Tamari Lattices for Parabolic Quotients of the Symmetric Group

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Catalan Objects

- **Catalan numbers**: $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$
- **Catalan objects**:
  - 231-avoiding permutations of $[n]$
  - triangulations of a $(n+2)$-gon
  - noncrossing set partitions of $[n]$
  - nonnesting set partitions of $[n]$
  - ...
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The Symmetric Group $\mathfrak{S}_n$

- **symmetric group** $\mathfrak{S}_n$: group of permutations of $[n]$
- generators: $s_i = (i\; i{+}1), i \in [n-1]$
- $S = \{s_1, s_2, \ldots, s_{n-1}\}$
- **inversion set**: $\mathrm{inv}(w) = \bigl\{(i,j) \mid i < j, w_i > w_j\bigr\}$

Parabolic Quotients of $\mathfrak{S}_n$

- **(standard) parabolic subgroup**:
  subgroup $(\mathfrak{S}_n)_J$ generated by $J \subseteq S$
- **(standard) parabolic quotient**:
  $\mathfrak{S}_n^J = \{w \in \mathfrak{S}_n \mid \mathrm{inv}(w) \subsetneq \mathrm{inv}(ws) \text{ for all } s \in J\}$

Parabolic Quotients of $\mathfrak{S}_n$

- $J = S \setminus \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$
- one-line notation for $w \in \mathfrak{S}_n^J$:
  $w_1 < \cdots < w_{i_1} | w_{i_1+1} < \cdots < w_{i_2} | \cdots | w_{i_k+1} < \cdots < w_n$

Outline

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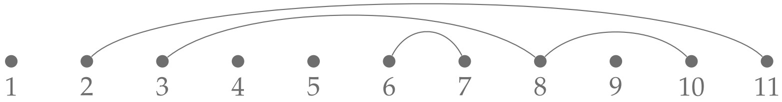

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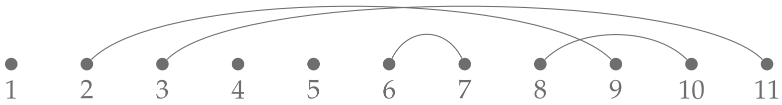

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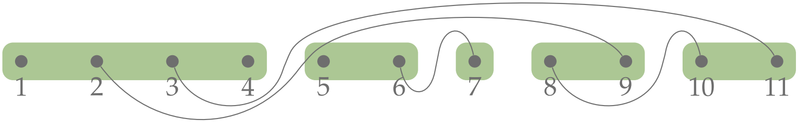

A Bijection

Theorem (& Williams, 2015)

For $n > 0$ and $J \subseteq S$, we have $\bigl\lvert NC_n^J \bigr\rvert = \bigl\lvert \mathfrak{S}_n^J(231) \bigr\rvert$.

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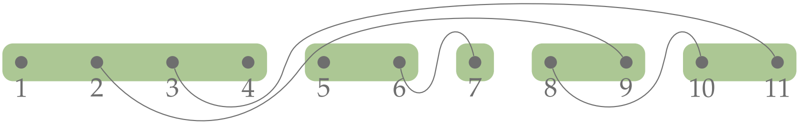

$$w \;=\; ?\;\;?\;\;?\;\;?\;\Big|\;?\;\;?\;\Big|\;?\;\Big|\;?\;\;?\;\Big|\;?\;\;?$$

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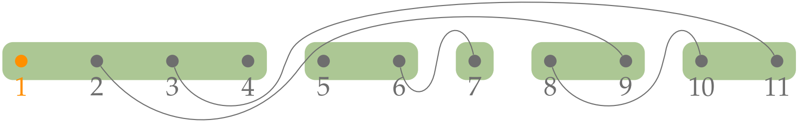

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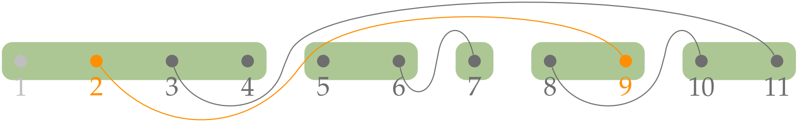

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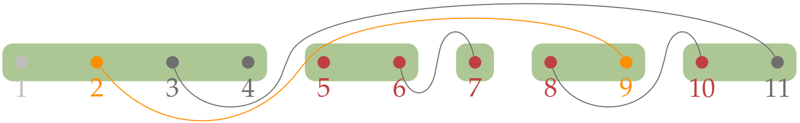

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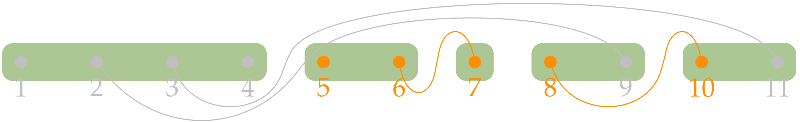

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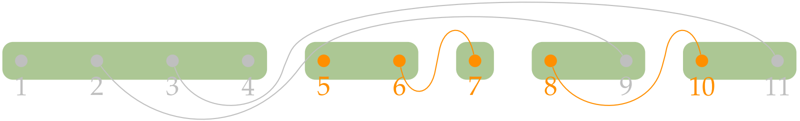

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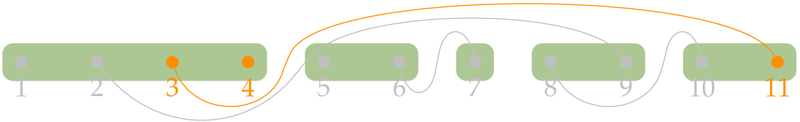

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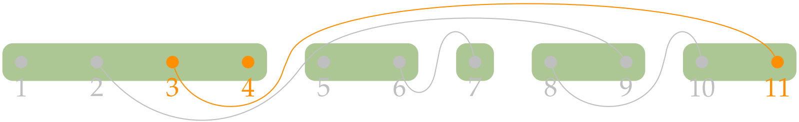

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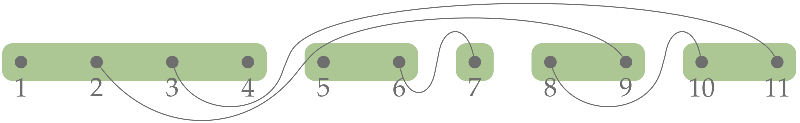

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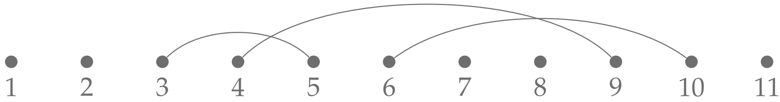

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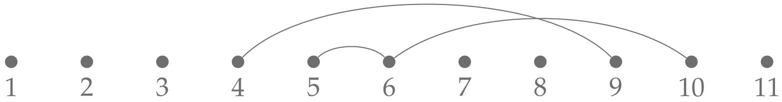

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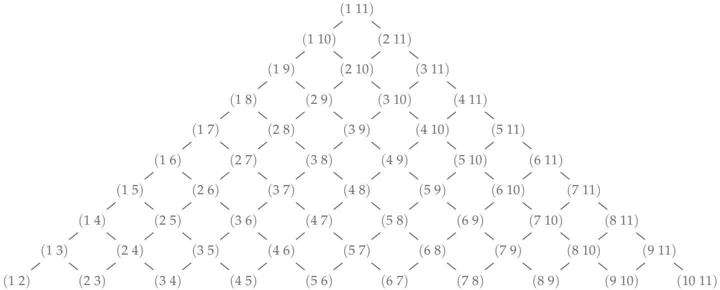

