Parabolic Cataland

A Type-A Story

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June 28, 2019

International Seminar, TU Dresden

Moral of the Story

- **Catalan numbers**: $\mathrm{Cat}(n) \overset{\text{def}}{=} \frac{1}{n+1}\binom{2n}{n}$
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- we replace n by a composition α of n and generalize

Outline

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Parabolic 231-Avoiding Permutations

Parabolic quotients

- $w \in \mathfrak{S}_n$

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Parabolic 231-Avoiding Permutations

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- **$(\alpha, 231)$-pattern**: a triple (i,j,k) with $i < j < k$ in different α-regions such that $w(i) < w(j)$ and (i,k) is a descent
- **$(\alpha, 231)$-avoiding**: does not have an $(\alpha, 231)$-pattern $\rightsquigarrow \mathfrak{S}_\alpha(231)$

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Parabolic Noncrossing Partitions

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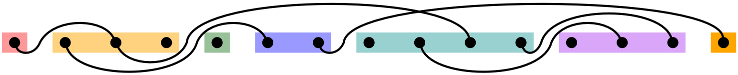

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- **noncrossing**: no bumps cross in the diagram $\rightsquigarrow NC_\alpha$

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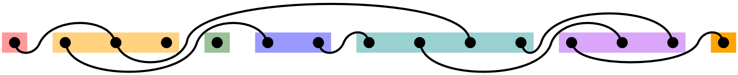

Parabolic Dyck Paths

- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ composition of n
- **Dyck path**: lattice path from $(0,0)$ to (n,n) with unit steps N and E that never goes below the main diagonal

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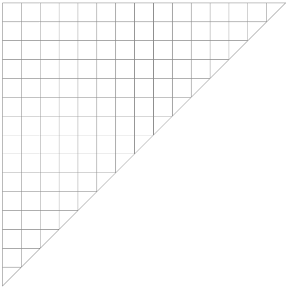

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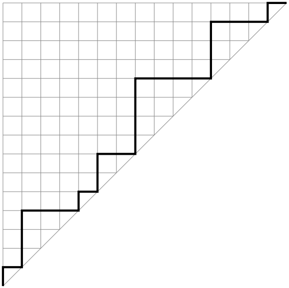

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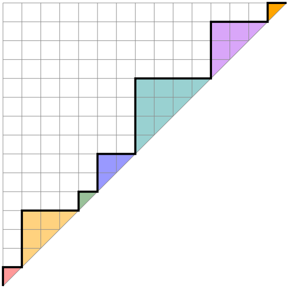

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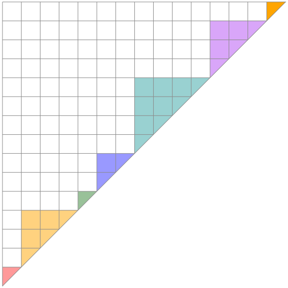

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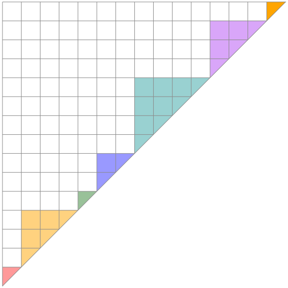

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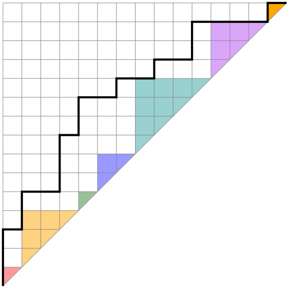

Left-Aligned Colorable Trees

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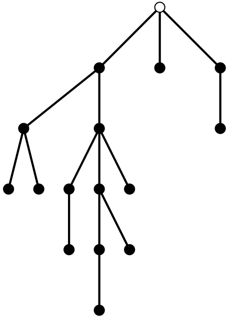

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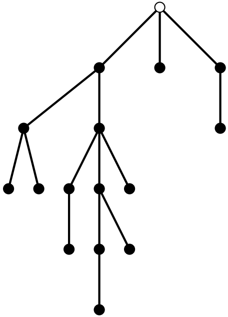
- ***α*-tree**: plane rooted tree with $n+1$ nodes colorable by the following algorithm $\rightsquigarrow \mathbb{T}_\alpha$



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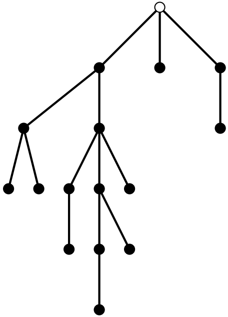

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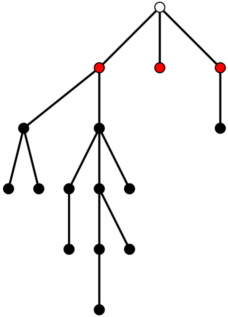

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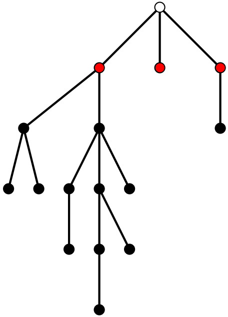
$3 < 4 \rightsquigarrow$ Failure!



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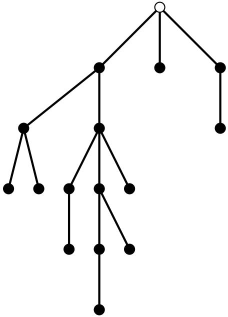

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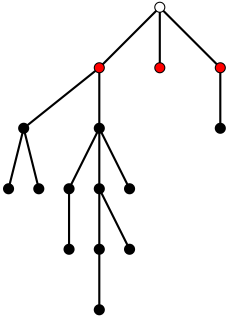

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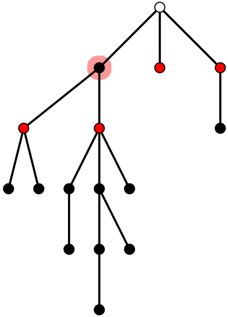

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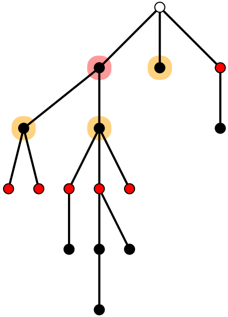

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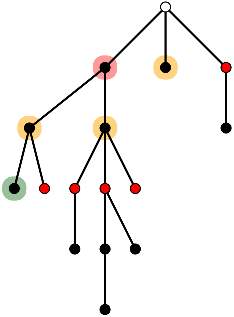

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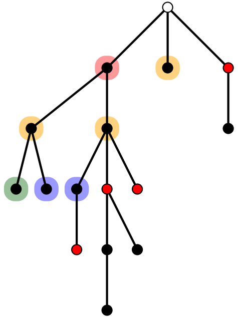

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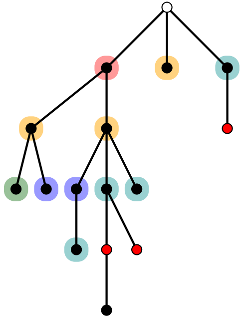

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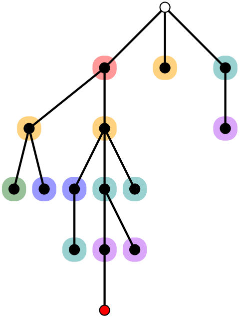

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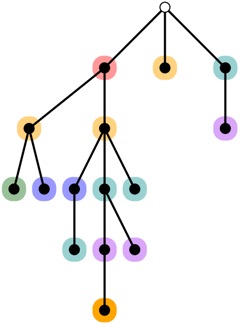
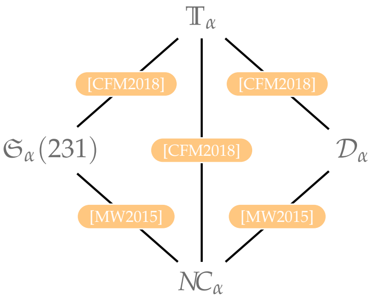


It is all Connected

Theorem (C. Ceballos, W. Fang, H. Mühle, N. Williams; 2015–2018)

For every composition α, the sets $\mathfrak{S}_\alpha(231)$, NC_α, $\mathcal{D}_\alpha$ and $\mathbb{T}_\alpha$ are in bijection.



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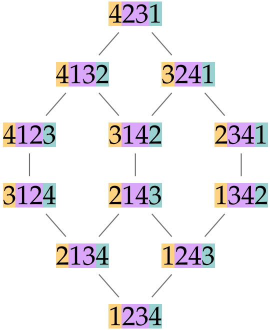

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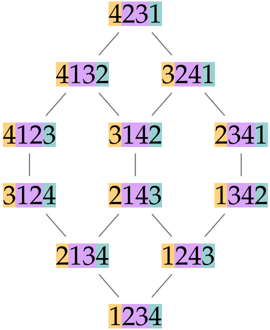

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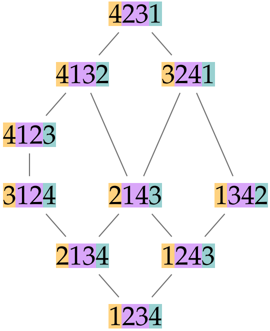

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Theorem (, N. Williams; 2015)

For every integer composition α, the poset $\mathcal{T}_\alpha$ is a quotient lattice of $(\mathfrak{S}_\alpha, \leq_L)$.

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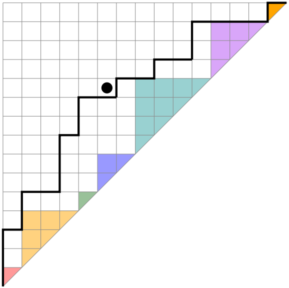

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- **valley**: coordinate preceded by E and followed by N
- **rotation** at valley: exchange east step with subpath subject to a distance condition $\rightsquigarrow \lessdot_\alpha$

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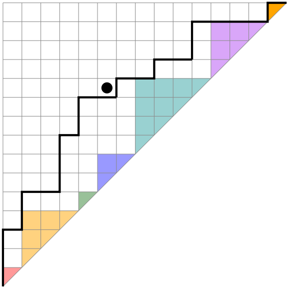

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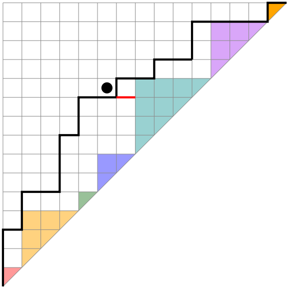

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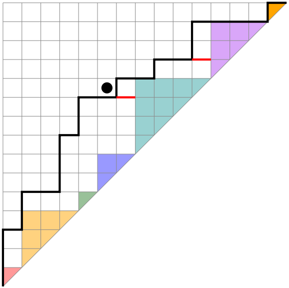

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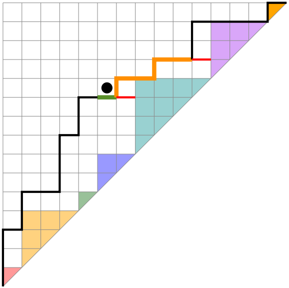

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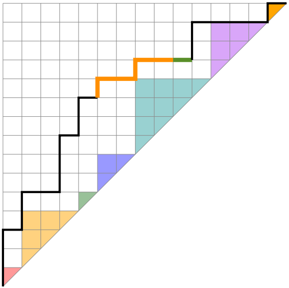

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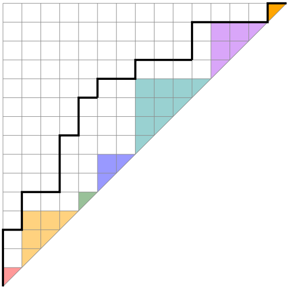

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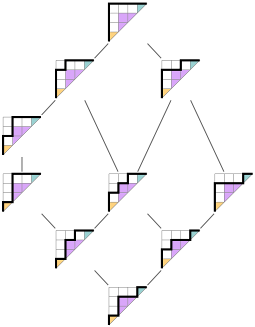

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Holds for arbitrary Dyck paths ν.

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Galois graphs

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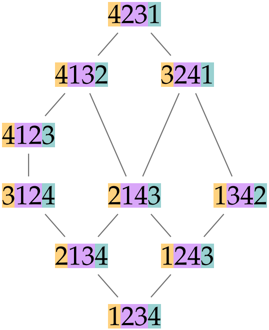

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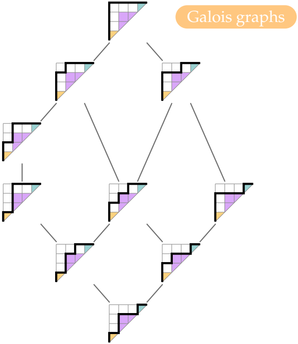

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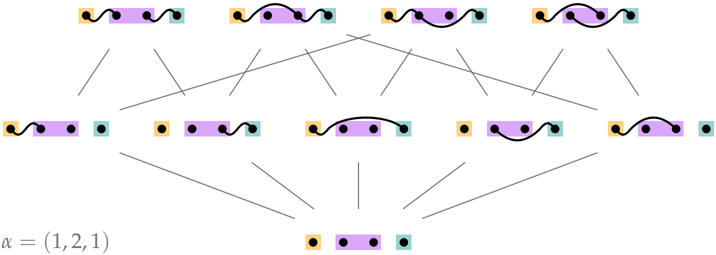

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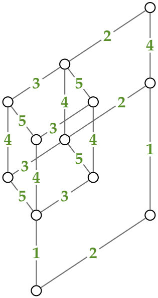

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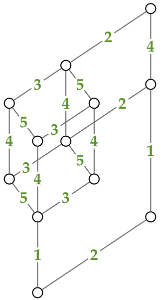

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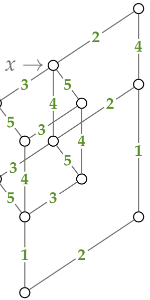

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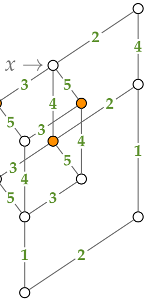

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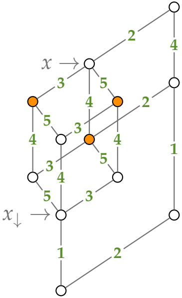

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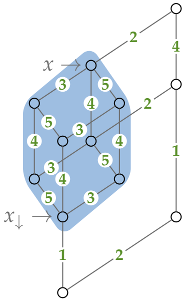

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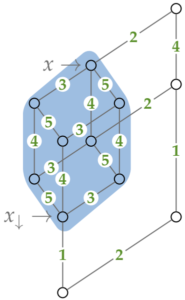

