Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

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Sperner's Theorem

- $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$
- **antichain**: set of pairwise incomparable subsets of $[n]$

Theorem (E. Sperner, 1928)

The maximal size of an antichain of $[n]$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's Theorem

- ***k*-family**: family of subsets of $[n]$ that can be written as a union of at most k antichains

Theorem (P. Erdős, 1945)

The maximal size of a k-family of $[n]$ is the sum of the k largest binomial coefficients.

A Generalization

- poset perspective:
  - antichain of $[n]$ $\longleftrightarrow$ antichain in the Boolean lattice $\mathcal{B}_n$
  - binomial coefficients $\longleftrightarrow$ rank numbers of $\mathcal{B}_n$

- $\mathcal{P}$ .. graded poset of rank n
- ***k*-Sperner**: size of a k-family does not exceed sum of k largest rank numbers
- **strongly Sperner**: k-Sperner for all $k \leq n$

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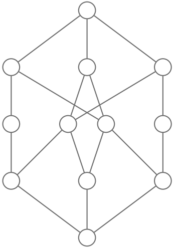
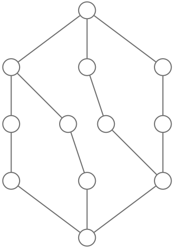
Examples

- a strongly Sperner poset



Examples

- a Sperner poset that is not 2-Sperner



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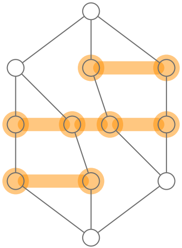

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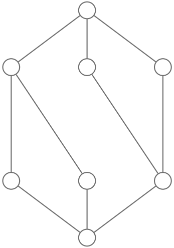

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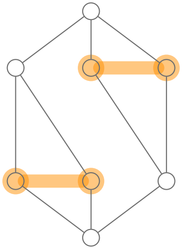

Examples

- strongly Sperner posets:
  - Boolean lattices
  - divisor lattices
  - lattices of noncrossing set partitions
  - Bruhat posets of finite Coxeter groups
  - weak order lattice of H_3
- non-Sperner posets:
  - lattices of set partitions
  - geometric lattices

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Outline

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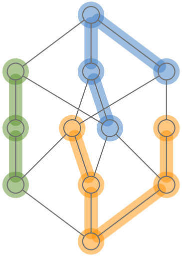
Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
- **decomposition**



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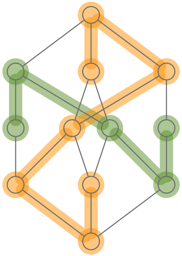

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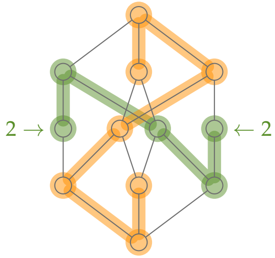

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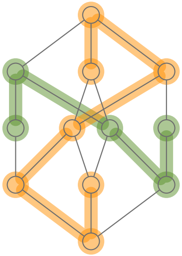

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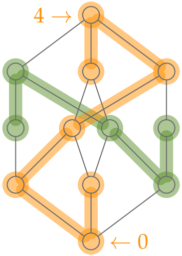

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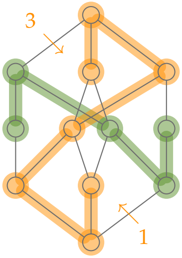

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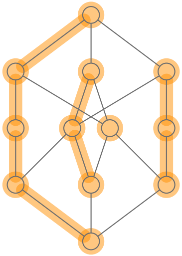

Symmetric Chain Decompositions

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Theorem (Folklore)

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is strongly Sperner.

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If $\mathcal{P}$ and $\mathcal{Q}$ admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$.

Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n; N_i .. size of i^{th} rank
- **rank-symmetric**: $N_i = N_{n-i}$
- **rank-unimodal**: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

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If $\mathcal{P}$ and $\mathcal{Q}$ are Peck, then so is $\mathcal{P} \times \mathcal{Q}$.