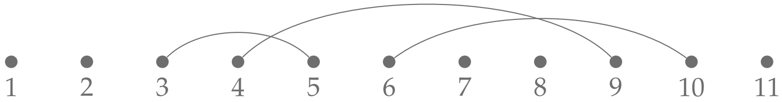

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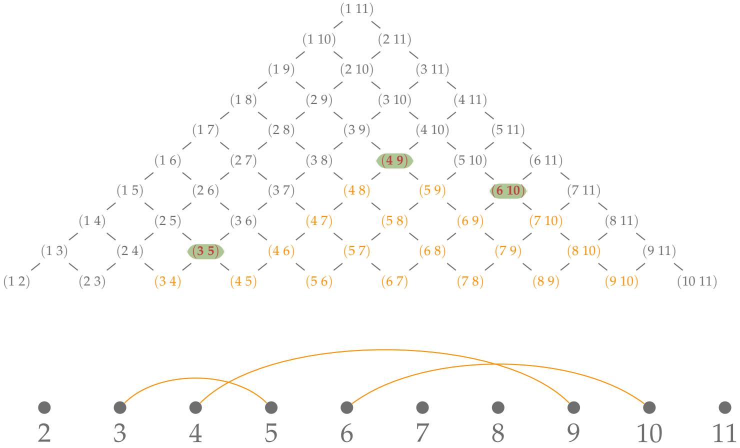

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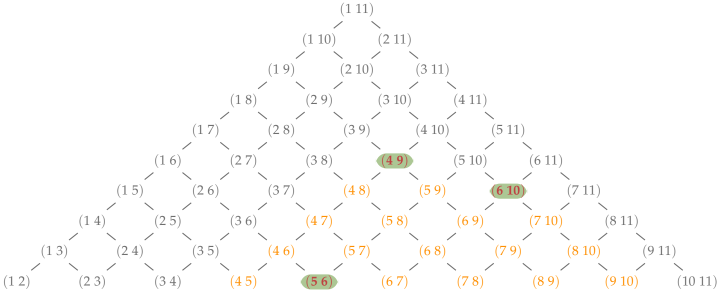

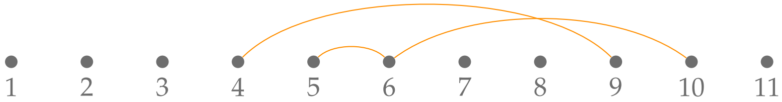

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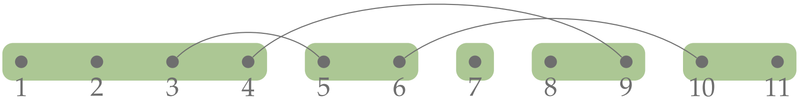

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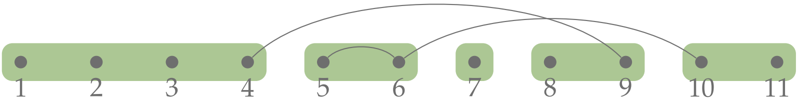

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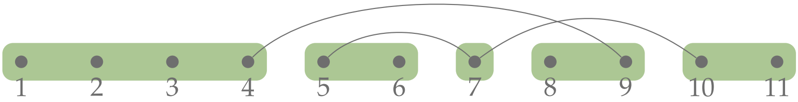

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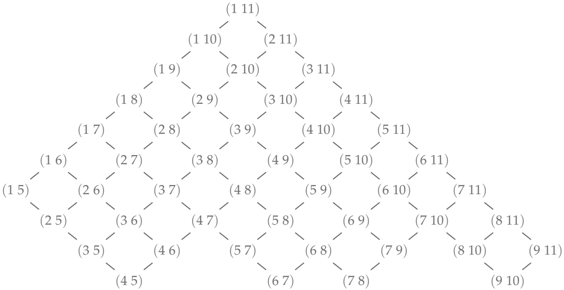

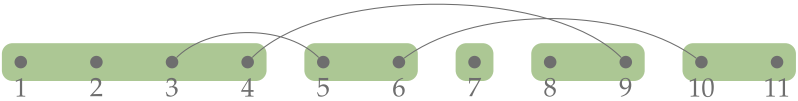

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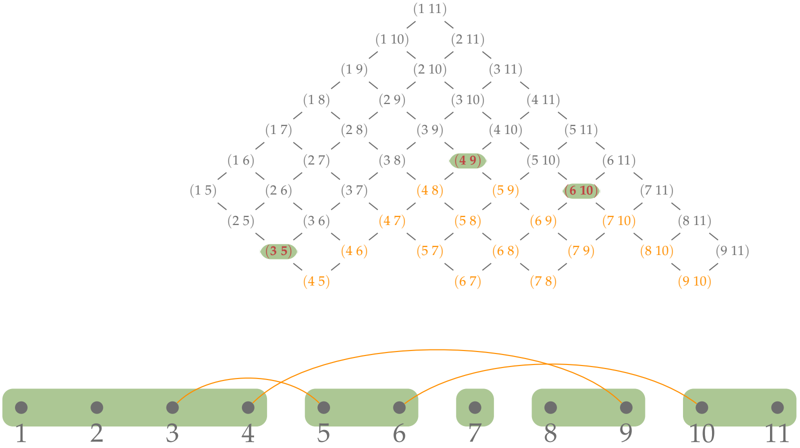

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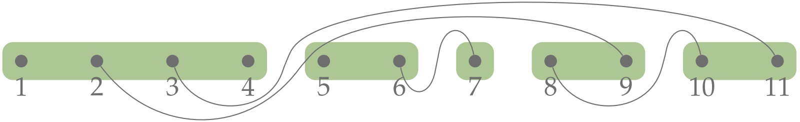

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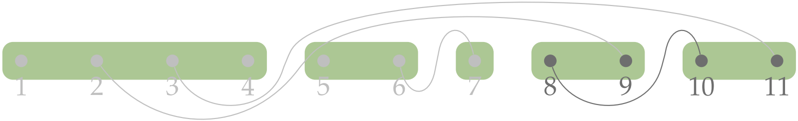

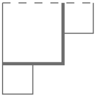

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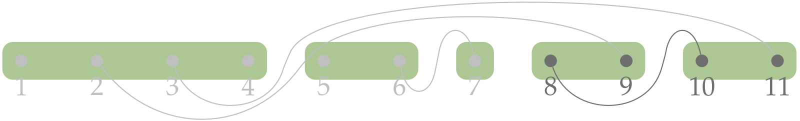

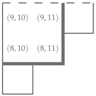

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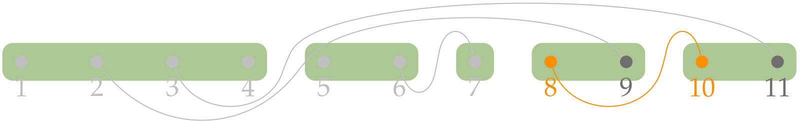

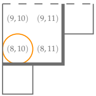

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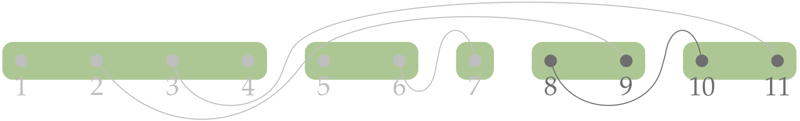