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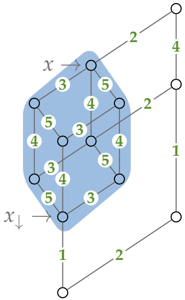

$\Psi_{\lambda}(x) = \{3, 4, 5\}$

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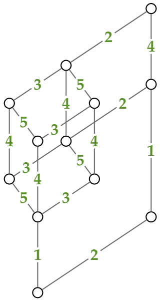

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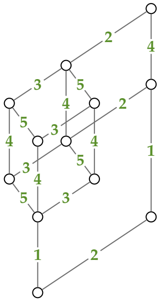

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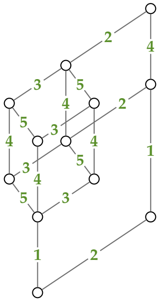

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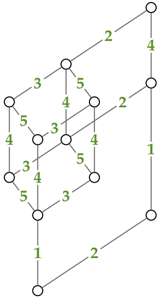

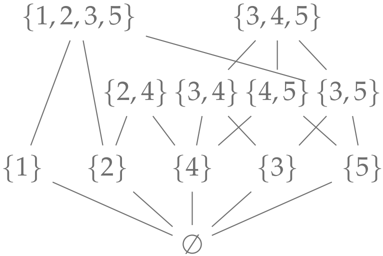

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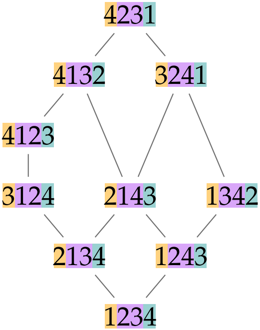

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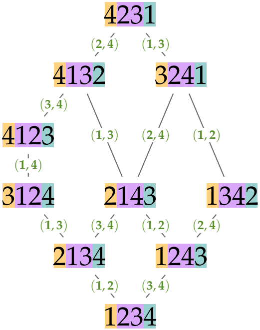

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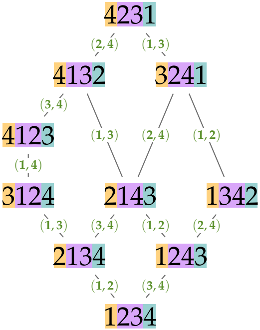

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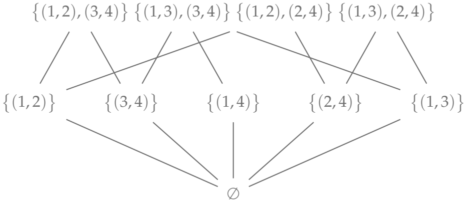

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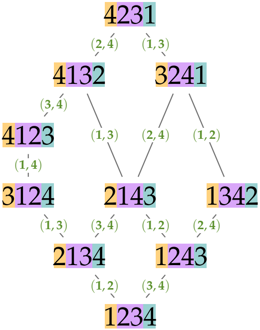

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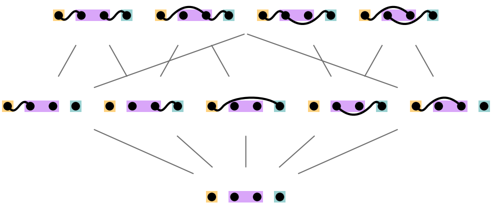

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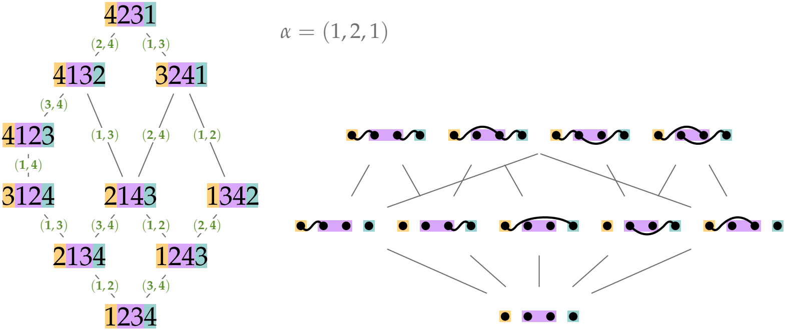

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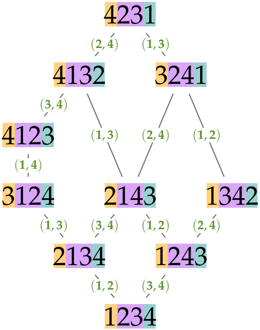

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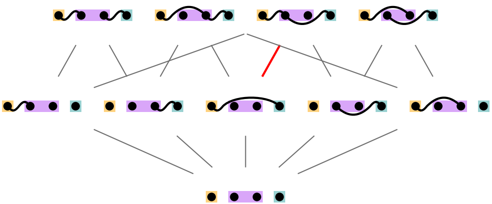

Outline

Decomposition of Permutations

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$$w = 8 \bullet 6457 \bullet 3 \bullet 12$$

Decomposition of Permutations

- $w \in \mathfrak{S}_n$

- **global split**: $k \in [n]$ such that $w\bigl([k]\bigr) = [n] \setminus [n-k]$
- **atomic**: permutation whose only global split is n
- unique decomposition of w into atomic permutations

$$w = 1 \bullet 3124 \bullet 1 \bullet 12$$

Pipe Dreams

- **pipe dream**: filling of a triangular shape with elbows ⌟⌜ and crosses $+$
- **reduced**: every pair of pipes crosses at most once
- technical requirement: elbow in top-left cell $\rightsquigarrow \Pi_n$

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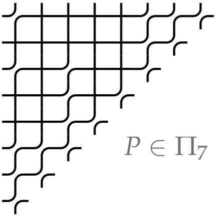

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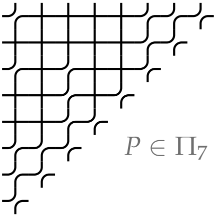

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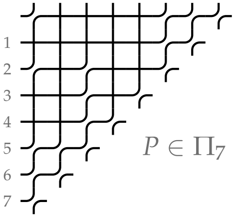

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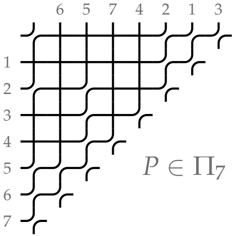

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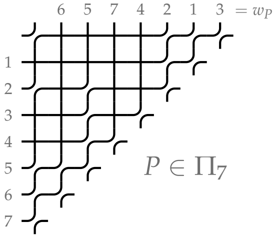

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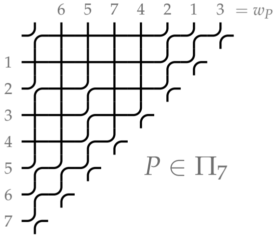
- consider the graded vector space $\mathbf{k}\Pi \overset{\text{def}}{=} \bigoplus_{n\geq 0} \mathbf{k}\Pi_n$



A Product on Pipe Dreams

- $P \in \Pi_m, Q \in \Pi_n$
- **P/Q-shuffle**: word with m letters p and n letters q such that the number of p's weakly before any pq is a global split of w_P and vice versa

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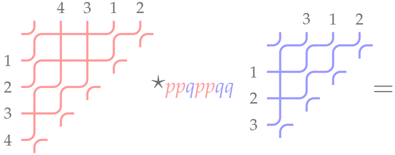

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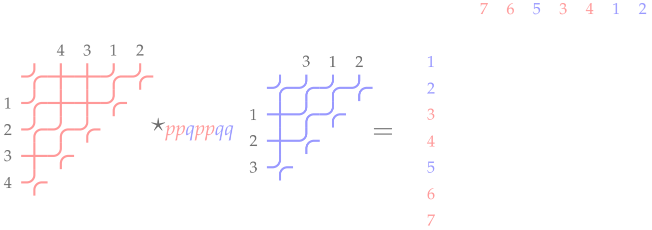

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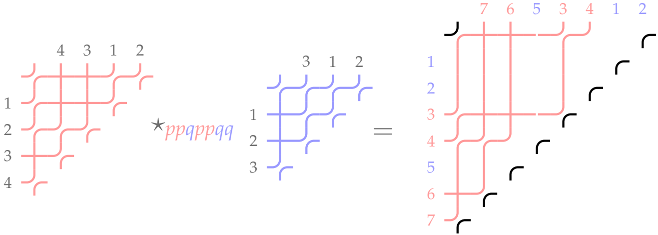

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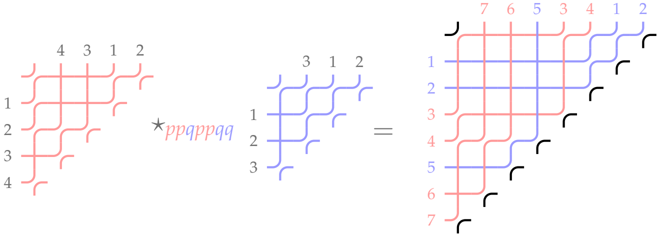

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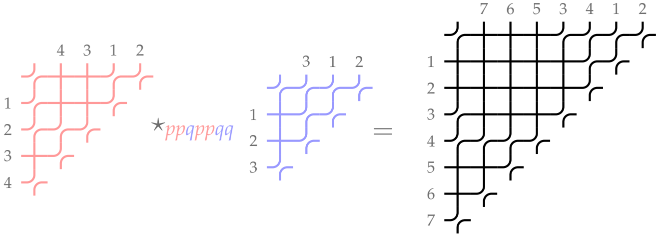

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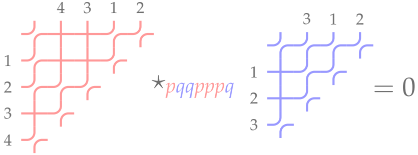

A Coproduct on Pipe Dreams

- $P \in \Pi_n$; k global split of w_P
- **untangling**: $\Delta_{k,n-k}(P) = P_1 \otimes P_2$, where P_1 restricts to pipes labeled $k, k+1, \ldots, n$ and P_2 restricts to pipes labeled $1, 2, \ldots, k-1$; $\Delta_{a,b}(P) = 0$ otherwise
- **coproduct**: $\Delta \stackrel{\text{def}}{=} \sum_{a,b \in \mathbb{N}} \Delta_{a,b}$
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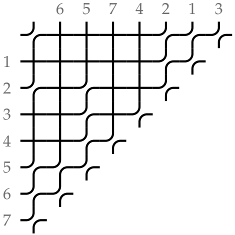

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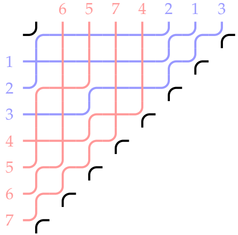

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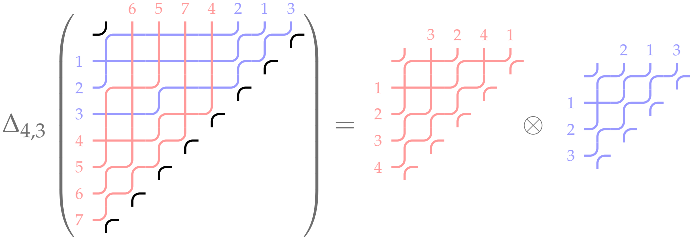

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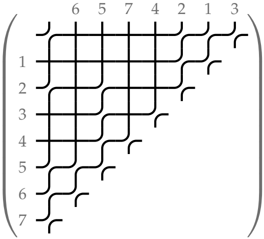
$\Delta_{5,2}$  $= 0$

A Hopf Algebra on Pipe Dreams

Theorem (N. Bergeron, C. Ceballos, V. Pilaud; 2018)

The product $\cdot$ and coproduct Δ endow the family of all pipe dreams with a graded, connected Hopf algebra structure.

The Graded Dimension of $\mathbf{k}\Pi$

- $\Pi_n\langle 1, 12, 123 \ldots\rangle$: set of pipe dreams whose exit permutation factors into identity permutations
- **$\nwarrow\uparrow\searrow$-walk**: a lattice walk in the positive quadrant starting at the origin, ending on the x-axis, and using $2n$ steps from the set $\{(-1,1),(1,-1),(0,1)\}$

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Theorem (C. Ceballos, W. Fang, ; 2018)

For $n \geq 0$, the dimension of $\mathbf{k}\Pi_n\langle 1, 12, 123, \ldots\rangle$ equals the number of $\nwarrow\uparrow\searrow$-walks of length $2n$.

Theorem (C. Ceballos, W. Fang, ; 2018)

For $n \geq 0$ and $k \in [n]$, the set of pipe dreams whose exit permutation factors into k identity permutations is in bijection with the set of $\nwarrow\!\uparrow\!\searrow$-walks of length $2n$ with exactly k north-steps.