Outline

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The Groups $G(d,d,n), d,n \geq 1$

- $G(d,d,n)$: group of monomial $(n \times n)$-matrices, where
  - non-zero entries are d^{th} roots of unity
  - product of non-zero entries is 1

$n = 5, d = 1$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Observation

For $n \geq 1$, the group $G(1,1,n)$ is isomorphic to the symmetric group $\mathfrak{S}_n$.

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- $G(d,d,n)$: group of monomial $(n \times n)$-matrices, where
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Observation

For $d,n \geq 1$ the group $G(d,d,n)$ is a normal subgroup of index d in $\boldsymbol{\mu}_d \wr \mathfrak{S}_n$, where $\boldsymbol{\mu}_d$ is the cyclic group of d^{th} roots of unity.

The Groups $G(d,d,n), d,n \geq 1$

- subgroups of $\mathfrak{S}_{dn}$, permuting elements of

$$\left\{1^{(0)},\ldots,n^{(0)},1^{(1)},\ldots,n^{(1)},\ldots,1^{(d-1)},\ldots,n^{(d-1)}\right\}$$

- $w \in G(d,d,n)$ satisfies $w\left(k^{(s)}\right) = \pi(k)^{(s+t_k)}$
  - $\sum_{k=1}^{n} t_k \equiv 0 \pmod{d}$
  - $\pi \in \mathfrak{S}_n$, and t_k depends on w and k

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$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{rcl} \pi & = & [5,1,3,2,4]\\ t & = & (0,0,0,1,1) \end{array}$$

The Groups $G(d,d,n), d,n \geq 1$

- elements can be decomposed into "cycles":

$$\left(\left(k_1^{(t_1)}\ \ldots\ k_r^{(t_r)}\right)\right) = \left(k_1^{(t_1)}\ \ldots\ k_r^{(t_r)}\right)\left(k_1^{(t_1+1)}\ \ldots\ k_r^{(t_r+1)}\right) \cdots \left(k_1^{(t_1+d-1)}\ \ldots\ k_r^{(t_r+d-1)}\right),$$

and

$$\left[k_1^{(t_1)}\ \ldots\ k_r^{(t_r)}\right]_s = \left(k_1^{(t_1)}\ \ldots\ k_r^{(t_r)}\ k_1^{(t_1+s)}\ \ldots\ k_r^{(t_r+s)}\ \ldots\ k_1^{(t_1(d-1)s)}\ \ldots\ k_r^{(t_r+(d-1)s)}\right).$$

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For $d,n \geq 1$ the group $G(d,d,n)$ is generated by

$$T = \Big\{ \Big(\Big(i^{(0)}\; j^{(s)} \Big)\Big) \mid 1 \leq i < j \leq n, 0 \leq s < d \Big\}.$$

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- ℓ_T .. minimal length of decomposition into elements of T

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A Partial Order

- **absolute order**: $u \leq_T v$ if and only if
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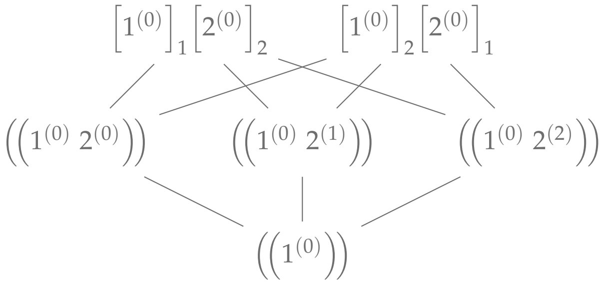
$\big(G(3,3,2), \leq_T\big)$



Outline

Noncrossing Partitions Lattices

- $\mathcal{NC}_{G(1,1,n)}$: interval $[e,c]_T$ in $\big(G(1,1,n),\leq_T\big)$ for $c = (1\ 2\ \ldots\ n)$

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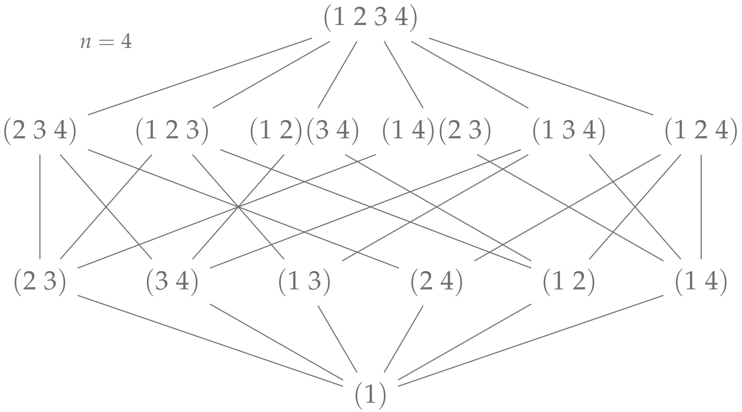