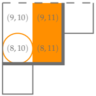

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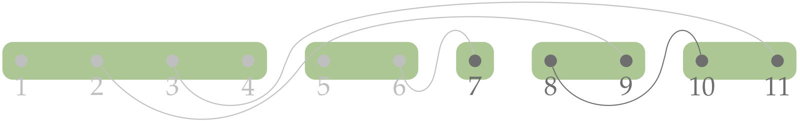

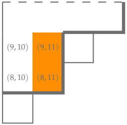

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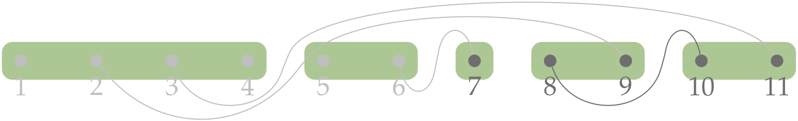

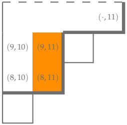

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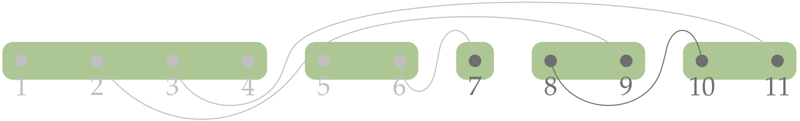

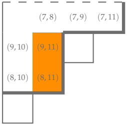

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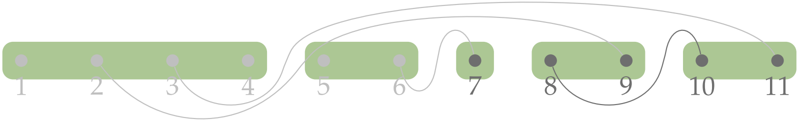

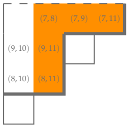

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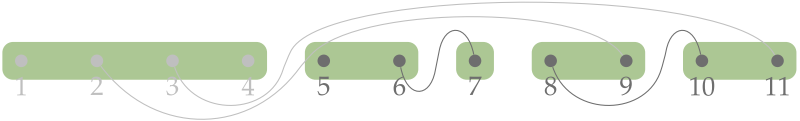

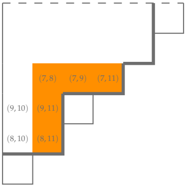

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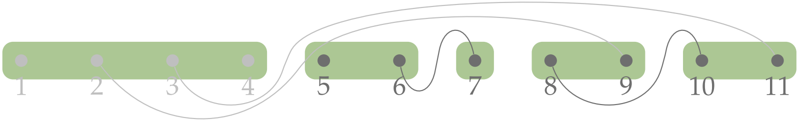

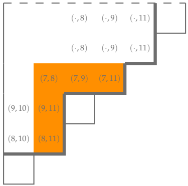

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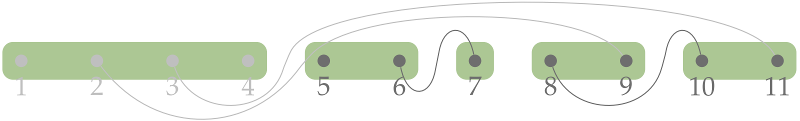

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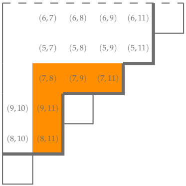

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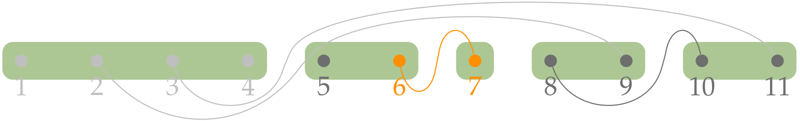
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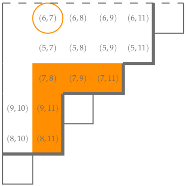
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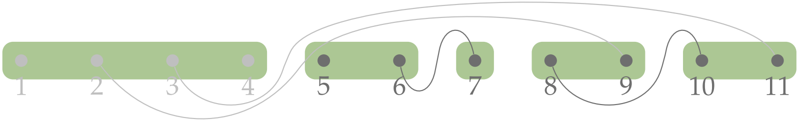
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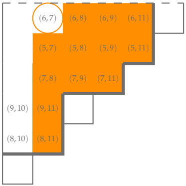
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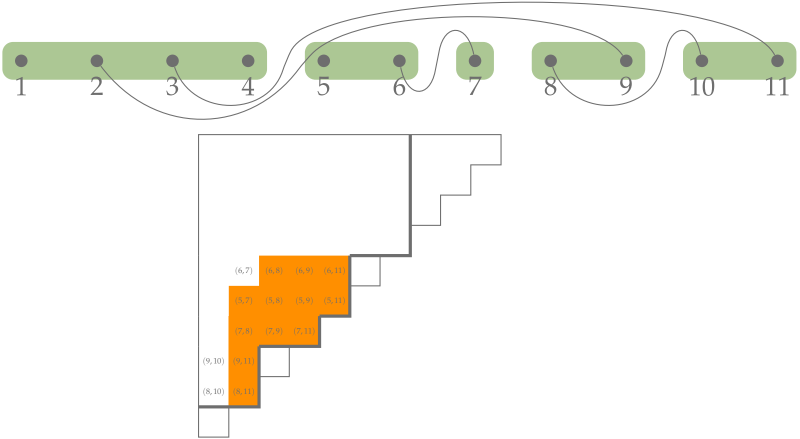