The Scenic Route Through Cataland

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$$w = 1 \bullet 123 \bullet 1 \bullet 12 \bullet 1234 \bullet 123 \bullet 1$$

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15	12	13	14	11	9	10	5	6	7	8	2	3	4	1

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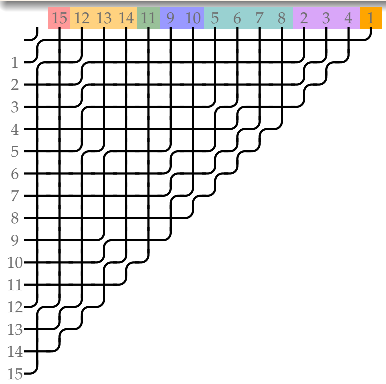

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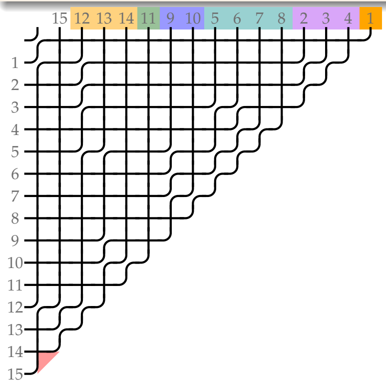

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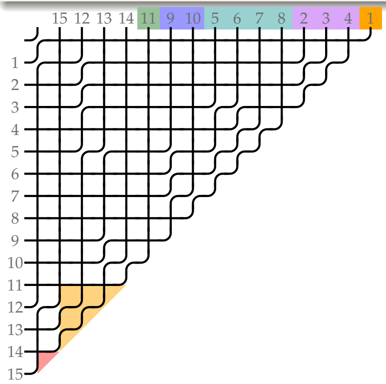

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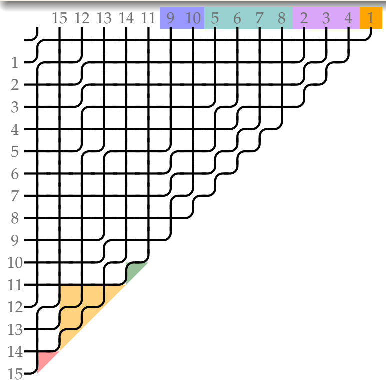

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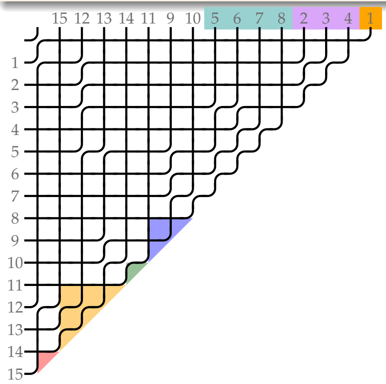

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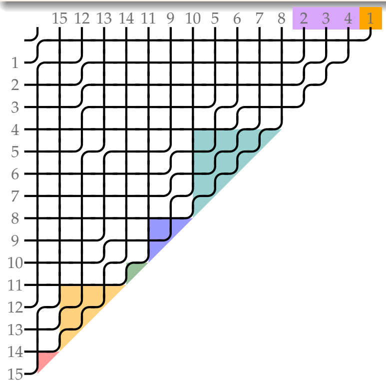

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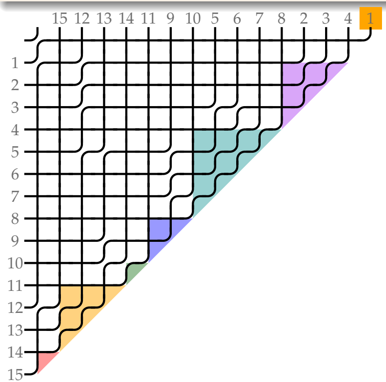

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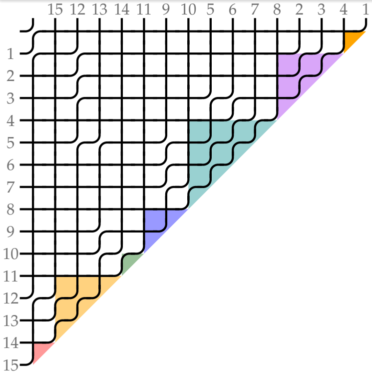

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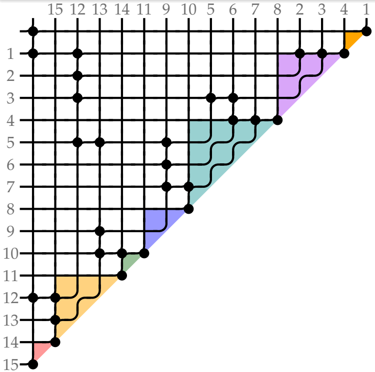

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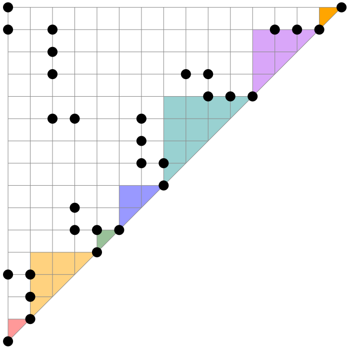

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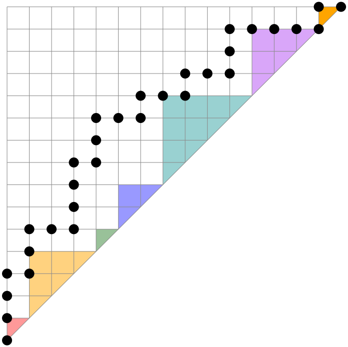

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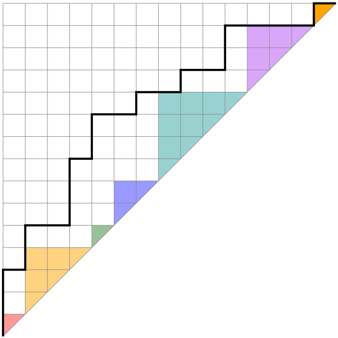

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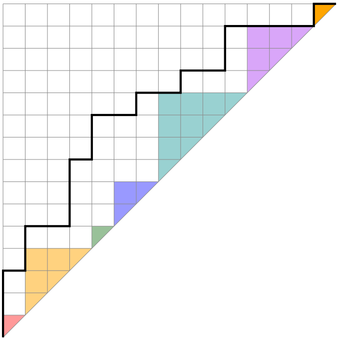

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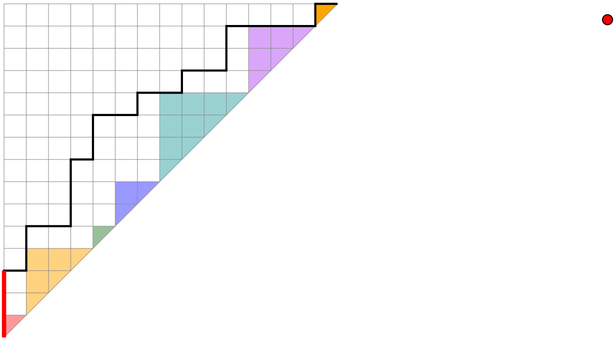

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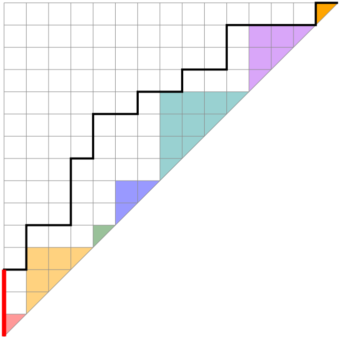

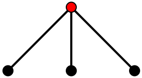

The Scenic Route Through Cataland

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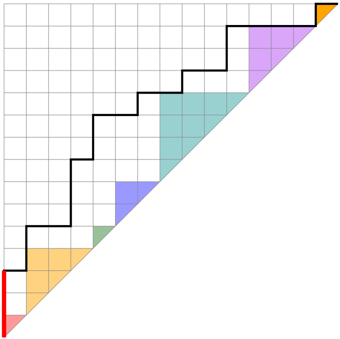

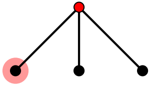

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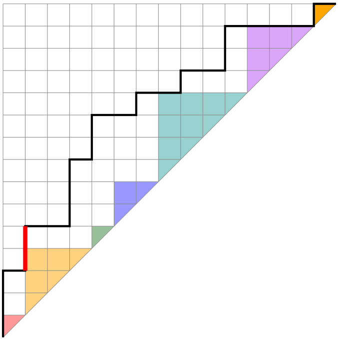

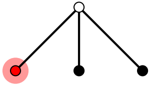

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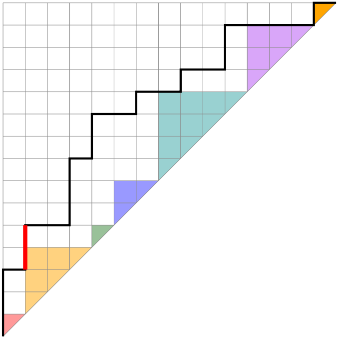

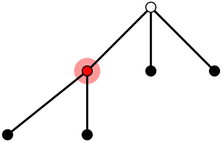

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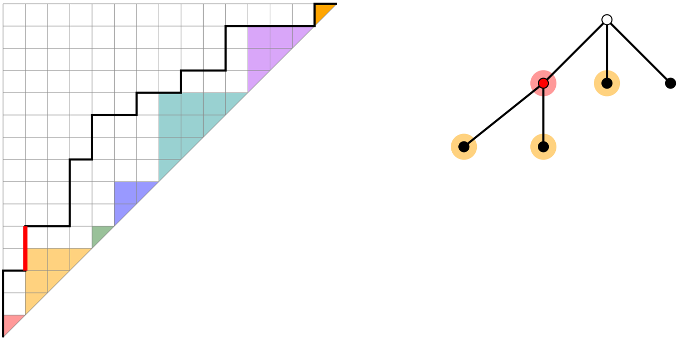

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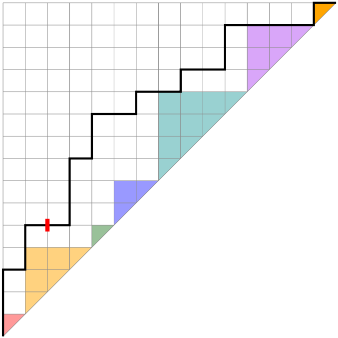

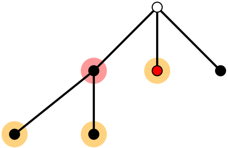

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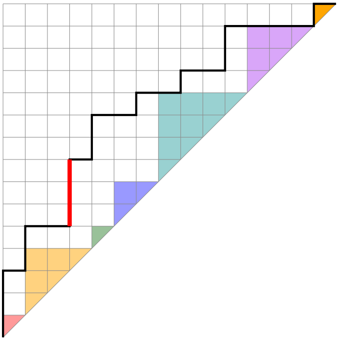

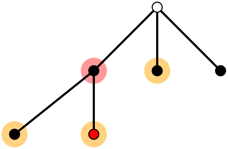

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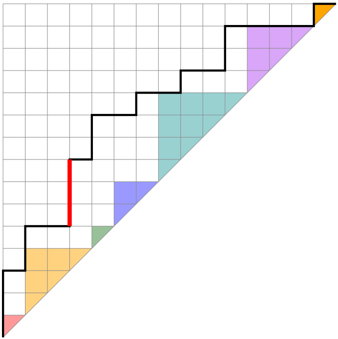

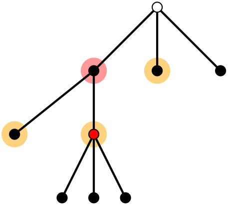

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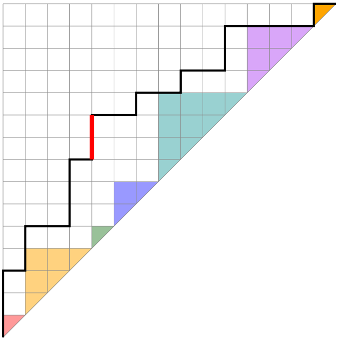

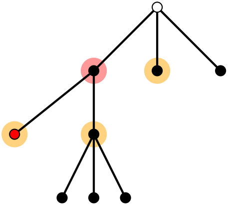

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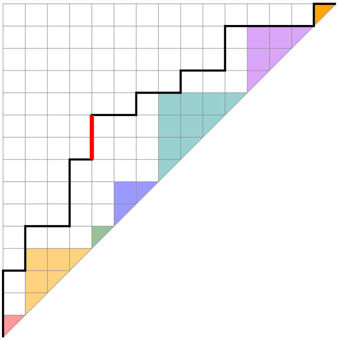

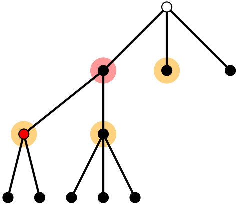

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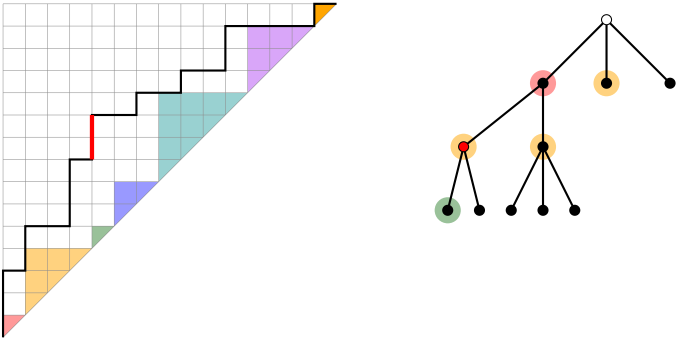

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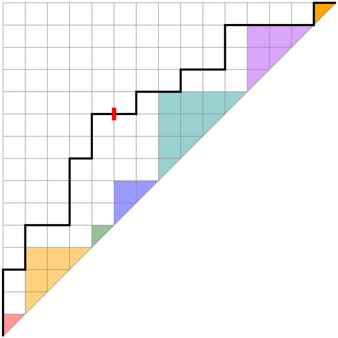

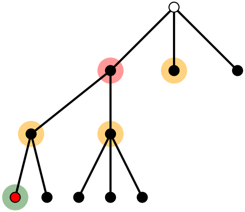

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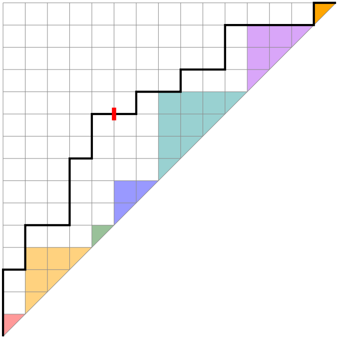

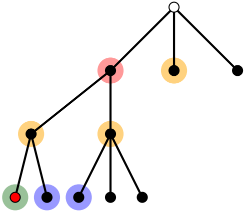

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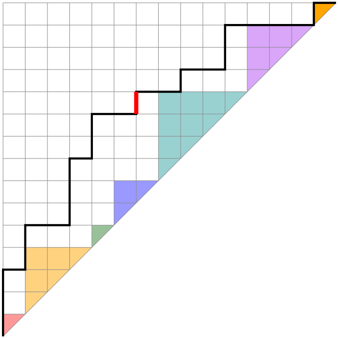

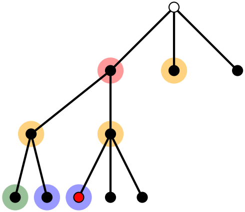

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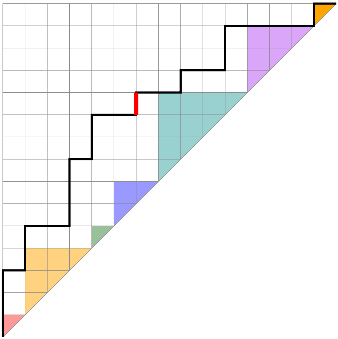

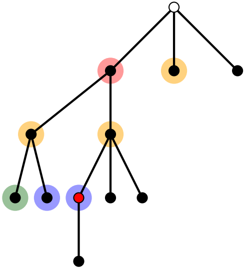

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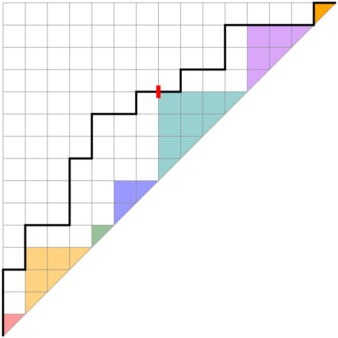

