_{\mathfrak{c}'})$ be two chains
- if (R, T) is a proper merging of $\mathfrak{c}$ and $\mathfrak{c}'$, then R and T must "fit together"



The Bijection

- plane partition π: a rectangular array which is weakly decreasing along rows and columns
- part of π: an entry $\pi_{i,j}$ in the array

- given a proper merging (R,T) of $\mathfrak{c}$ and $\mathfrak{c}'$, define a plane partition π with m rows, n columns and largest part at most 2 as follows:

$$\pi_{i,j} = \begin{cases} 2, & \text{if } (c_i, c'_{n-j+1}) \in R \\ 0, & \text{if } (c'_{n-j+1}, c_i) \in T \\ 1, & \text{otherwise} \end{cases}$$

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- this is in fact a bijection!

The Enumeration

- the enumeration of plane partitions is classical

Theorem (MacMahon)

The number $\pi(m,n,l)$ of plane partitions with m rows, n columns and largest part at most l is given by

$$\pi(m,n,l) = \prod_{i=1}^{m}\prod_{j=1}^{n}\prod_{k=1}^{l}\frac{i+j+k-1}{i+j+k-2}.$$

The Enumeration

- in view of the bijection from before, we obtain the following result

Theorem

The number $F_{\mathfrak{c}}(m,n)$ of proper mergings of an m-chain and an n-chain is given by

$$F_{\mathfrak{c}}(m,n)=\pi(m,n,2)=\frac{1}{m+n+1}\binom{m+n+1}{m+1}\binom{m+n+1}{m}.$$

- $F_{\mathfrak{c}}(m,n)=\mathrm{Nar}(m+n+1,m+1)$

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Outline

1. Motivation
2. Characterization
3. Enumeration
   - Proper Mergings of Two Chains
   - Proper Mergings of Two Antichains
   - Proper Mergings of an Antichain and a Chain

Preparation

- let $A=\{a_1,a_2,\ldots,a_n\}$ be a set and define $a_i=_{\mathfrak{a}}a_j$ if and only if $i=j$

- thus, the extents and intents of $\neq_{\mathfrak{a}}$ are precisely the subsets of A

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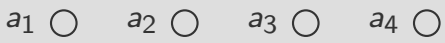

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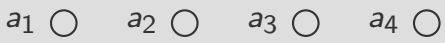

$=_{\mathfrak{a}}$	a_1	a_2	a_3	a_4
a_1	×			
a_2		×		
a_3			×	
a_4				×

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a_1	×			
a_2		×		
a_3			×	
a_4				×

$\neq_{\mathfrak{a}}$	a_1	a_2	a_3	a_4
a_1		×	×	×
a_2	×		×	×
a_3	×	×		×
a_4	×	×	×	

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a_1 ○ a_2 ○ a_3 ○ a_4 ○

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The Idea

- unfortunately, we have no idea of a bijection between proper mergings of two antichains and some other mathematical objects
- but, we have an idea for a generating function approach

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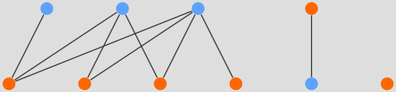

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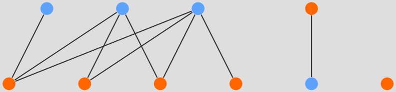

- every connected component of such a Hasse diagram can occur in two variations, unless this component is just a single node

The Generating Function

- let $B(x,y)$ denote the bivariate exponential generating function for bipartite graphs, and let $B_c(x,y)$ denote the bivariate exponential generating function for connected bipartite graphs
- we clearly have

$$B(x,y)=\sum_{n\geq 0}\sum_{m\geq 0}2^{mn}\frac{x^m y^n}{m!\,n!}$$

- since every bipartite graph can be considered as a collection of connected bipartite graphs, we obtain

$$B(x,y)=\exp\bigl(B_c(x,y)\bigr)$$

The Generating Function

- let $G(x,y)$ denote the bivariate exponential generating function for proper mergings of two antichains
- we obtain

$$\begin{aligned} G(x,y) &= \exp\big(2\cdot B_c(x,y) - x - y\big)\\ &= \exp\big(2\cdot \log B(x,y) - x - y\big)\\ &= B(x,y)^2 - \exp(x) - \exp(y)\\ &= \sum 2^{n_1n_2+m_1m_2}(-1)^{k_1}(-1)^{k_2}\\ &\qquad\cdot \frac{x^{n_1+m_1+k_1}}{n_1!\ m_1!\ k_1!}\cdot\frac{y^{n_2+m_2+k_2}}{n_2!\ m_2!\ k_2!} \end{aligned}$$