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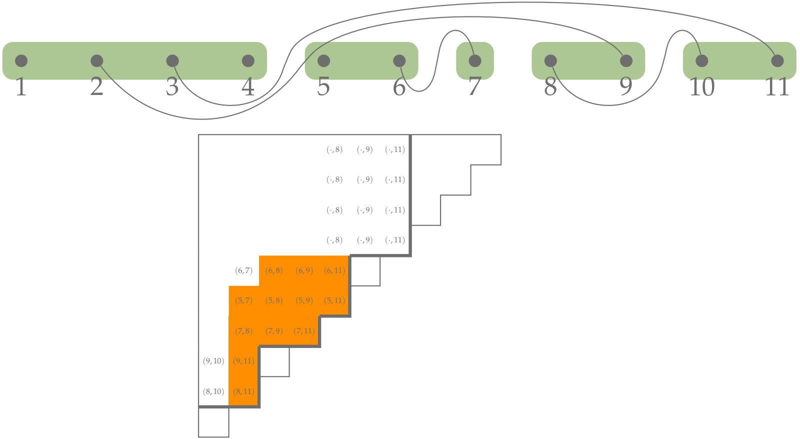
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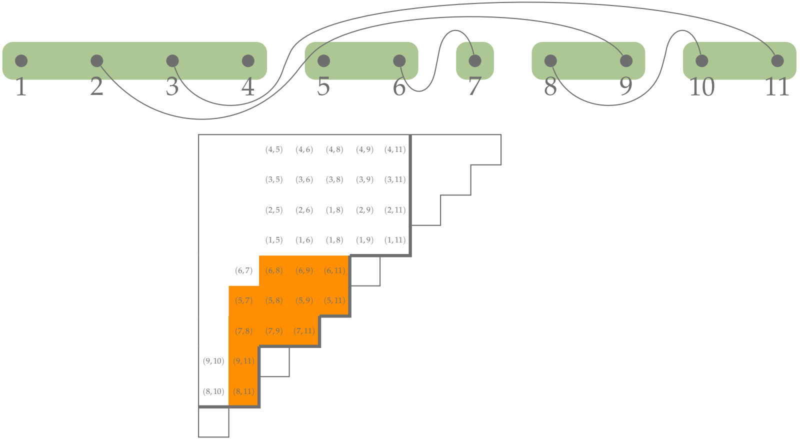
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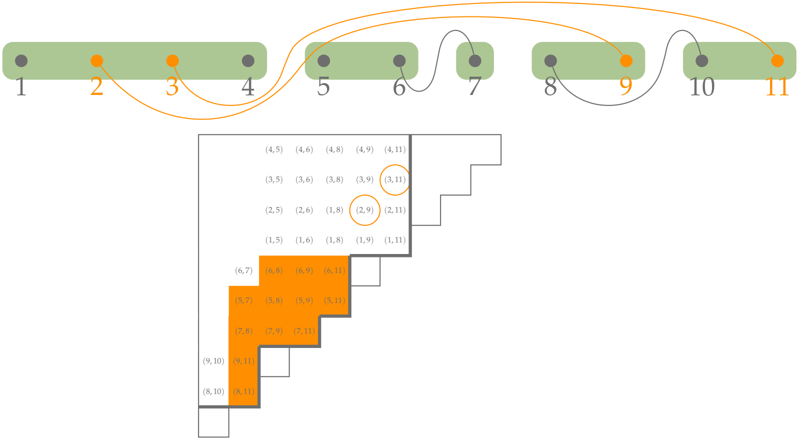
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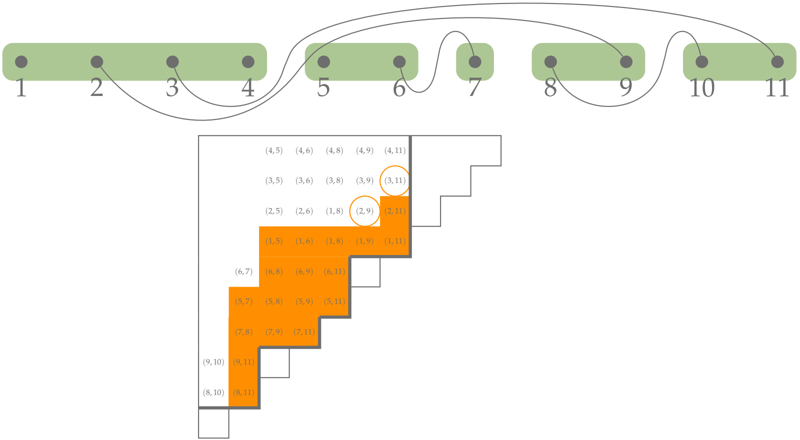

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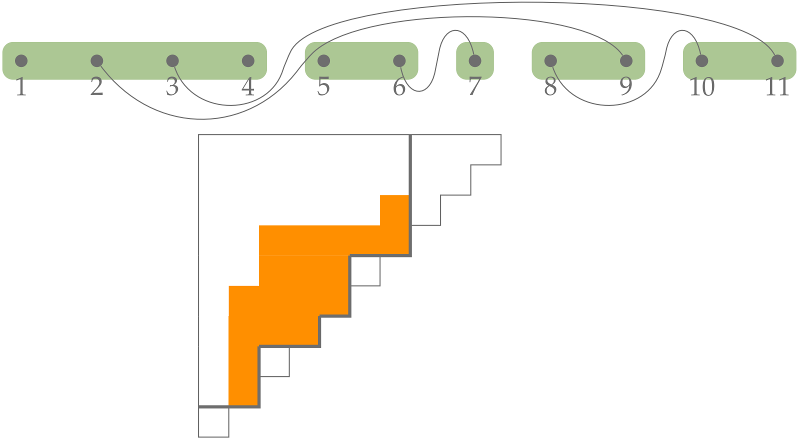
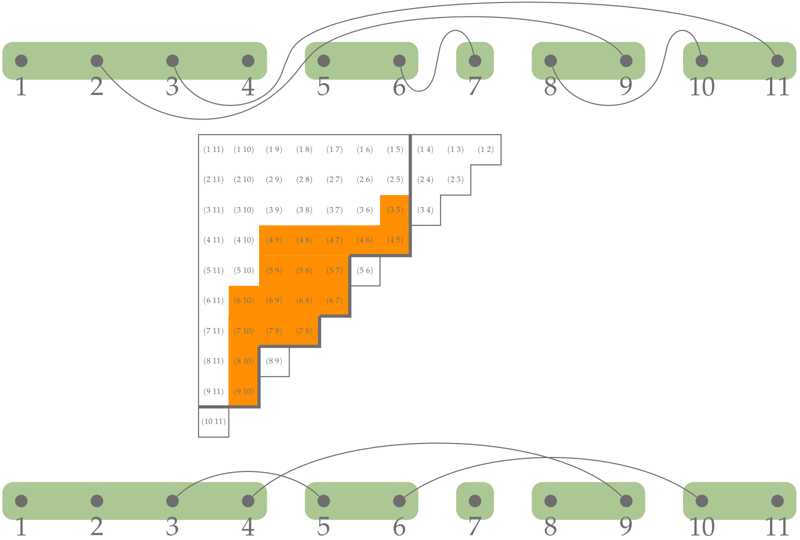


Example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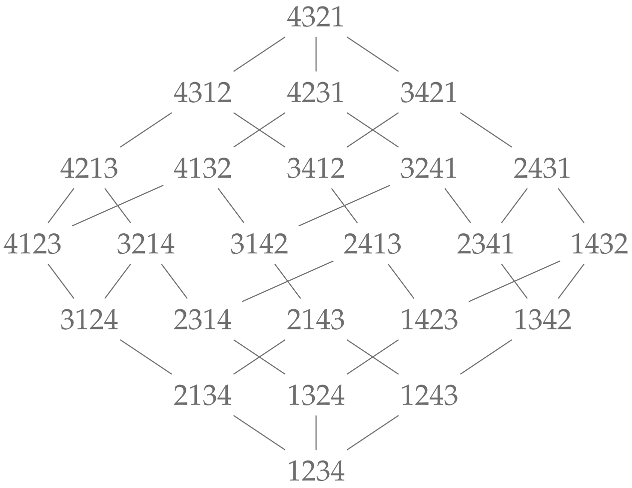
Outline

Weak Order

- **inversion set**: $\operatorname{inv}(w) = \bigl\{(i,j) \mid i < j, w_i > w_j\bigr\}$
- **weak order**: $u \leq_S v$ if and only if $\operatorname{inv}(u) \subseteq \operatorname{inv}(v)$

  $\rightsquigarrow \operatorname{Weak}(\mathfrak{S}_n)$
- **longest element**: $w_o = n \cdots 21$

Example: Weak$(\mathfrak{S}_4)$



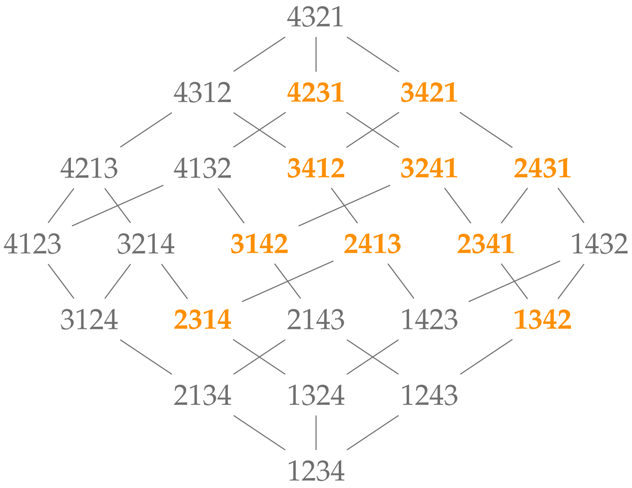
The Tamari Lattices

Theorem (Björner & Wachs, 1997)