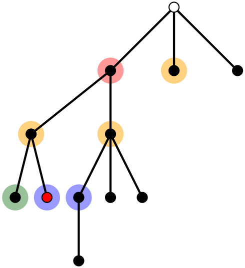

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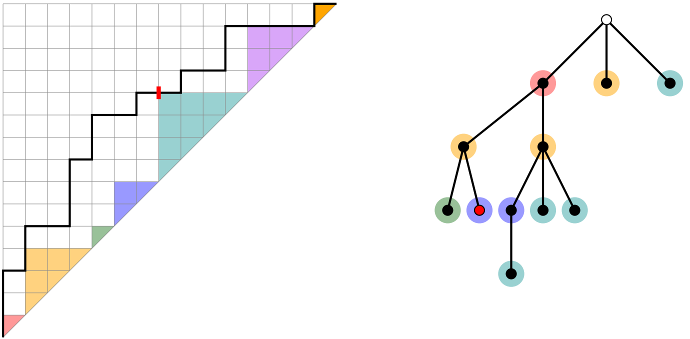

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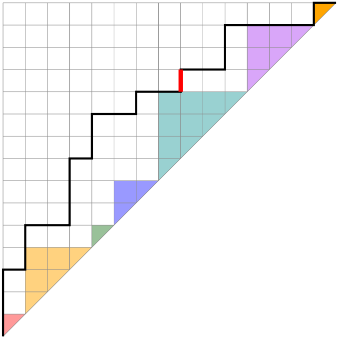

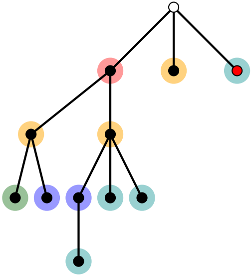

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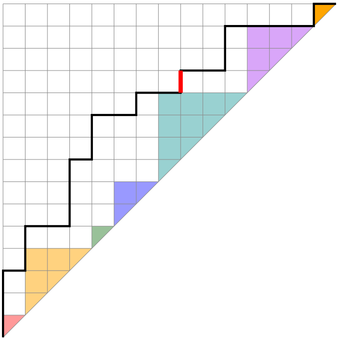

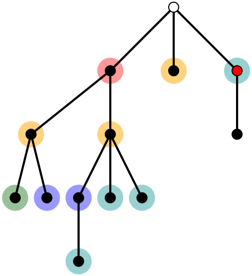

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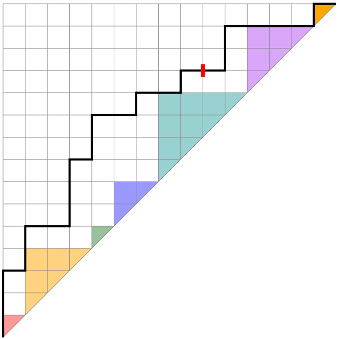

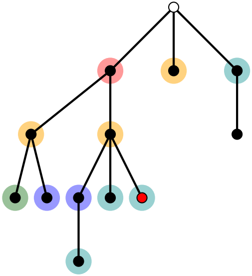

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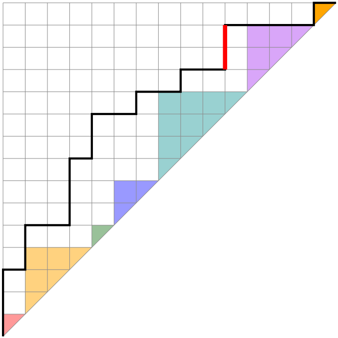

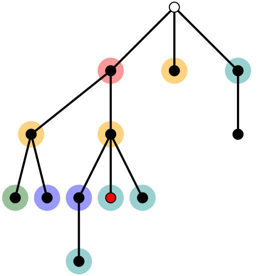

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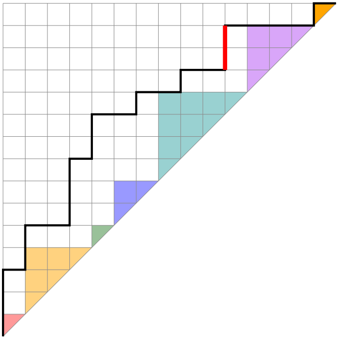

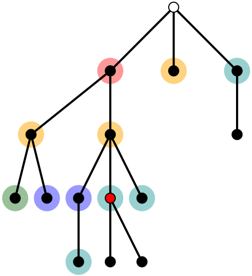

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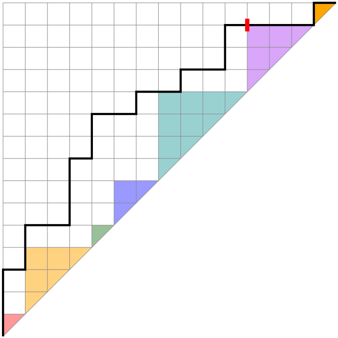

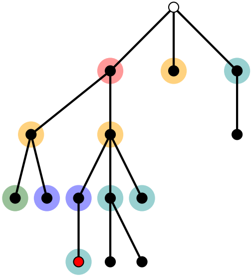

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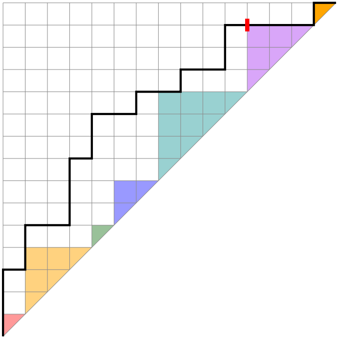

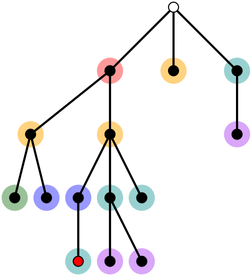

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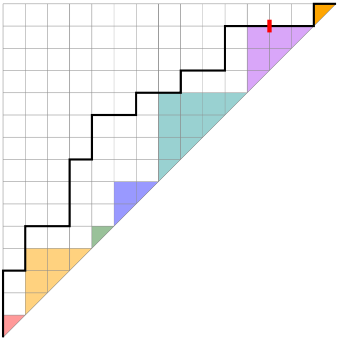

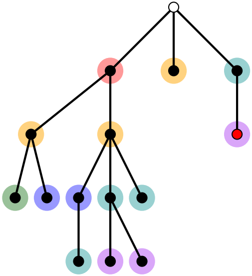

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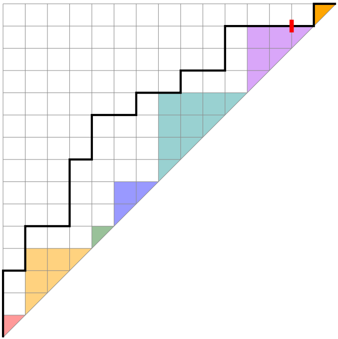

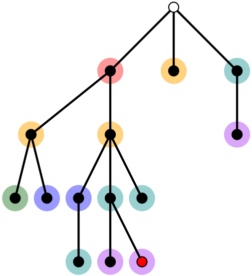

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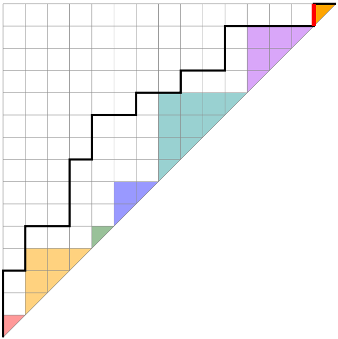

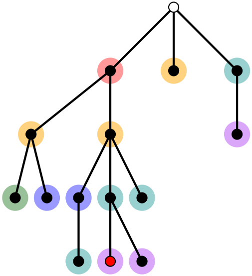

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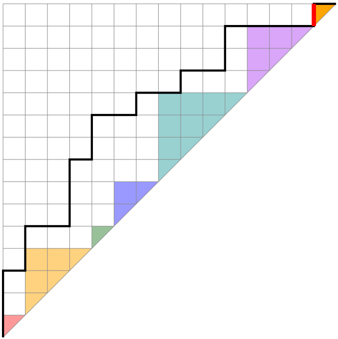

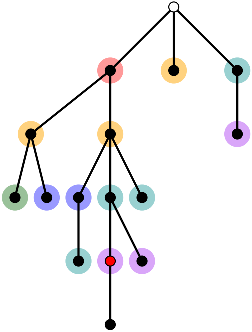

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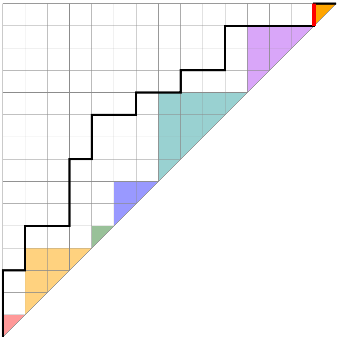

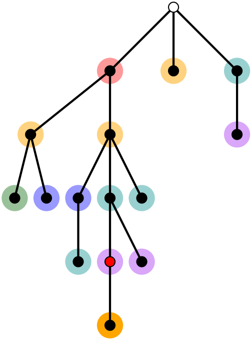

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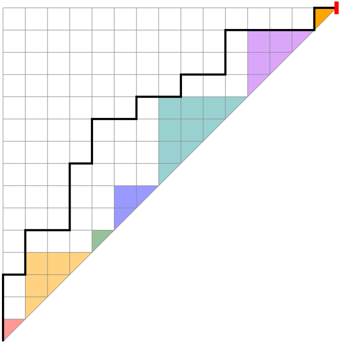

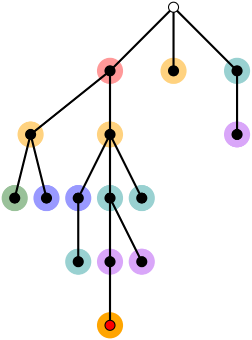

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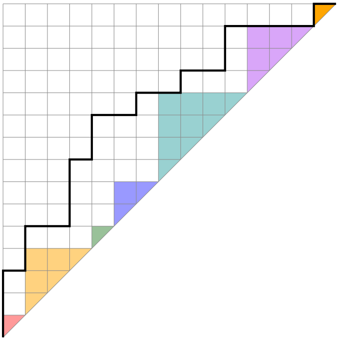

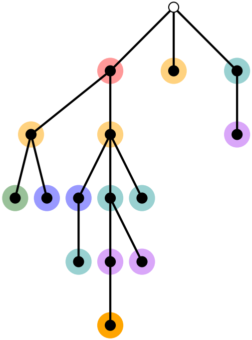

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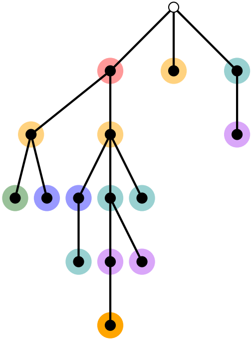

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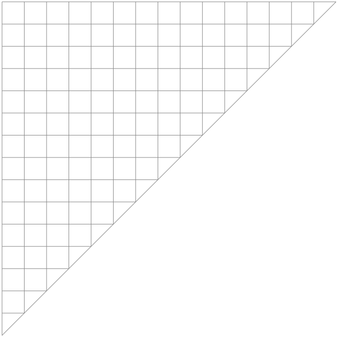

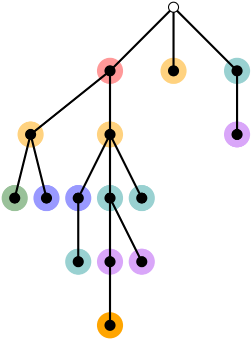

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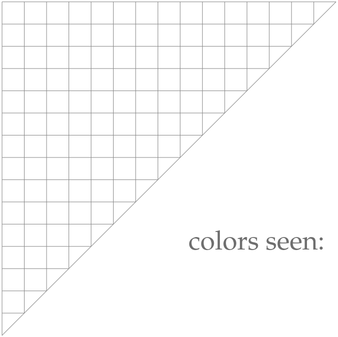

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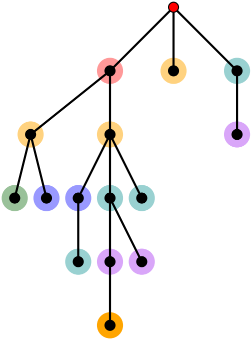

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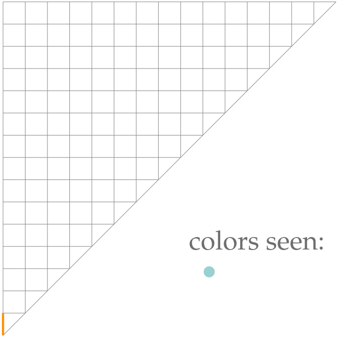

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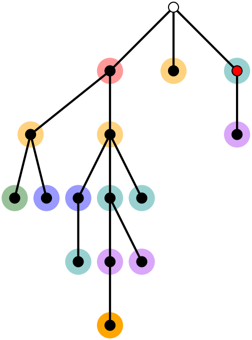

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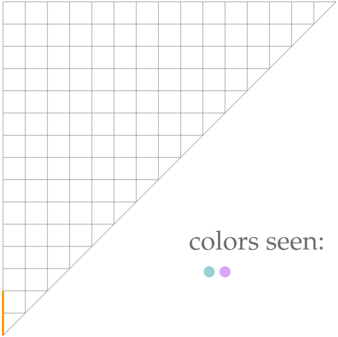

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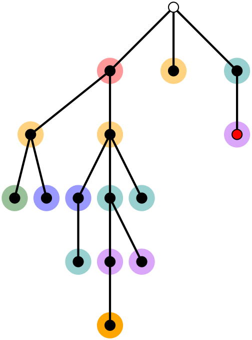

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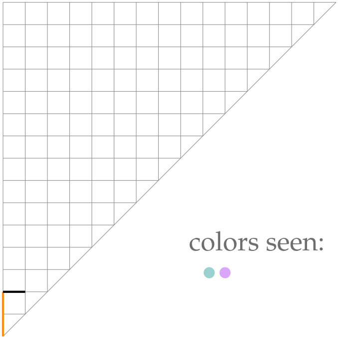

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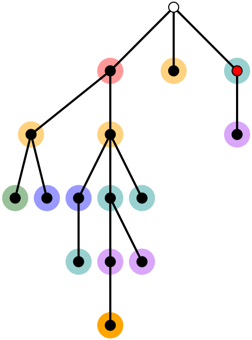