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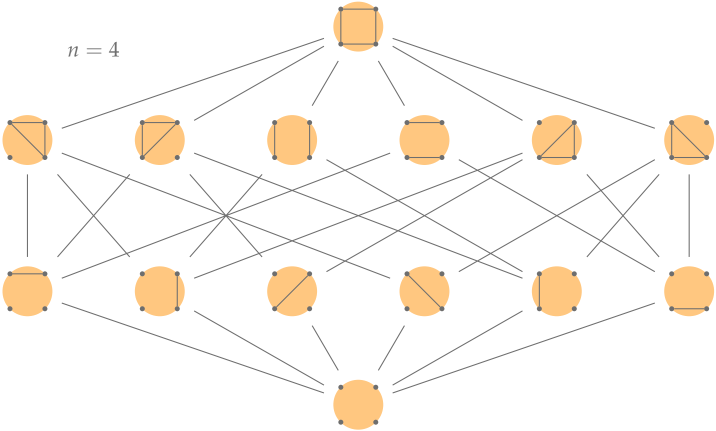
$n=4$



Noncrossing Partitions Lattices

- $\mathcal{NC}_{G(d,d,n)}$: interval $[e,\gamma]_T$ in $\big(G(d,d,n),\leq_T\big)$ for

$$\gamma=\left[1^{(0)}\;2^{(0)}\;\ldots\;(n-1)^{(0)}\right]_1\left[n^{(0)}\right]_{d-1}$$

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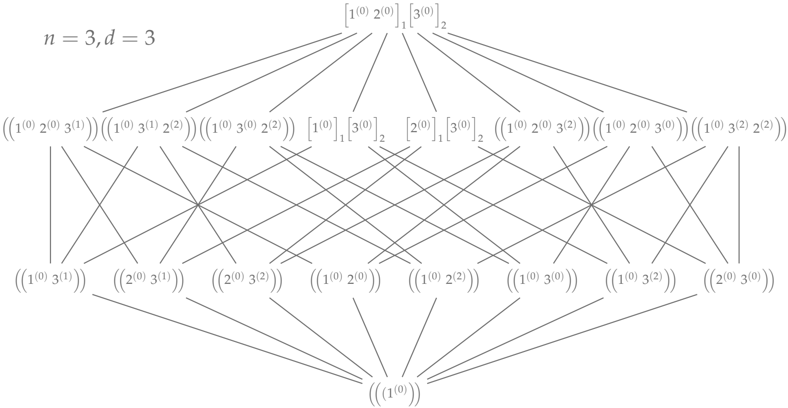
$n = 3, d = 3$



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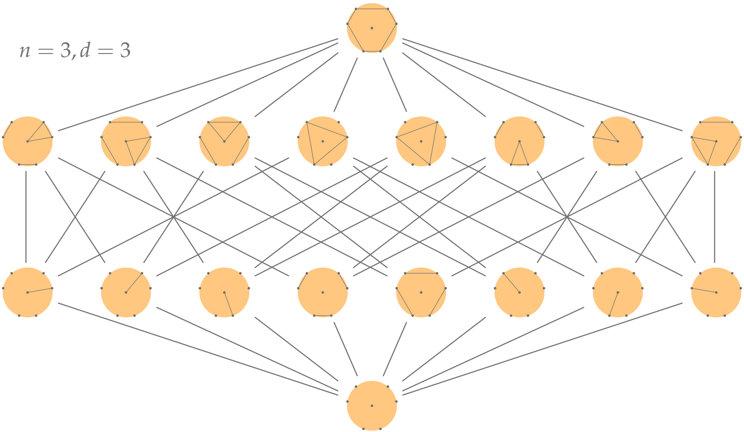

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

Theorem (R. Simion & D. Ullmann, 1991)

The lattice $\mathcal{NC}_{G(1,1,n)}$ admits a symmetric chain decomposition for each $n \geq 1$.