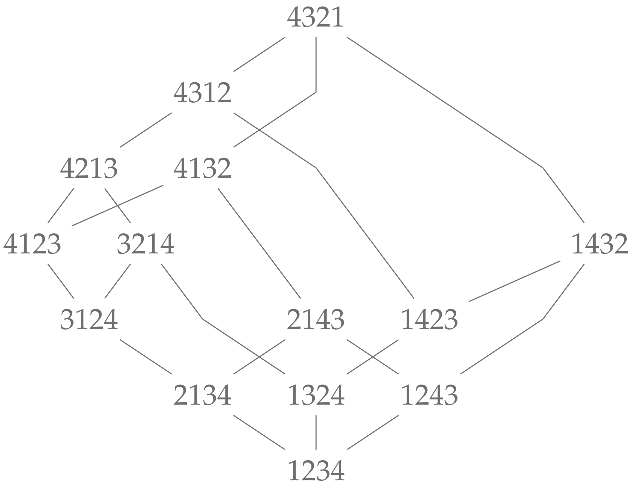
For $n > 0$ the Tamari lattice $\mathcal{T}_n$ is isomorphic to the weak order on the 231-avoiding permutations of $\mathfrak{S}_n$, i.e. $\mathcal{T}_n \cong \mathit{Weak}\big(\mathfrak{S}_n(231)\big)$.

- $\mathcal{T}_n$ is a sublattice and a quotient lattice of $\mathrm{Weak}\big(\mathfrak{S}_n\big)$

Example: Weak$(\mathfrak{S}_4)$



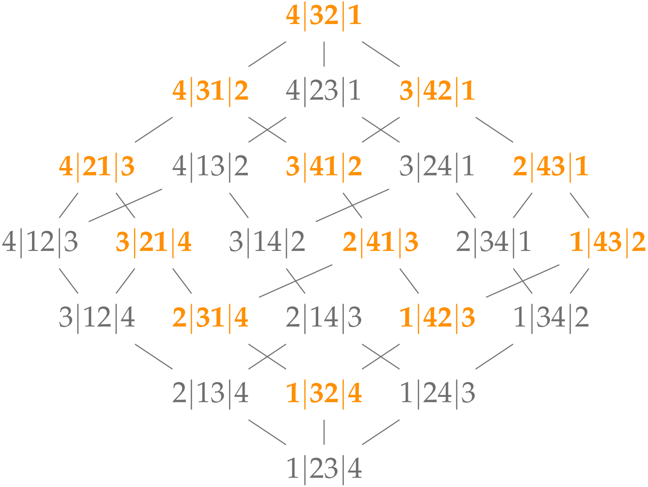
Example: $\mathcal{T}_4$



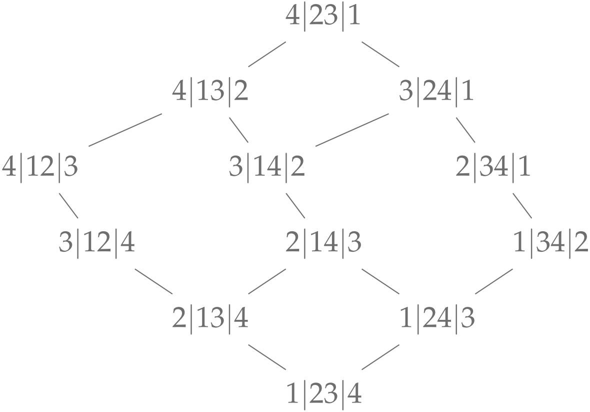
Parabolic Weak Order

- **parabolic weak order**: restrict $\mathrm{Weak}\big(\mathfrak{S}_n\big)$ to $\mathfrak{S}_n^J$ $\rightsquigarrow \mathrm{Weak}\big(\mathfrak{S}_n^J\big)$
- $\mathrm{Weak}\big(\mathfrak{S}_n^J\big) \cong \mathrm{Weak}\big(e, w_o^J\big)$

Example: Weak$(\mathfrak{S}_4)$



Example: Weak$\left(\mathfrak{S}_4^{\{s_2\}}\right)$



Parabolic Tamari Lattices

Theorem (& Williams, 2015)

*For $n > 0$ and $J \subseteq S$, the poset $\mathrm{Weak}\bigl(\mathfrak{S}_n^J(231)\bigr)$ is a lattice, the **parabolic Tamari lattice** $\mathcal{T}_n^J$.*

- for any $w \in \mathfrak{S}_n^J$ there is a unique maximal $w' \in \mathfrak{S}_n^J(231)$ with $w' \leq_S w$

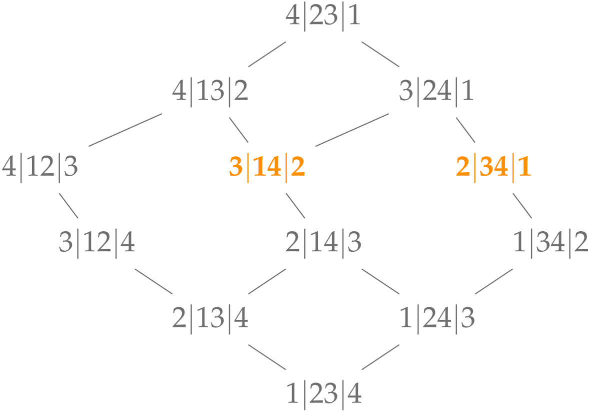
Parabolic Tamari Lattices

Theorem (& Williams, 2015)

For $n > 0$ and $J \subseteq S$, the poset $\mathrm{Weak}\big(\mathfrak{S}_n^J(231)\big)$ is a lattice, the ***parabolic Tamari lattice*** *$\mathcal{T}_n^J$. It is a quotient lattice, but not a sublattice of $\mathrm{Weak}\big(\mathfrak{S}_n^J\big)$.*

- for any $w \in \mathfrak{S}_n^J$ there is a unique maximal $w' \in \mathfrak{S}_n^J(231)$ with $w' \leq_S w$

Example: Weak$\left(\mathfrak{S}_4^{\{s_2\}}\right)$



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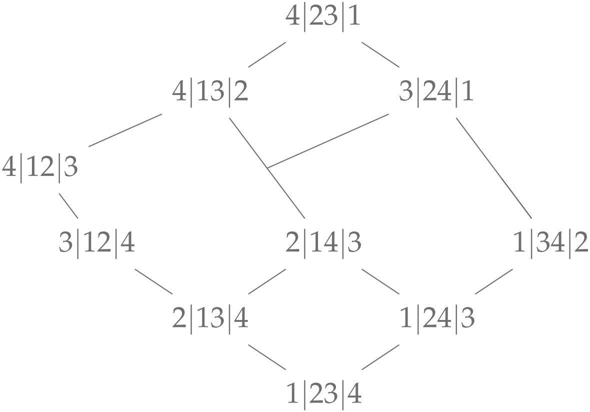