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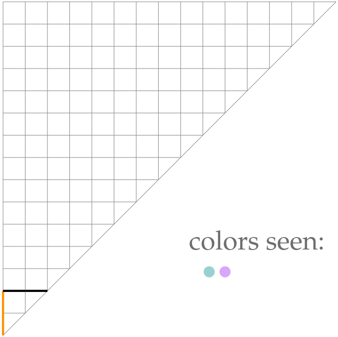

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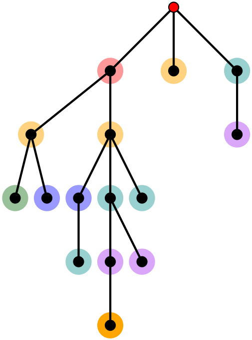

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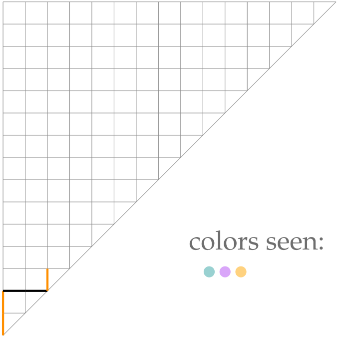

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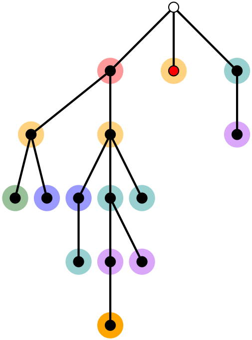

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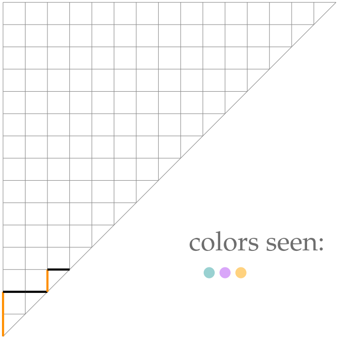

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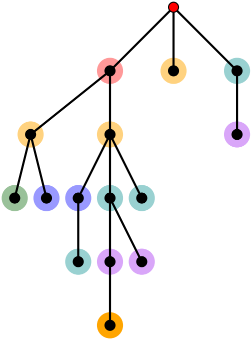

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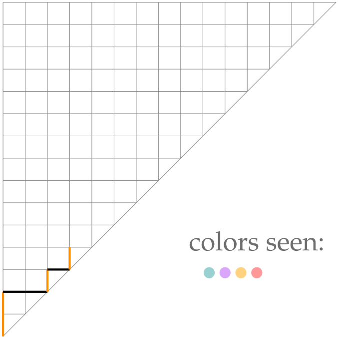

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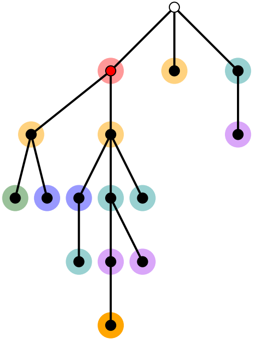

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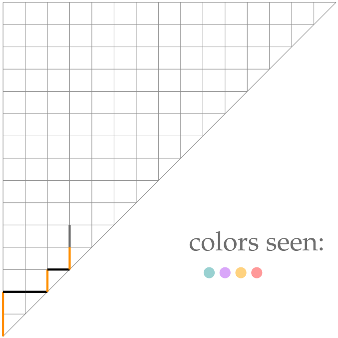

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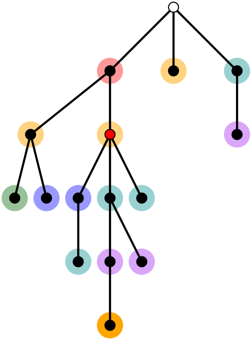

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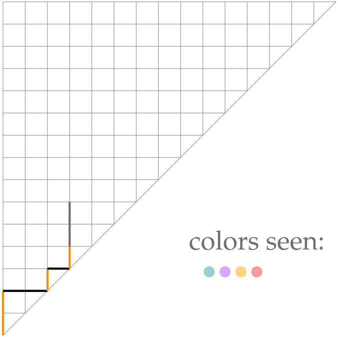

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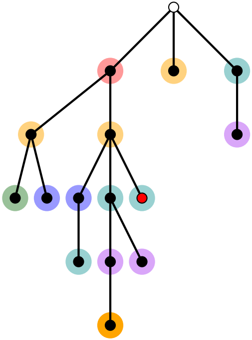

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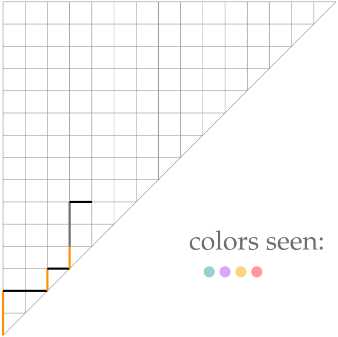

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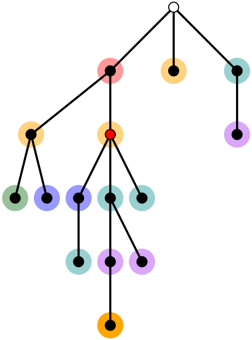

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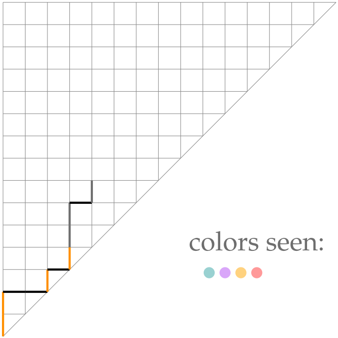

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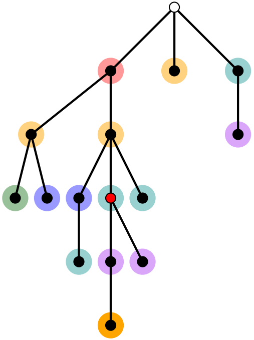

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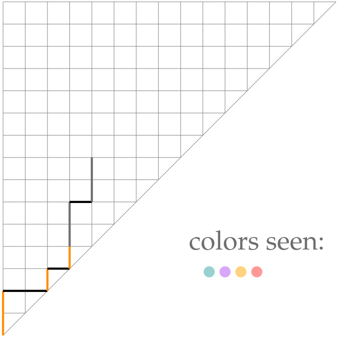

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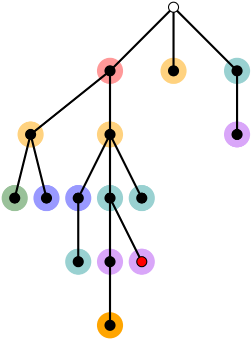

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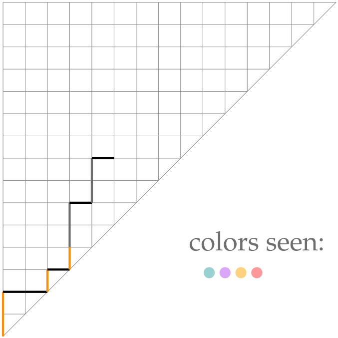

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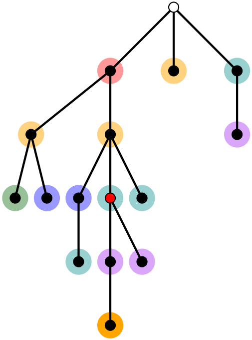

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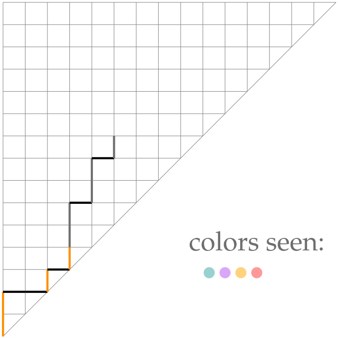

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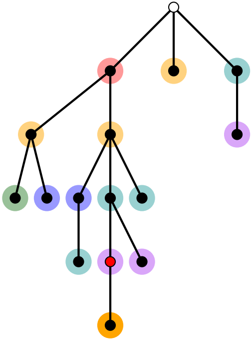

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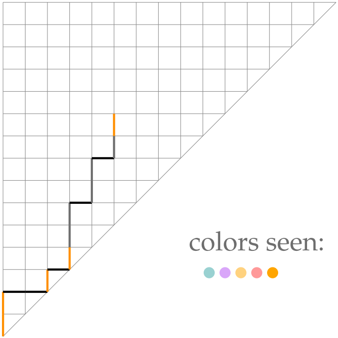

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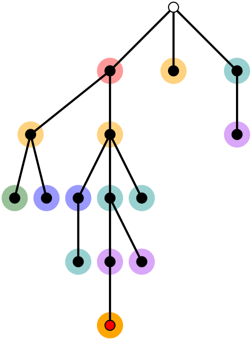

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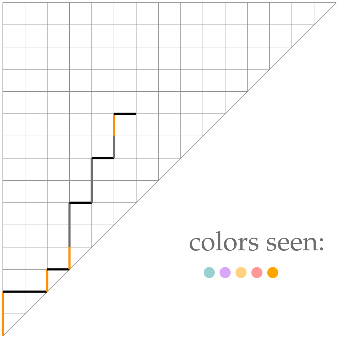

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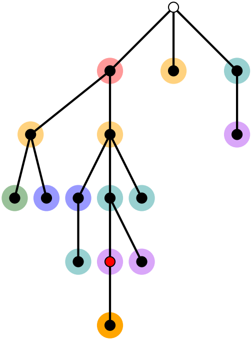

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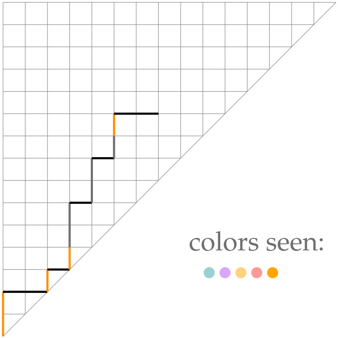

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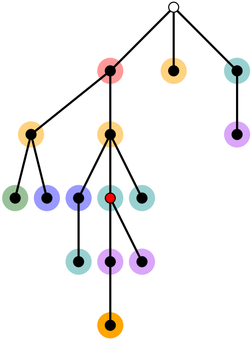

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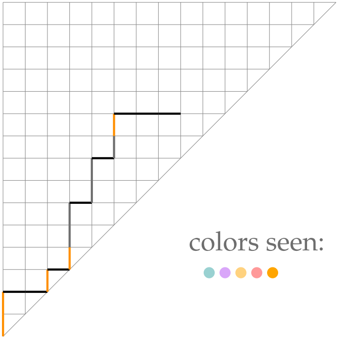

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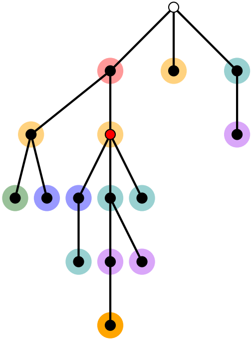

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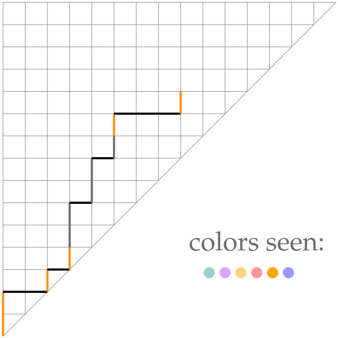

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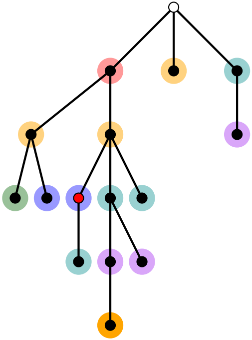

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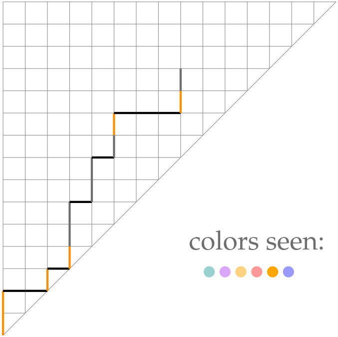

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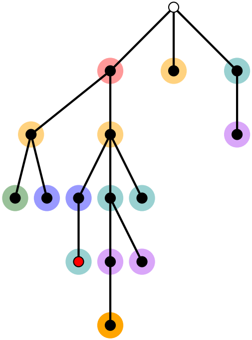

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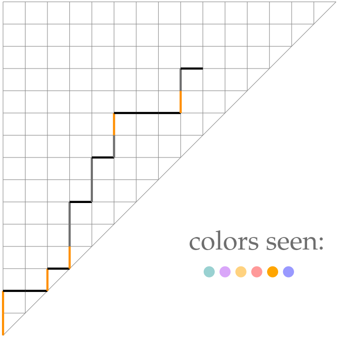

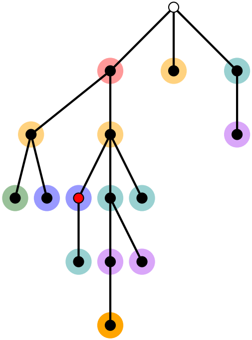

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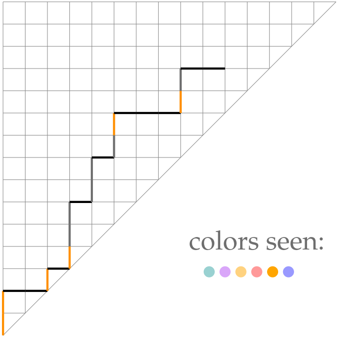

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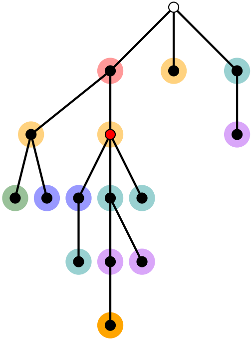

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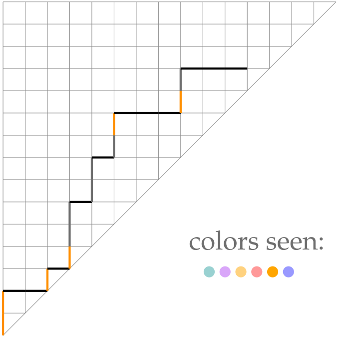

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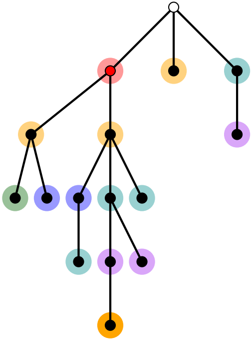