- $R_k = \{w \leq_T c \mid w(1) = k\}$, $\mathcal{R}_k = (R_k, \leq_T)$
- $\uplus$ .. disjoint set union; $\mathbf{2}$ .. 2-chain

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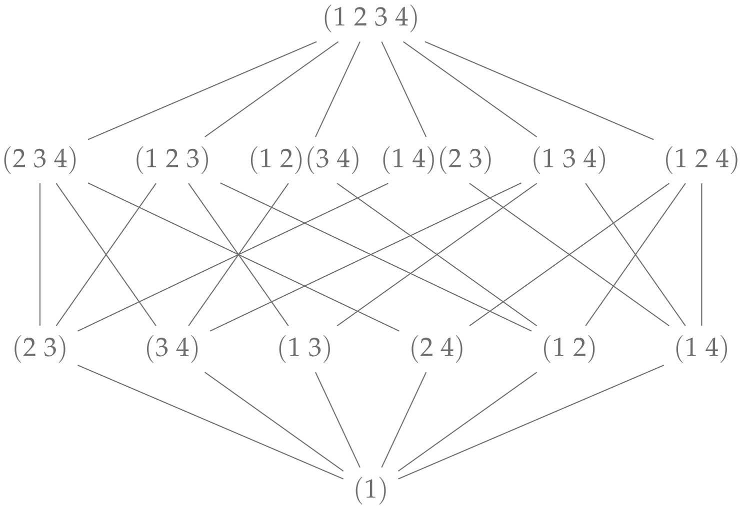
Lemma (R. Simion & D. Ullmann, 1991)

We have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, and $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$ whenever $3 \leq i \leq n$. Moreover, this decomposition is symmetric.

Example: $\mathcal{NC}_{G(1,1,4)}$



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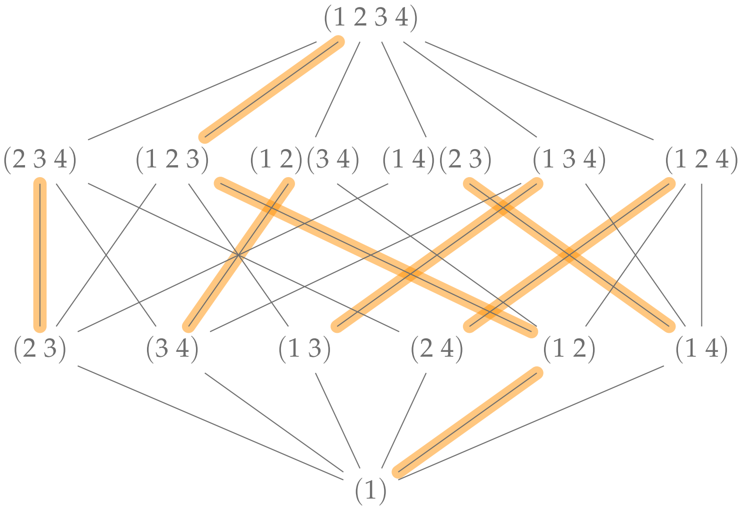

Outline

A First Decomposition

- $R_k^{(s)} = \left\{ w \leq_T \gamma \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

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Lemma (🌬, 2015)

The sets $R_1^{(s)}$ and $R_k^{(s')}$ are empty for $2 \leq s < d$ as well as $2 \leq k < n$ and $1 \leq s' < d-1$.

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Lemma (, 2015)

The poset $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$. Moreover, its least element has length 0, and its greatest element has length n.

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Lemma (, 2015)

The poset $\mathcal{R}_n^{(s)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-1)}$ for $0 \leq s < d$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

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Lemma (, 2015)

The poset $\mathcal{R}_i^{(0)}$ is isomorphic to $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

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The poset $\mathcal{R}_i^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

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The poset $\mathcal{R}_1^{(1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 2, and its greatest element has length $n-1$.