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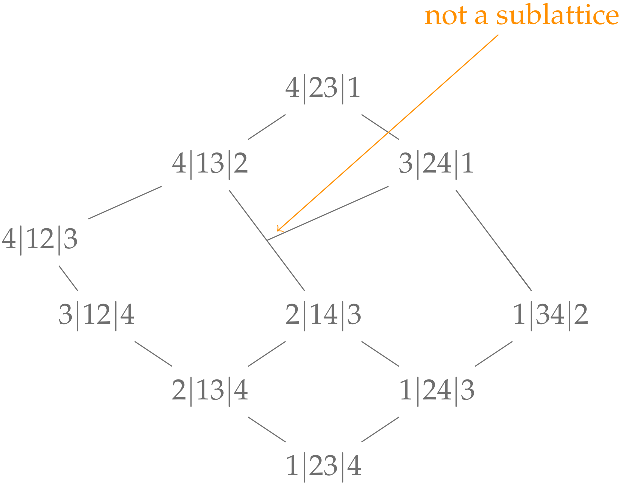

Connections

- recent work by Préville-Ratelle and Viennot relates $\mathcal{T}_n^J$ to intervals in $\mathcal{T}_{2n+2}$
  - by relating the shape of the parabolic root poset to the "canopy" of binary trees

Outlook

- more Catalan objects:
  - subword complexes $\rightsquigarrow$ sortable elements
- generalize to all Coxeter groups

Thank You.

Another Bijection

- Reading recently gave an explicit bijection between noncrossing diagrams and permutations

$$w = 1\ 8\ 10\ 11\ 2\ 4\ 3\ 6\ 7\ 5\ 9$$

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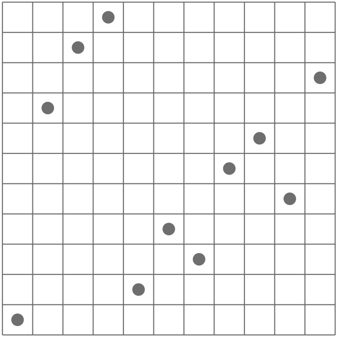

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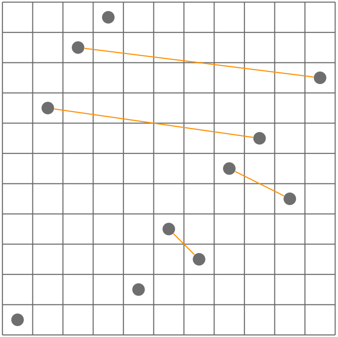

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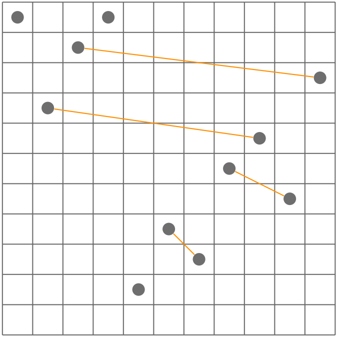

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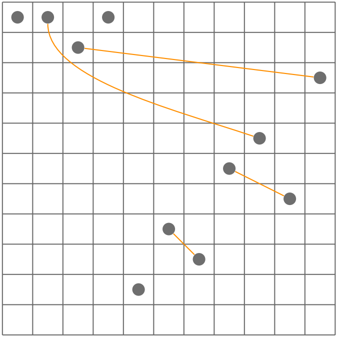

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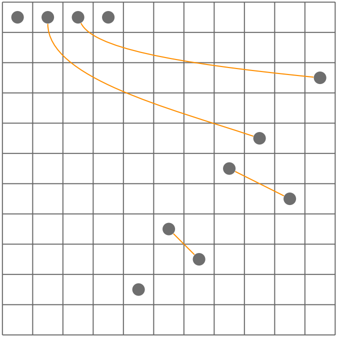

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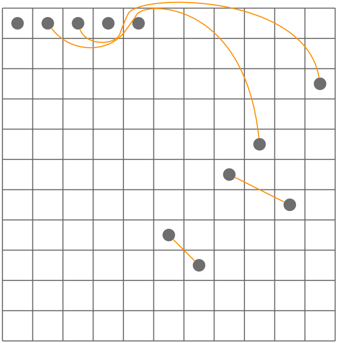

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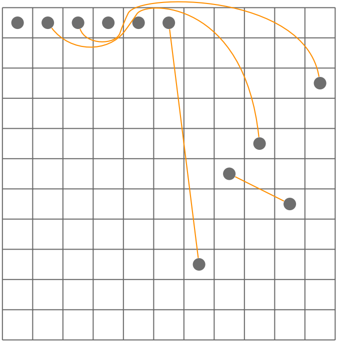

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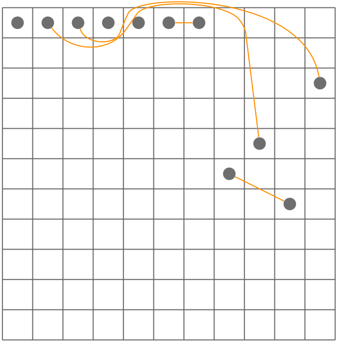

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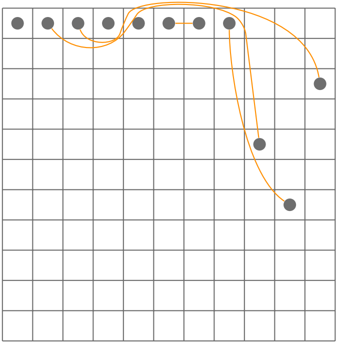

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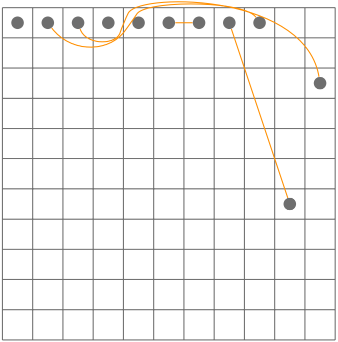

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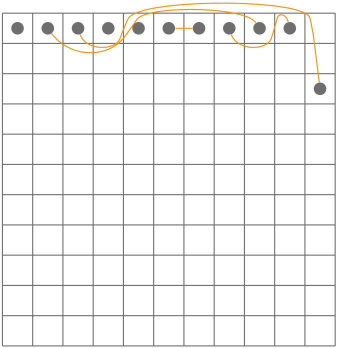

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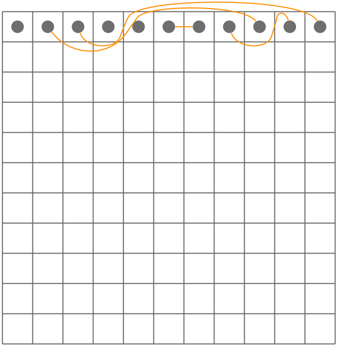

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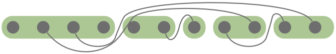

Subword Complexes

- **subword complex** $\rightsquigarrow \mathcal{S}(Q,w)$

- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4321$
-

1	2	3	4	5	6	7	8	9
s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1

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- $(1,2,3)$

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**
s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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- $(1,2,3)$, $(2,3,4)$

1	2	3	4	**5**	**6**	**7**	**8**	**9**
$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	3	4	5	**6**	**7**	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	4	5	6	**7**	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	**4**	5	6	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	**4**	**5**	6	7	8	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_1}$

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1	**2**	**3**	**4**	**5**	6	**7**	8	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	s_1

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- $(1,2,3), (2,3,4), (3,4,5), (4,5,6), (5,6,7), (6,7,8), (6,8,9), (4,6,9)$

1	**2**	**3**	4	**5**	6	**7**	**8**	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