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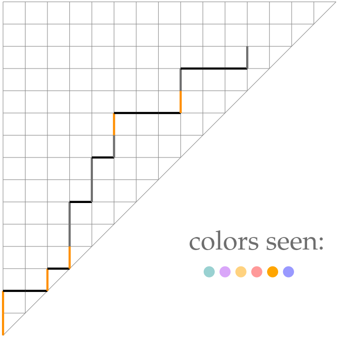

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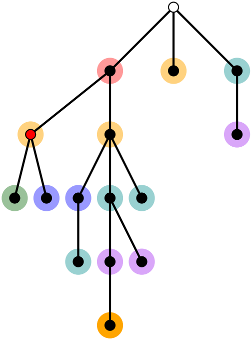

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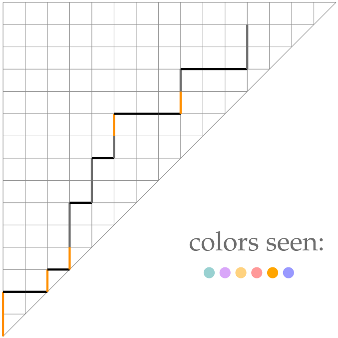

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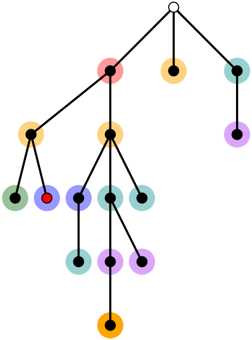

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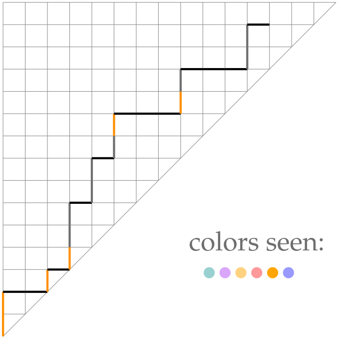

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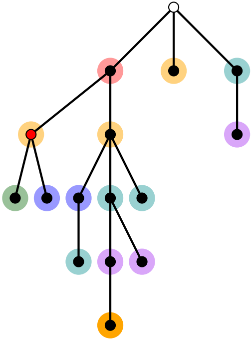

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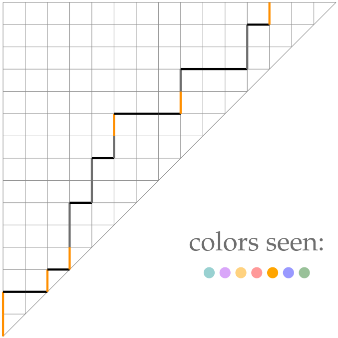

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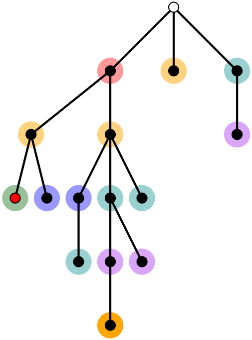

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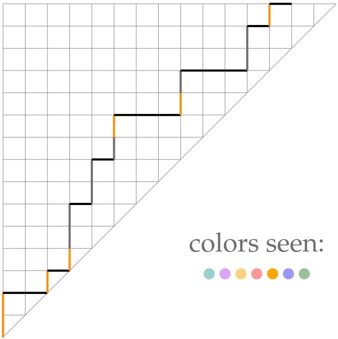

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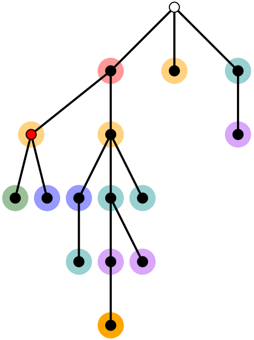

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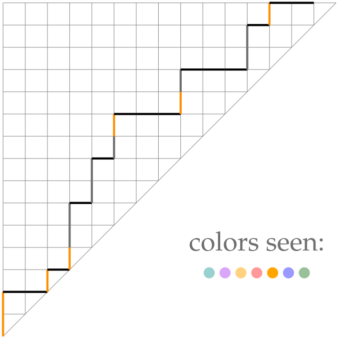

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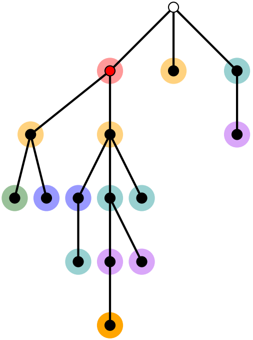

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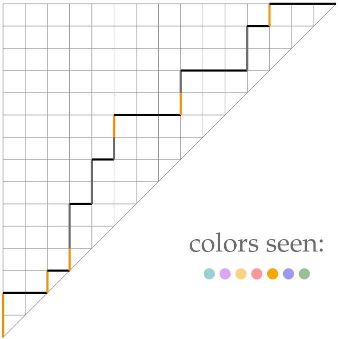

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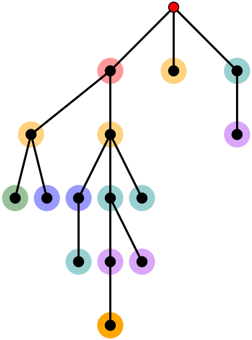

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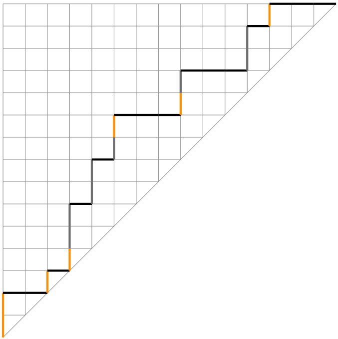

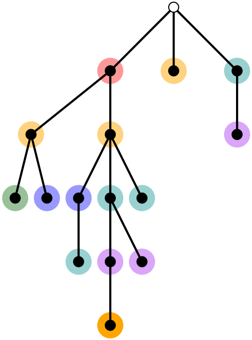

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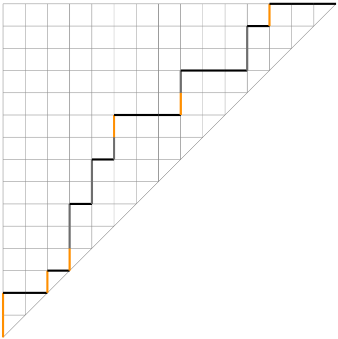

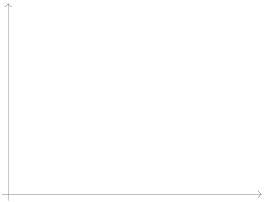

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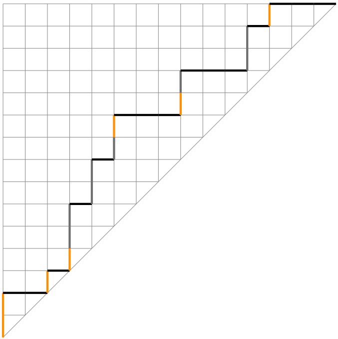

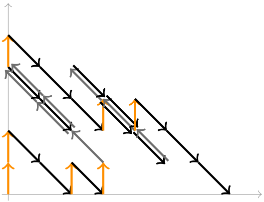

Outline

Diagonal Coinvariants

- $X \stackrel{\text{def}}{=} \{x_1, x_2, \ldots, x_n\}$, $Y \stackrel{\text{def}}{=} \{y_1, y_2, \ldots, y_n\}$
- **diagonal action**: $\sigma \cdot f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$
- **polarized power sum**: $p_{h,k} \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_i^h y_i^k$

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Theorem (H. Weyl; 1949)

The ring $\mathbb{Q}[X, Y]^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariant polynomials is generated by the polarized power sums.

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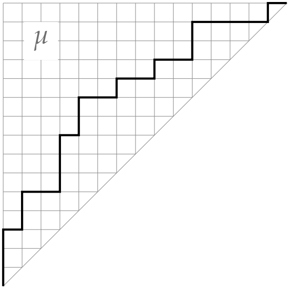

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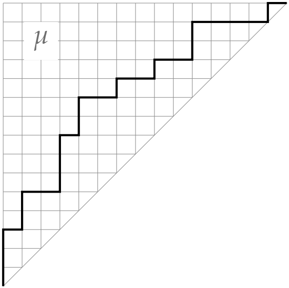

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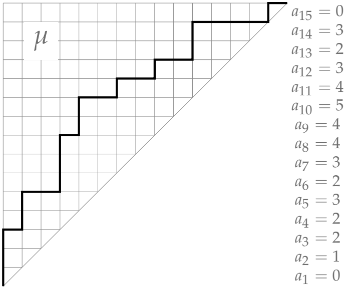

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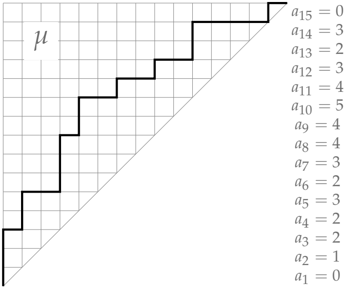

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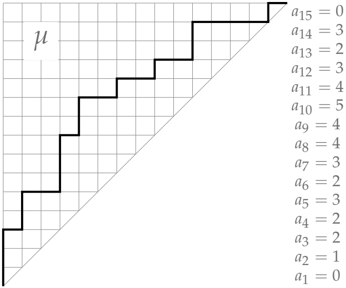

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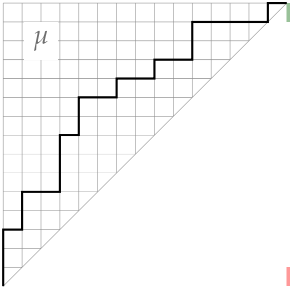

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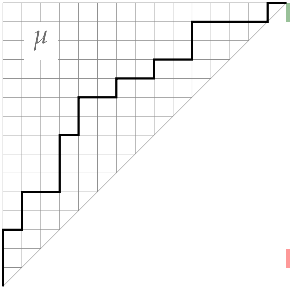

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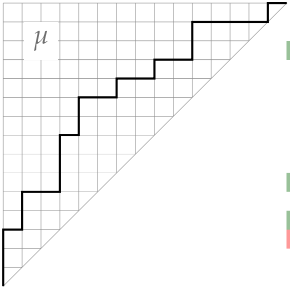

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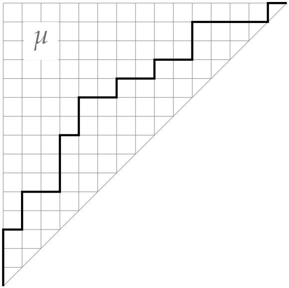

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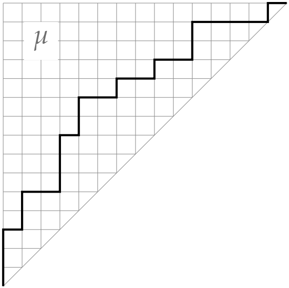

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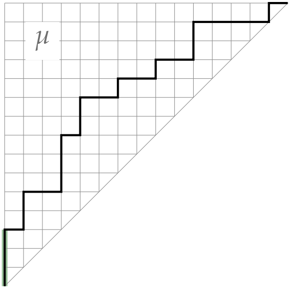

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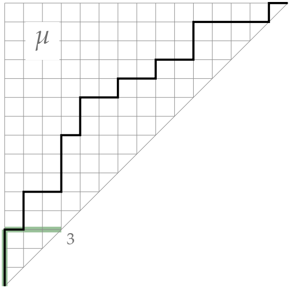

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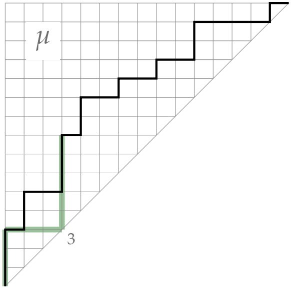

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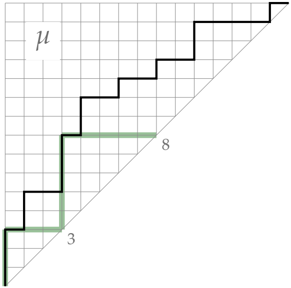

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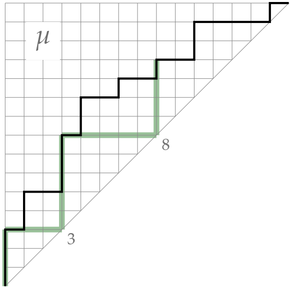

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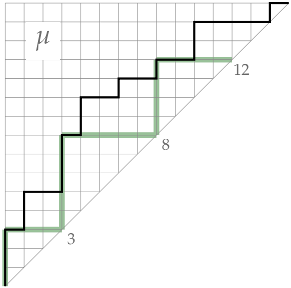

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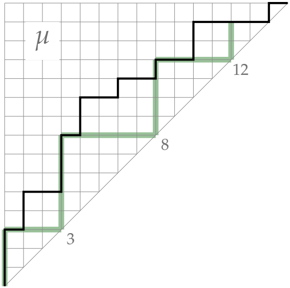

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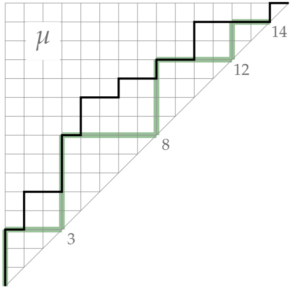

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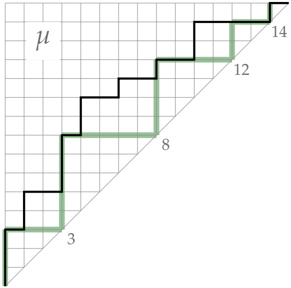

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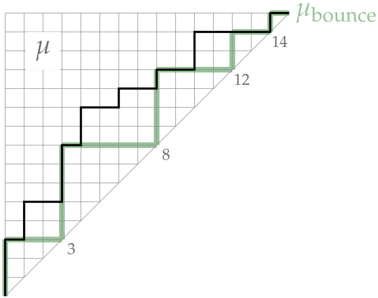

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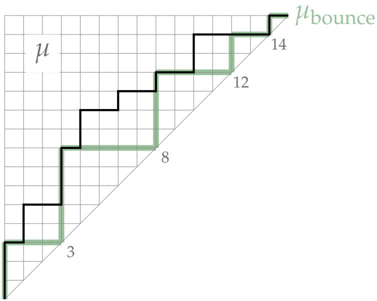
- **bounce parameters**: b_i is i-th contact of μ_{bounce} with diagonal
- **bounce**: $\text{bounce}(\mu) \overset{\text{def}}{=} \sum (n - b_i)$