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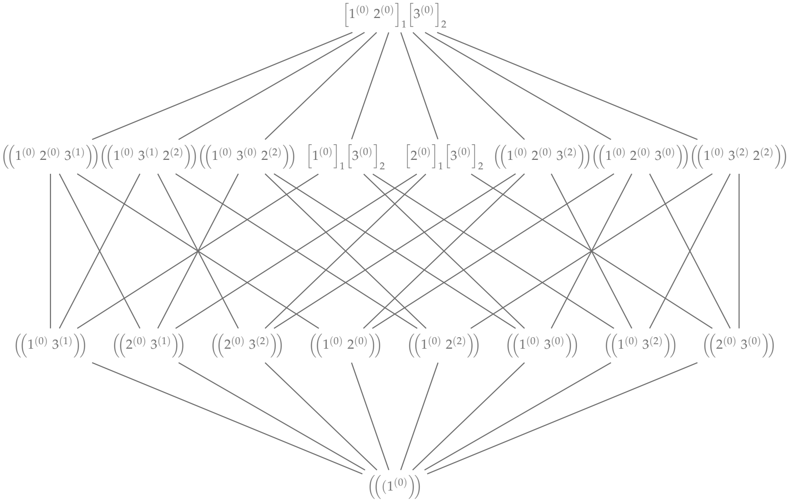
- $R_k^{(s)} = \left\{w \leq_T \gamma \mid w\left(1^{(0)}\right) = k^{(s)}\right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$

Lemma (, 2015)

The poset $\mathcal{R}_2^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-2$.

Example: $\mathcal{NC}_{G(3,3,3)}$



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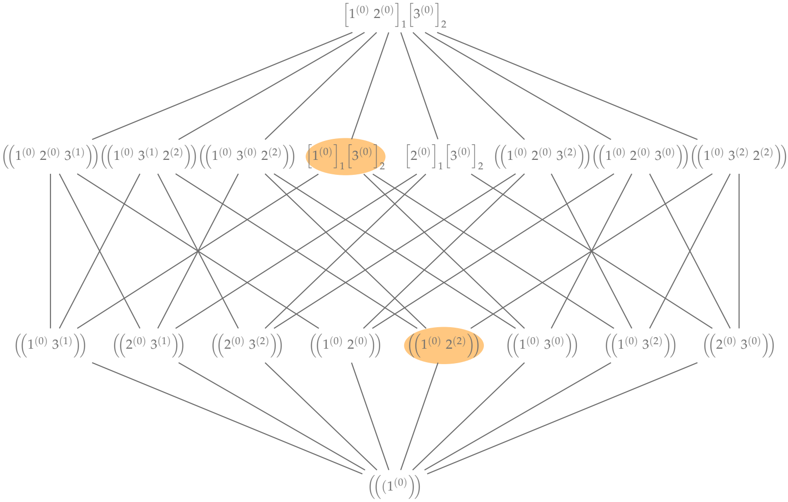

Outline

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

A Second Decomposition

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- consider the map

$$f_1 : R_1^{(1)} \to NC_{G(d,d,n)}(\gamma), \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$$

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- consider the map
  $f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$
- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\Big(n^{(d-1)}\Big) = 1^{(0)}$

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  $\Big[\Big(\big(1^{(0)}\ n^{(d-1)}\big)\Big), \Big(\big(1^{(0)}\ n^{(d-1)}\big)\Big)\Big(\big(2^{(0)}\ \ldots\ (n-1)^{(0)}\big)\Big)\Big]_T$

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Lemma (, 2015)

The interval $\left(f_1\left(R_1^{(1)}\right), \leq_T\right)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

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- consider the map
  $f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$
- define $D_1 = R_1^{(1)} \uplus f_1\Big(R_1^{(1)}\Big)$, and $\mathcal{D}_1 = (D_1, \leq_T)$

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Lemma (, 2015)

The poset $\mathcal{D}_1$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  $f_2 : R_2^{(d-1)} \to NC_{G(d,d,n)}(\gamma), \quad x \mapsto \Big(\big(2^{(0)}\ n^{(0)}\big)\Big)x$

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- its image is the interval

$$\Big[\Big(\big(1^{(0)}\ n^{(d-1)}\ 2^{(d-1)}\big)\Big), \Big(\big(1^{(0)}\ n^{(d-1)}\ 2^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\big)\Big)\Big]_T$$