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- $(1,2,3)$, $(2,3,4)$, $(3,4,5)$, $(4,5,6)$, $(5,6,7)$, $(6,7,8)$, $(6,8,9)$, $(4,6,9)$, $(2,4,9)$

1	2	**3**	4	**5**	**6**	**7**	**8**	9
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

Subword Complexes

- **subword complex** $\rightsquigarrow \mathcal{S}(Q,w)$

- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4321$
- $(1,2,3)$, $(2,3,4)$, $(3,4,5)$, $(4,5,6)$, $(5,6,7)$, $(6,7,8)$, $(6,8,9)$, $(4,6,9)$, $(2,4,9)$, $(1,2,9)$

1	2	**3**	**4**	**5**	**6**	**7**	**8**	9
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

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1	**2**	**3**	**4**	**5**	**6**	**7**	8	9
s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

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1	**2**	**3**	**4**	**5**	**6**	7	8	**9**
s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

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1	**2**	3	**4**	**5**	**6**	7	**8**	**9**
s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	3	**4**	5	**6**	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o, w = w_o^J$, where $c = s_1 s_2 \cdots s_{n-1}$ and $w_o = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_1 \cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1, w = 4|23|1$
-

1	2	3	4	5	6	7	8	9
s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1

Subword Complexes

- **parabolic subword complex**: $Q = cw_o, w = w_o^J$, where $c = s_1 s_2 \cdots s_{n-1}$ and $w_o = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_1 \cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1, w = 4|23|1$
- $(1,2,3,7)$

1	2	3	4	**5**	**6**	7	**8**	**9**
s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1, w = 4|23|1$
- $(1,2,3,7), (2,3,4,7)$

1	2	3	4	**5**	**6**	7	**8**	**9**
$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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- $(1,2,3,7), (2,3,4,7), (3,4,5,7)$

1	**2**	3	4	5	**6**	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

Subword Complexes

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1	**2**	**3**	4	5	6	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	4	**5**	6	7	8	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_1}$

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- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$

1	**2**	**3**	4	**5**	6	**7**	8	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	s_1

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1	2	**3**	4	**5**	**6**	**7**	8	9
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

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- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$

1	2	**3**	4	**5**	**6**	7	8	**9**
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

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- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1, w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$, $(1,2,7,8)$

1	2	**3**	4	**5**	**6**	7	8	**9**
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

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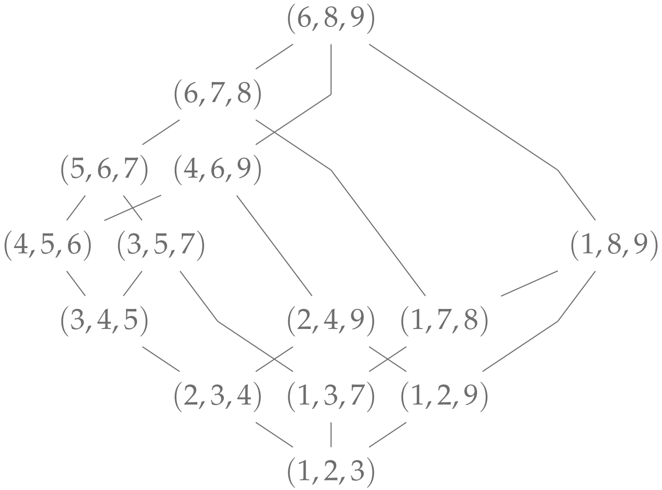
$\rightsquigarrow \mathcal{S}_n^J$

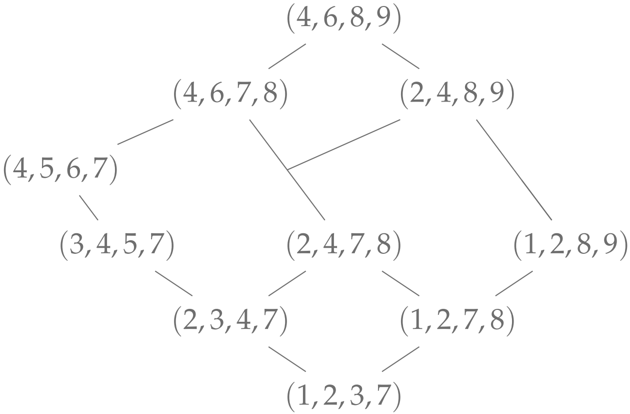
- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1, w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$, $(1,2,7,8)$, $(1,2,8,9)$

1	2	**3**	4	**5**	**6**	**7**	8	9
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

Example: $(\mathcal{S}_4^{\emptyset}, \leq_{\text{flip}})$



Example: $(\mathcal{S}_4^{\{s_2\}}, \leq_{\text{flip}})$



A Bijection

Theorem (Serrano & Stump, 2011; Williams, 2013)

For $n > 0$ and $J \subseteq S$, we have $\left|\mathcal{S}_n^J\right| = \left|NN_n^J\right|$.

- Edelman-Greene insertion on positions of subword
- slight modification of the recording tableau

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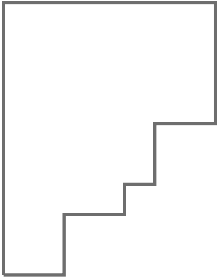
- slight modification of the recording tableau

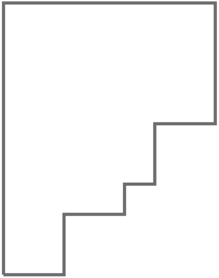
Example

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1

Example

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_7$	$\mathbf{s}_8$	$\mathbf{s}_9$	$\mathbf{s}_{10}$	s_1	s_2	$\mathbf{s}_3$
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_7$	$\mathbf{s}_8$	$\mathbf{s}_9$	s_{10}	s_1	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
$\mathbf{s}_7$	s_8	s_9	s_1	s_2	$\mathbf{s}_3$	s_4	s_5	$\mathbf{s}_6$	s_7	$\mathbf{s}_8$	$\mathbf{s}_1$	$\mathbf{s}_2$
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_7$	s_1	s_2	$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_1$	$\mathbf{s}_2$
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_1$	s_2	$\mathbf{s}_3$	s_1	$\mathbf{s}_2$	s_1





Example

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

4	5	6	7	8	9	10
13	14	15	16	17	18	19
22	23	24	25	26	27	37
32	35	40	41	42	43	44
38	39	48	49	50		
47	52	53	54	55		
51	57	58	59			
56	62					
60	64					

Example

1	1	1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	2	2	**2**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
2	**2**	**2**	**2**	**2**	**2**	2	3	**3**	**3**	**3**	**3**	**3**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
3	3	3	4	4	**4**	4	4	**4**	4	**4**	**5**	**5**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
5	**5**	**5**	**5**	**5**	6	6	**6**	**6**	**6**	**6**	**7**	**7**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
7	**7**	**7**	**8**	**8**	**8**	**8**	**9**	9	**9**	10	**10**	11
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	5	5	5	5	5
5	5	6	6	6		
6	7	7	7	7		
7	8	8	8			
8	9					
9	10					