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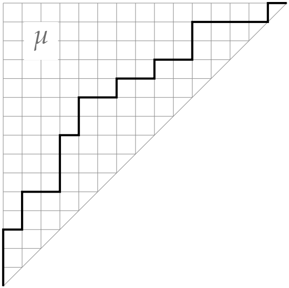

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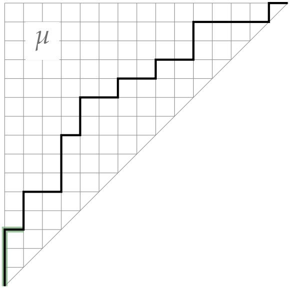

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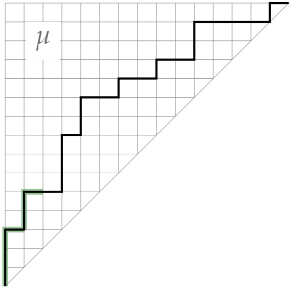

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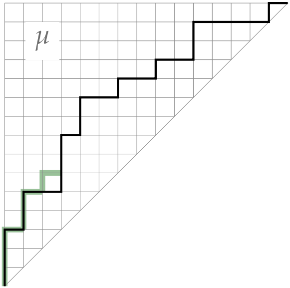

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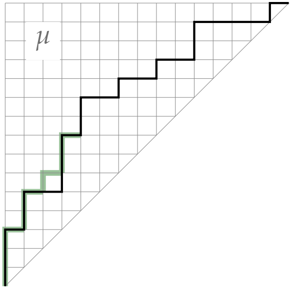

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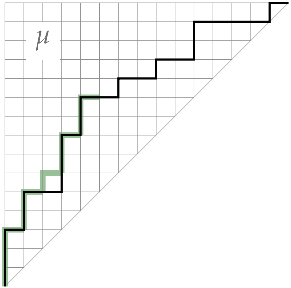

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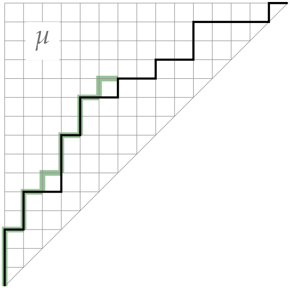

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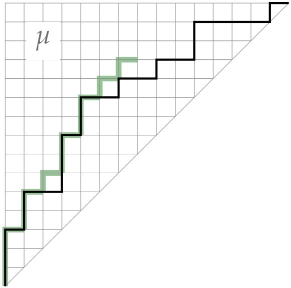

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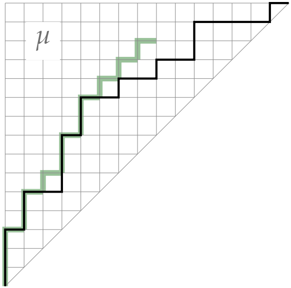

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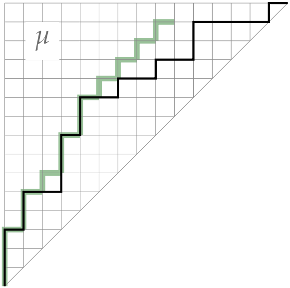

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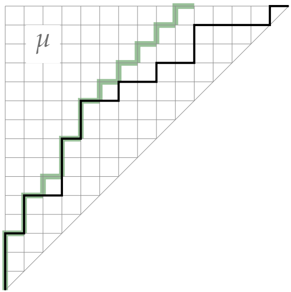

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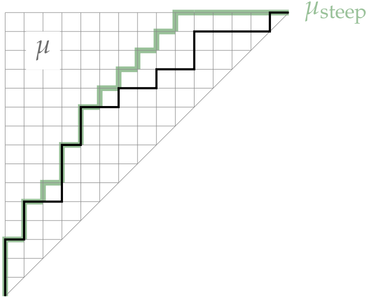

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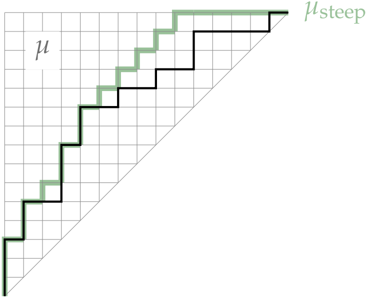
- **steep**: number of east-steps at the end of μ_{steep}



$$\begin{aligned} \text{area}(\mu) &= 39 \\ \text{dinv}(\mu) &= 28 \\ \text{bounce}(\mu) &= 23 \\ \text{steep}(\mu) &= 6 \end{aligned}$$

The Zeta Map

Theorem (A. Garsia, J. Haglund, M. Haiman; 2000s)

For $n \geq 0$, we have

$$\mathcal{H}_n^{\epsilon}(q,t) = \sum_{\mu \in \mathcal{D}_n} q^{\mathrm{area}(\mu)} t^{\mathrm{bounce}(\mu)}.$$

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- the first equality is proven via a detour through q,t-Catalan numbers
- the second equality is proven via an explicit bijection; the **zeta map** ζ

The Steep-Bounce Zeta Map

Theorem (C. Ceballos, W. Fang, ; 2018)

For every $n > 0$ and every $r \in [n]$, there exists an explicit bijection Γ from

- *the set of nested pairs $(\mu_1, \mu_2) \in \mathcal{D}_n^2$, where μ_2 is a steep path ending in r east-steps, to*
- *the set of nested pairs $(\mu_1', \mu_2') \in \mathcal{D}_n^2$, where μ_1' is a bounce path that touches the diagonal $r + 1$ times.*

The Steep-Bounce Zeta Map

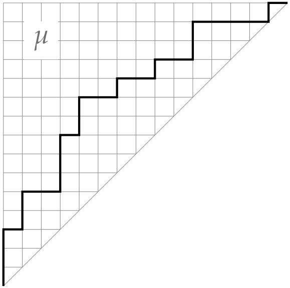
Theorem (C. Ceballos, W. Fang, ; 2018)

For every $n > 0$, the map Γ restricts to a bijection from

- *the set of pairs $(\mu, \mu_{\text{steep}})$, where $\mu \in \mathcal{D}_n$, to*
- *the set of pairs $(\nu_{\text{bounce}}, \nu)$, where $\nu \in \mathcal{D}_n$.*

Moreover, if $(\nu_{\text{bounce}}, \nu) = \Gamma(\mu, \mu_{\text{steep}})$, then $\nu = \zeta(\mu)$.

Recovering the Zeta Map



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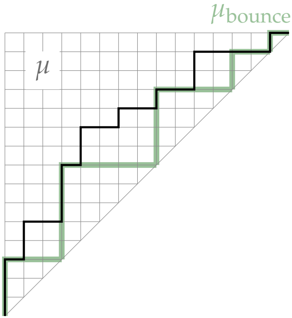

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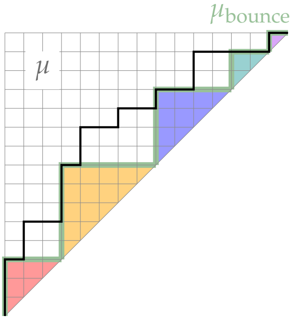

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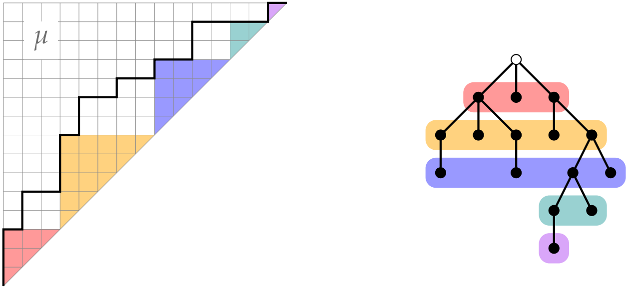

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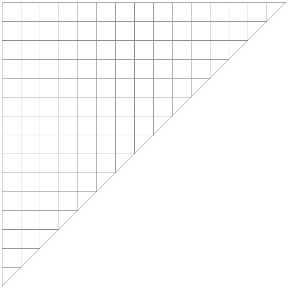

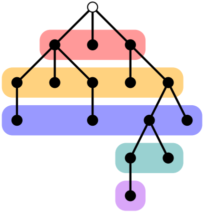

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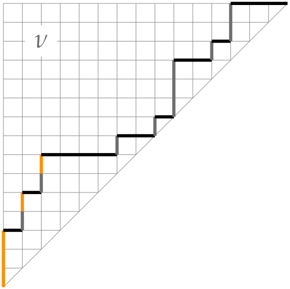

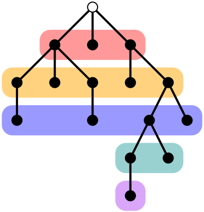

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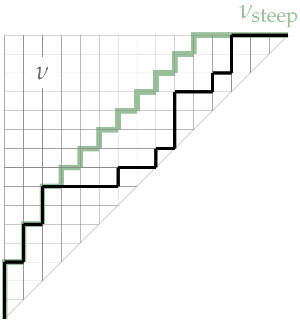

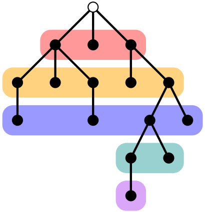

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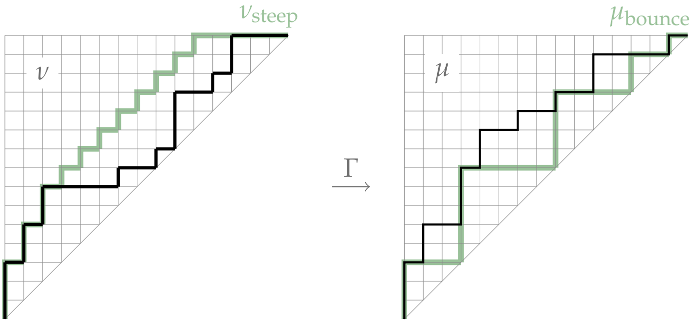

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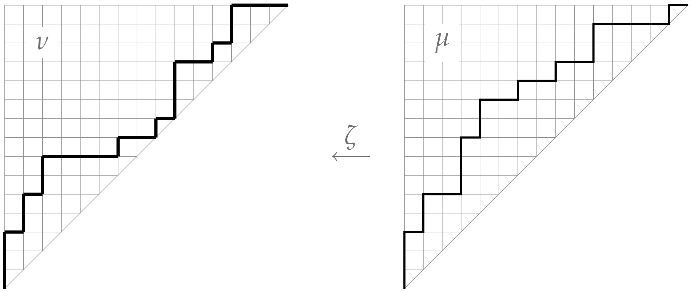

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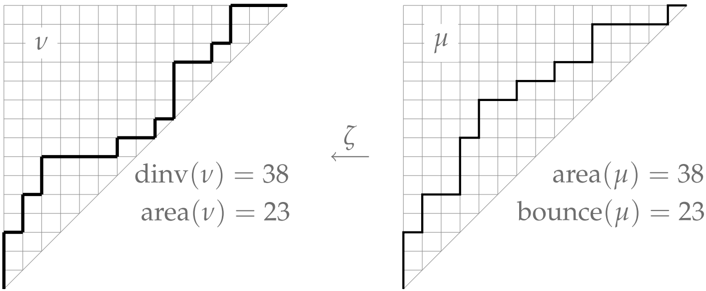

Thank You.

Outline

$\mathfrak{S}_\alpha(231) \cong NC_\alpha$

Theorem (, N. Williams; 2015)

For every composition α, there is an explicit bijection from $\mathfrak{S}_\alpha(231)$ to NC_α.

Back

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$\alpha = (1,3,1,2,4,3,1)$

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12	3	11	13	1	2	5	4	9	10	15	7	8	14	6

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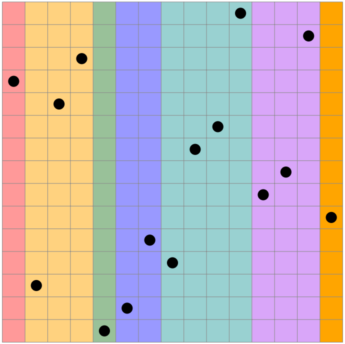

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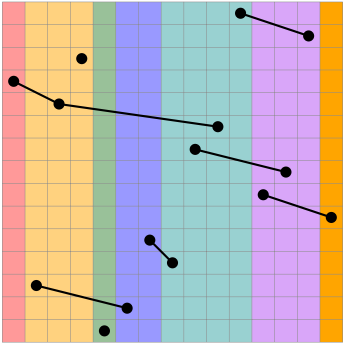

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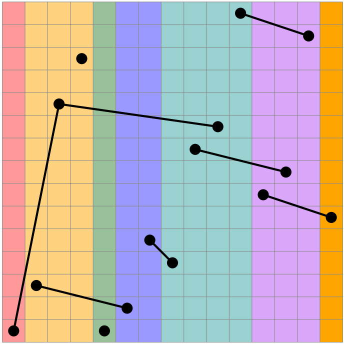

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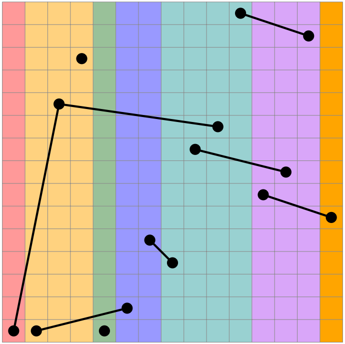

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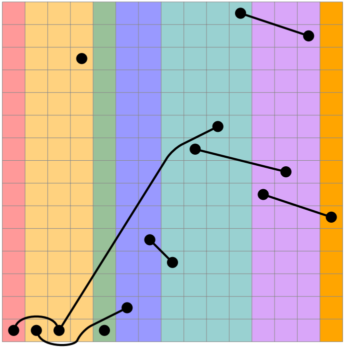

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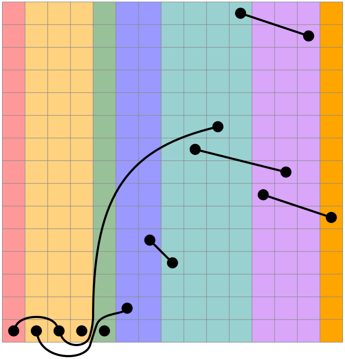

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Theorem (Mühle, N. Williams; 2015)

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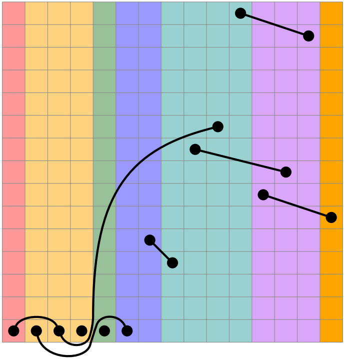