The Enumeration

- the number of proper mergings of an m-antichain and an n-antichain is given by the coefficient of $\frac{x^m y^n}{m!\,n!}$ in $G(x,y)$

Theorem

The number $F_{\mathfrak{a}}(m,n)$ of proper mergings of an m-antichain and an n-antichain is given by

$$F_{\mathfrak{a}}(m,n) = \sum_{k_1+m_1+n_1=m} \binom{m}{k_1, m_1, n_1} (-1)^{k_1} \left(2^{m_1} + 2^{n_1} - 1\right)^n.$$

Outline

The Idea

- let $A=\{a_1,a_2,\ldots,a_m\}$ and $C=\{c_1,c_2,\ldots,c_n\}$ be sets, and let $\mathfrak{a}=(A,=_{\mathfrak{a}})$ be an m-antichain and $\mathfrak{c}=(C,\leq_{\mathfrak{c}})$ be an n-chain
- recall that the intents and extents of $\neq_{\mathfrak{a}}$ are just subsets of A, and the extents of $\not\geq_{\mathfrak{c}}$ are of the form $\{c_1,c_2,\ldots,c_k\}$ and the intents are of the form $\{c_k,c_{k+1},\ldots,c_n\}$

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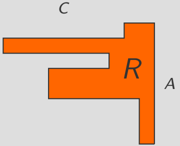

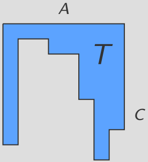

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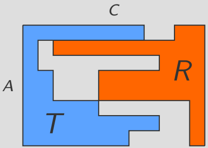

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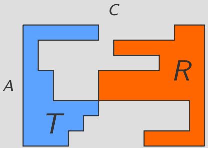

The Bijection

- complete bipartite digraph $\vec{K}_{m,n}$: a bipartite digraph with vertex set $V = V_1 \uplus V_2$, where $|V_1| = m$ and $|V_2| = n$, and edge set $\vec{E} = V_1 \times V_2$
- monotone coloring of a digraph: a map $\gamma : V \to \mathbb{N}$ with the property: if $(v_1, v_2) \in \vec{E}$, then $\gamma(v_1) \leq \gamma(v_2)$
- given a proper merging (R, T) of $\mathfrak{a}$ and $\mathfrak{c}$, define a monotone $(n+1)$-coloring γ of $\vec{K}_{m,m}$ as follows:

$$\gamma(v_i) = k \quad \text{if and only if} \quad \begin{cases} v_i \in V_1 & \text{and } (a_i, c_j) \in R \\ & \text{for all } n + 2 - k \leq j \leq n \\ v_i \in V_2 & \text{and } (c_j, a_i) \in T \\ & \text{for all } 1 \leq j \leq n + 1 - k \end{cases}$$

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The Enumeration

- the number of monotone n-colorings of $\vec{K}_{m_1,m_2}$ is known

Proposition (Jovović & Kilibarda, 2004)

Let $\eta_n(\vec{K}_{m_1,m_2})$ denote the number of monotone n-colorings of $\vec{K}_{m_1,m_2}$. Then,

$$\eta_n(\vec{K}_{m_1,m_2}) = \sum_{k=1}^{n}\Bigl((n+1-k)^{m_1} - (n-k)^{m_1}\Bigr)\cdot k^{m_2}$$
$$= \sum_{k=1}^{n}\Bigl((n+1-k)^{m_2} - (n-k)^{m_2}\Bigr)\cdot k^{m_1}.$$

The Enumeration

- in view of the bijection from before, we obtain the following result

Theorem

The number $F_{\mathfrak{a}}(m,n)$ of proper mergings of an m-antichain and an n-chain is given by

$$\begin{aligned} F_{\mathfrak{a}}(m,n) &= \eta_{n+1}(\vec{K}_{m,m}) \\ &= \sum_{k=1}^{n+1}\Big((n+2-k)^m-(n+1-k)^m\Big)\cdot k^m. \end{aligned}$$

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Thank You.