Example

1 s_1	1 s_2	1 s_3	**1** $\mathbf{s_4}$	**1** $\mathbf{s_5}$	**1** $\mathbf{s_6}$	**1** $\mathbf{s_7}$	**1** $\mathbf{s_8}$	**1** $\mathbf{s_9}$	**1** $\mathbf{s_{10}}$	2 s_1	2 s_2	**2** $\mathbf{s_3}$
2 $\mathbf{s_4}$	**2** $\mathbf{s_5}$	**2** $\mathbf{s_6}$	**2** $\mathbf{s_7}$	**2** $\mathbf{s_8}$	**2** $\mathbf{s_9}$	2 s_{10}	3 s_1	**3** $\mathbf{s_2}$	**3** $\mathbf{s_3}$	**3** $\mathbf{s_4}$	**3** $\mathbf{s_5}$	**3** $\mathbf{s_6}$
3 $\mathbf{s_7}$	3 s_8	3 s_9	4 s_1	4 s_2	**4** $\mathbf{s_3}$	4 s_4	4 s_5	**4** $\mathbf{s_6}$	4 s_7	**4** $\mathbf{s_8}$	**5** $\mathbf{s_1}$	**5** $\mathbf{s_2}$
5 $\mathbf{s_3}$	**5** $\mathbf{s_4}$	**5** $\mathbf{s_5}$	**5** $\mathbf{s_6}$	**5** $\mathbf{s_7}$	6 s_1	6 s_2	**6** $\mathbf{s_3}$	**6** $\mathbf{s_4}$	**6** $\mathbf{s_5}$	**6** $\mathbf{s_6}$	**7** $\mathbf{s_1}$	**7** $\mathbf{s_2}$
7 $\mathbf{s_3}$	**7** $\mathbf{s_4}$	**7** $\mathbf{s_5}$	**8** $\mathbf{s_1}$	**8** $\mathbf{s_2}$	**8** $\mathbf{s_3}$	**8** $\mathbf{s_4}$	**9** $\mathbf{s_1}$	9 s_2	**9** $\mathbf{s_3}$	10 s_1	**10** $\mathbf{s_2}$	11 s_1

$$
\begin{array}{ccccccc}
s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\
s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\
s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\
s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \\
s_5 & s_6 & s_7 & s_8 & s_9 & & \\
s_6 & s_7 & s_8 & s_9 & s_{10} & & \\
s_7 & s_8 & s_9 & s_{10} & & & \\
s_8 & s_9 & & & & & \\
s_9 & s_{10} & & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & & \\
0 & 1 & 1 & 1 & 1 & & \\
0 & 1 & 1 & 1 & & & \\
0 & 1 & & & & & \\
0 & 1 & & & & &
\end{array}
$$

Example

1 s_1	1 s_2	1 s_3	**1 s_4**	**1 s_5**	**1 s_6**	**1 s_7**	**1 s_8**	**1 s_9**	**1 s_{10}**	2 s_1	2 s_2	**2 s_3**
2 s_4	**2 s_5**	**2 s_6**	**2 s_7**	**2 s_8**	**2 s_9**	2 s_{10}	3 s_1	**3 s_2**	**3 s_3**	**3 s_4**	**3 s_5**	**3 s_6**
3 s_7	3 s_8	3 s_9	4 s_1	4 s_2	**4 s_3**	4 s_4	4 s_5	**4 s_6**	4 s_7	**4 s_8**	**5 s_1**	**5 s_2**
5 s_3	**5 s_4**	**5 s_5**	**5 s_6**	**5 s_7**	6 s_1	6 s_2	**6 s_3**	**6 s_4**	**6 s_5**	**6 s_6**	**7 s_1**	**7 s_2**
7 s_3	**7 s_4**	**7 s_5**	**8 s_1**	**8 s_2**	**8 s_3**	**8 s_4**	**9 s_1**	9 s_2	**9 s_3**	10 s_1	**10 s_2**	11 s_1

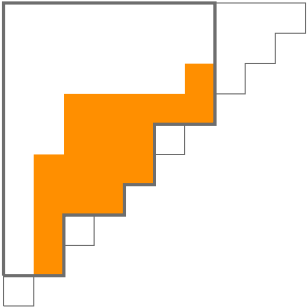
s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	1	1	1	1	1
0	0	1	1	1		
0	1	1	1	1		
0	1	1	1			
0	1					
0	1					

Example

1 s_1	1 s_2	1 s_3	**1 s_4**	**1 s_5**	**1 s_6**	**1 s_7**	**1 s_8**	**1 s_9**	**1 s_{10}**	2 s_1	2 s_2	**2 s_3**
2 s_4	**2 s_5**	**2 s_6**	**2 s_7**	**2 s_8**	**2 s_9**	2 s_{10}	3 s_1	**3 s_2**	**3 s_3**	**3 s_4**	**3 s_5**	**3 s_6**
3 s_7	3 s_8	3 s_9	4 s_1	4 s_2	**4 s_3**	4 s_4	4 s_5	**4 s_6**	4 s_7	**4 s_8**	**5 s_1**	**5 s_2**
5 s_3	**5 s_4**	**5 s_5**	**5 s_6**	**5 s_7**	6 s_1	6 s_2	**6 s_3**	**6 s_4**	**6 s_5**	**6 s_6**	**7 s_1**	**7 s_2**
7 s_3	**7 s_4**	**7 s_5**	**8 s_1**	**8 s_2**	**8 s_3**	**8 s_4**	**9 s_1**	9 s_2	**9 s_3**	10 s_1	**10 s_2**	11 s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					



A Conjecture

- let c be a Coxeter element, let $w_o(c)$ be the c-sorting word of w_o

Conjecture (Williams, 2013)

Let (W,S) be a finite Coxeter system. For any Coxeter element $c \in W$ and any $J \subseteq S$, the flip poset of $\mathcal{S}\big(cw_o(c), w_o^J\big)$ is a lattice.

- works for $W = A_n$ and for $J = S \setminus \{s\}$
- in the latter case, w_o^J is fully commutative and $\big(\mathcal{S}(cw_o(c), w_o^J), \leq_{\text{flip}}\big) \cong \text{Weak}(e, w_o^J)$

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A Different Perspective

- fix reduced word $\mathbf{w} = a_1 a_2 \cdots a_k$ for $w \in W$
- **inversion sequence**: $t_1 \prec_{\mathbf{w}} t_2 \prec_{\mathbf{w}} \cdots \prec_{\mathbf{w}} t_k$, where $t_i = a_1 a_2 \cdots a_i \cdots a_2 a_1$
- **cover reflection**: $t \in \mathrm{inv}(w)$ with $tw = ws$ for $s \in S$ $\rightsquigarrow \mathrm{cov}(w)$
- **w-aligned element**: $x \leq_S w$ with $t_{a\alpha+b\beta} \in \mathrm{cov}(x)$ and $t_\alpha \prec_{\mathbf{w}} t_{a\alpha+b\beta}$, then $t_\alpha \in \mathrm{inv}(x)$ $\rightsquigarrow \mathrm{Sort}(W, \mathbf{w})$

A Different Perspective

Conjecture (Williams, 2013)

Let (W, S) be a finite Coxeter system. For any Coxeter element and any $J \subseteq S$, the facets of $\mathcal{S}\big(cw_o(c), w_o^J\big)$ are in bijection with $Sort\big(W, w_o^J(c)\big)$.

A Different Perspective

Conjecture (& Williams, 2015)

Let (W,S) be a finite Coxeter system. For any Coxeter element c and any $w \in W$, the poset $\mathit{Weak}\big(\mathit{Sort}\big(W, w(c)\big)\big)$ is a lattice.

A Different Perspective

- it does not work for any reduced word
- $\mathrm{Weak}\big(\mathrm{Sort}\big(\mathfrak{S}_5, s_2s_1s_2s_3s_4s_2s_3s_1s_2s_1\big)\big)$ is *not* a lattice