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The interval $\Big(f_2\Big(R_2^{(d-1)}\Big), \leq_T\Big)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

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- define $D_2 = R_2^{(d-1)} \uplus f_2\Big(R_2^{(d-1)}\Big)$, and $\mathcal{D}_2 = (D_2, \leq_T)$

Lemma (, 2015)

The poset $\mathcal{D}_2$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- define $D = R_n^{(d-1)} \setminus \left(f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right)\right)$, and $\mathcal{D} = (D, \leq_T)$

Lemma (, 2015)

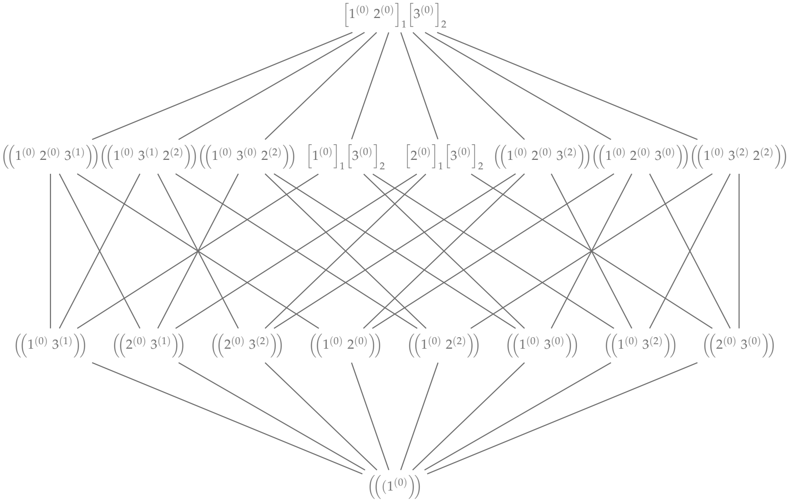
The poset $\mathcal{D}$ is isomorphic to $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$. Morever, its minimal elements have length 2, and its maximal elements have length $n-2$.

The Main Result

Theorem (, 2015)

For $d, n \geq 2$ the lattice $\mathcal{NC}_{G(d,d,n)}$ admits a symmetric chain decomposition. Consequently, it is Peck.

Example: $\mathcal{NC}_{G(3,3,3)}$



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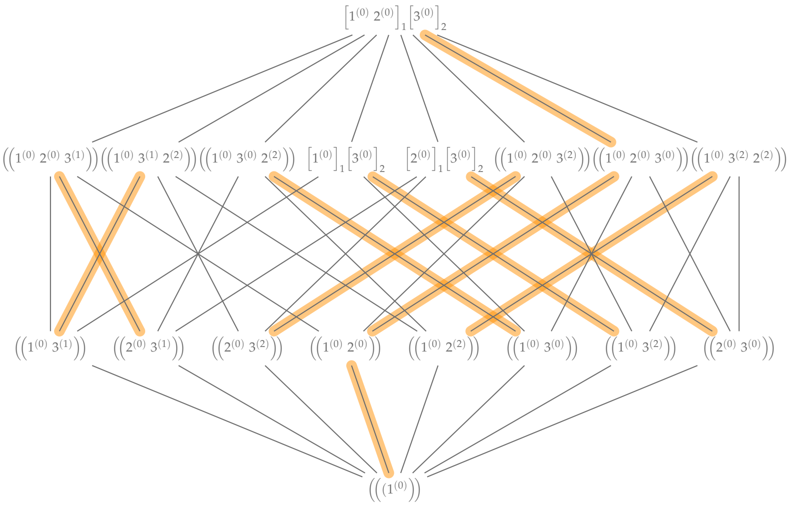

Thank You.

The General Setting

- $G(1,1,n)$ and $G(d,d,n)$ are well-generated irreducible complex reflection groups
- $(1\ 2\ \ldots\ n)$ and $\left[1^{(0)}\ 2^{(0)}\ \ldots\ (n{-}1)^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$ are Coxeter elements
- $\mathcal{NC}_W(c)$: interval $[e,c]_T$ in $(W,\leq_T)$ for some Coxeter element $c \in W$

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Theorem (, 2015)

The lattice $\mathcal{NC}_W$ is Peck for any well-generated complex reflection group W.

The Proof Strategy

- seen: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$
- remaining: $\mathcal{NC}_{G(d,1,n)}$ and exceptional groups
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \geq 2$ and $n \geq 1$

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Theorem (V. Reiner, 1997)

The lattice $\mathcal{NC}_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \geq 1$.

A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with i largest ranks removed

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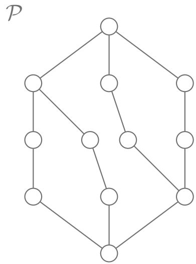

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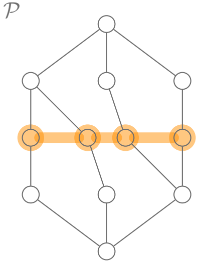

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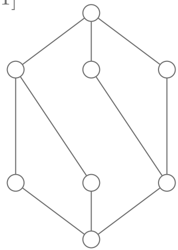
$\mathcal{P}[1]$



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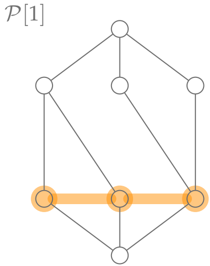

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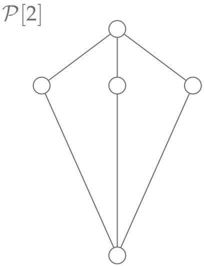

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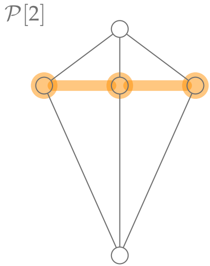

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$\mathcal{P}[4]$

A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with i largest ranks removed

Proposition (, 2015)

A graded poset $\mathcal{P}$ of rank n is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, \ldots, n\}$.

- antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for $s < i$

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A Decomposition Argument

- SAGE has a fast implementation to compute the size of the largest antichain of a poset

A Decomposition Argument

- SAGE has a fast implementation to compute the **width** of a poset

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The lattice $\mathcal{NC}_W$ is Peck for any well-generated exceptional complex reflection group W.

m-Divisible Noncrossing Partition Posets

- W .. well-generated complex reflection group; c .. Coxeter element of W
- **m-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$

$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$

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$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$
$$\partial(w)_m = [w_1; w_1^{-1}w_2, w_2^{-1}w_3, \ldots, w_{m-1}^{-1}w_m, w_m^{-1}c]$$

m-Divisible Noncrossing Partition Posets

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- ***m*-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$
- ***m*-delta sequence**: sequence of "differences" of elements in a multichain
- partial order: $(u)_m \leq (v)_m$ if and only if
$$\partial(u)_m \leq_T \partial(v)_m \qquad \rightsquigarrow \mathcal{NC}_W^{(m)}(c)$$

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

m-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$

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m-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$
- what about $m > 1$?
  - $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $\left(\mathcal{NC}_W\right)^m$
  - $\left(\mathcal{NC}_W\right)^m$ is Peck
  - $\mathcal{NC}_W^{(m)}$ is not rank-symmetric $\rightsquigarrow$ no symmetric chain decomposition

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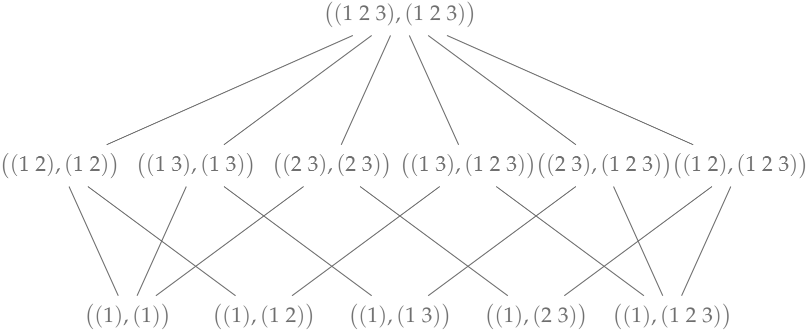
Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

Example: $\mathcal{NC}^{(2)}_{G(1,1,4)}$



Example: $\mathcal{NC}^{(2)}_{G(1,1,4)}$