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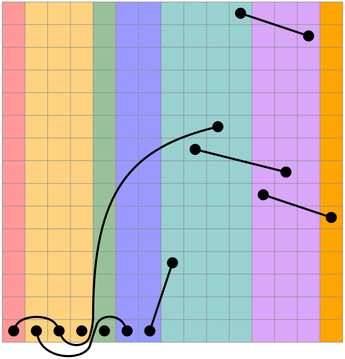

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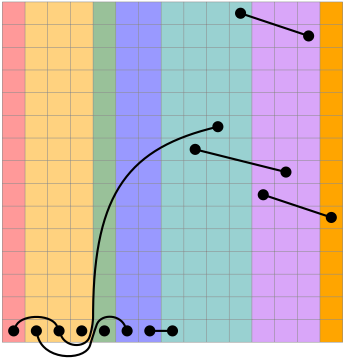

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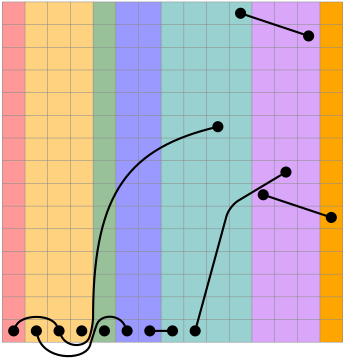

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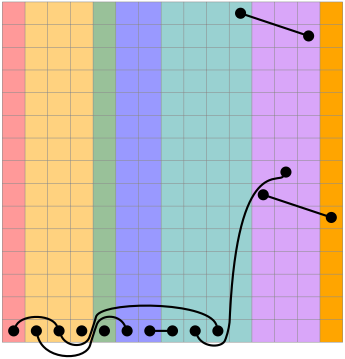

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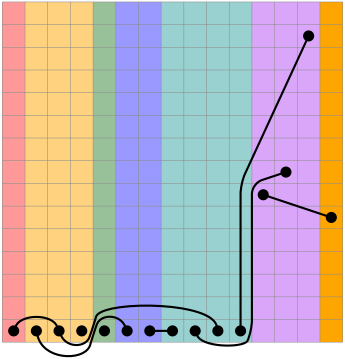

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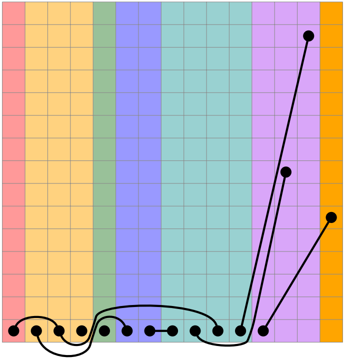

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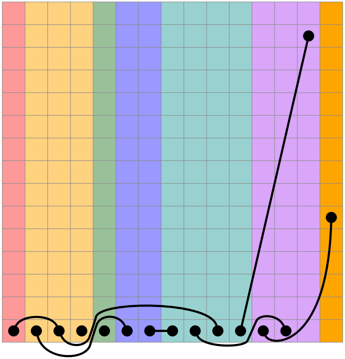

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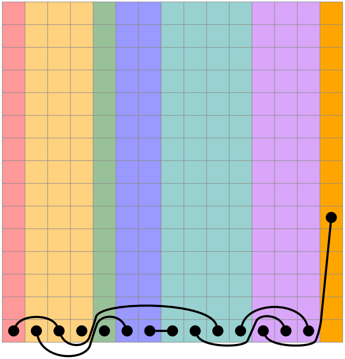

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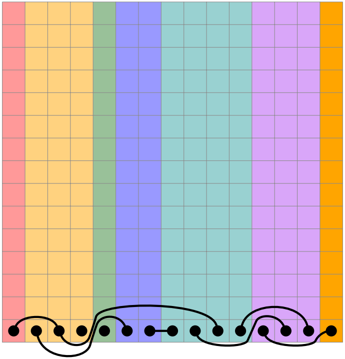

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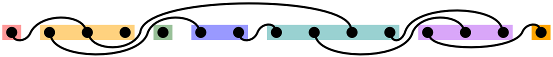

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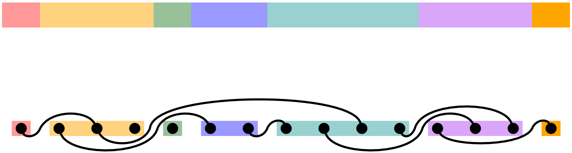

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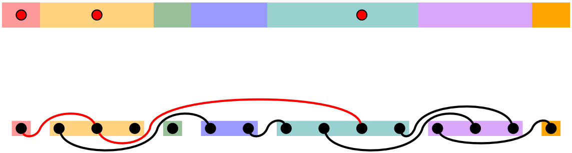

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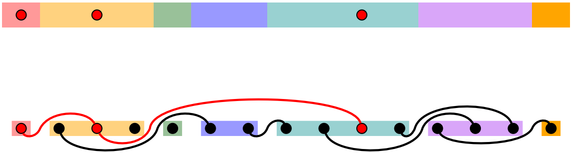

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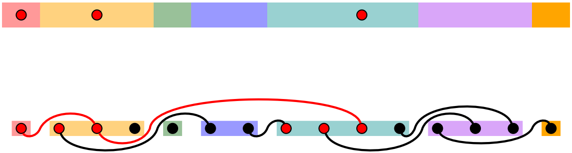

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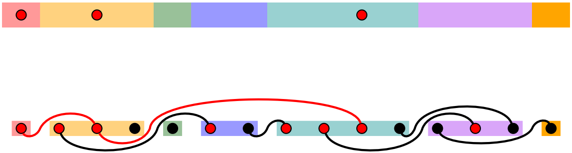

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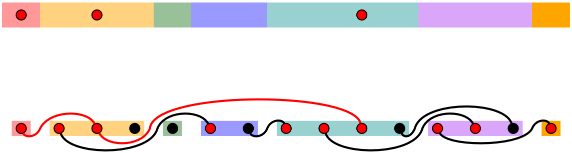

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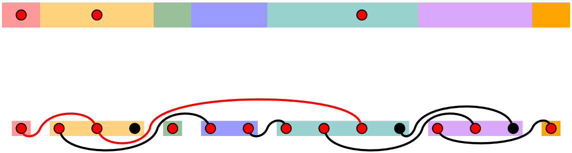

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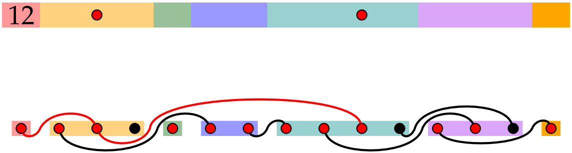

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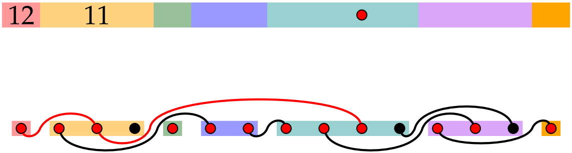

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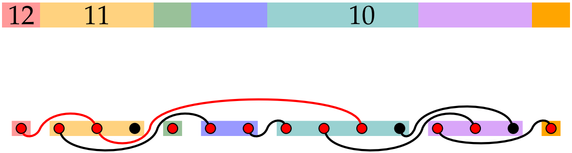

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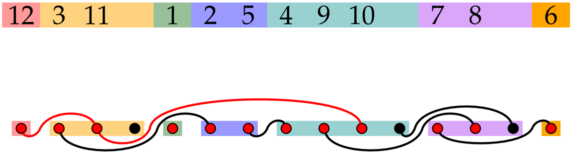

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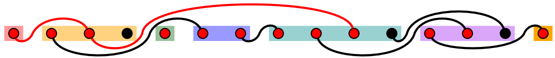

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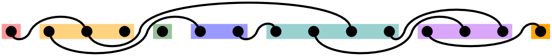

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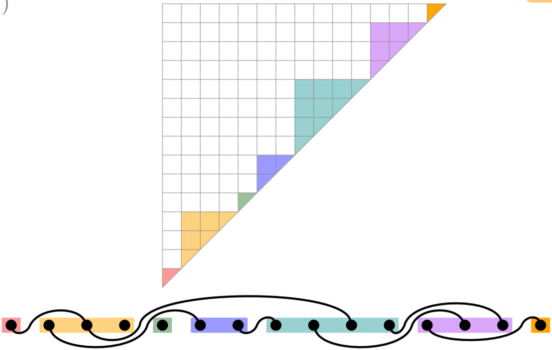

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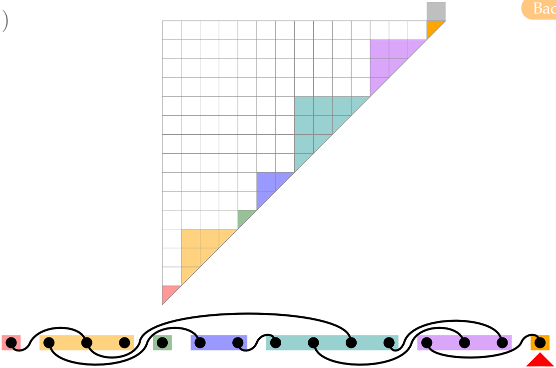

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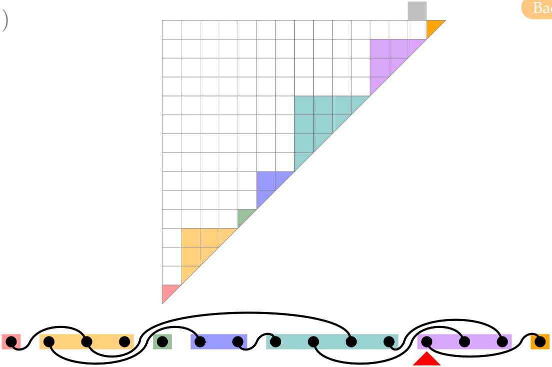

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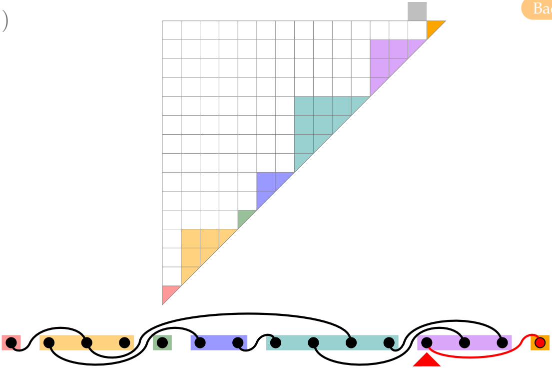

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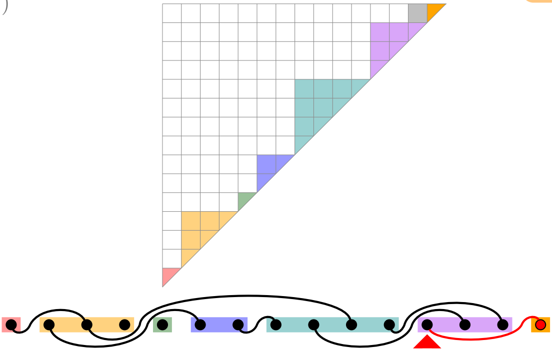

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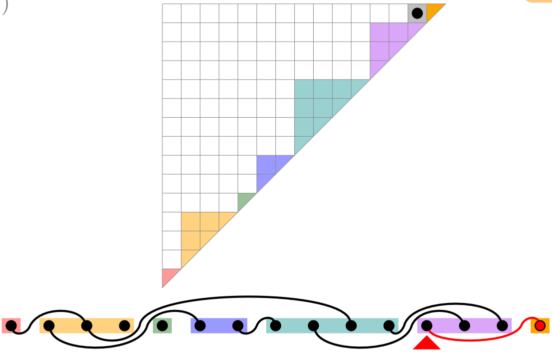

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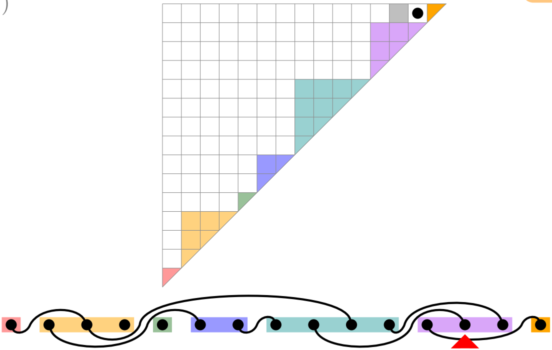

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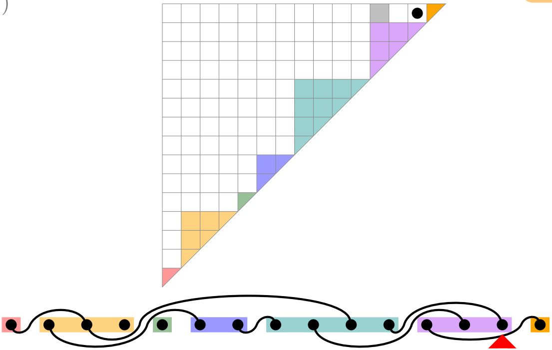

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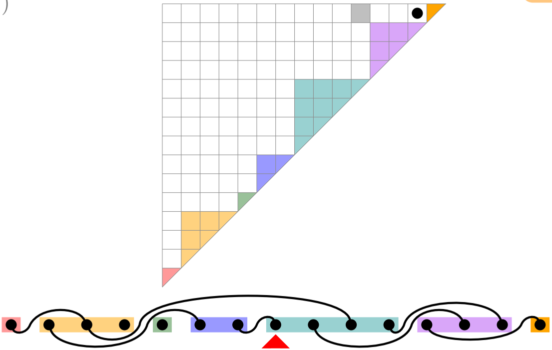

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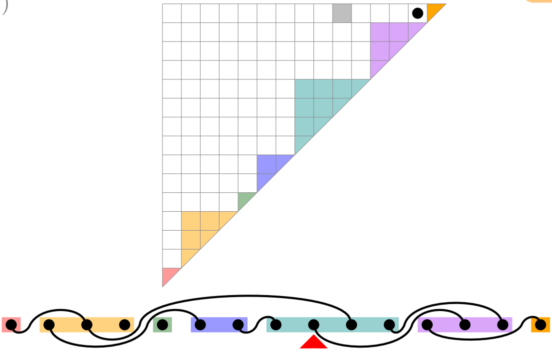

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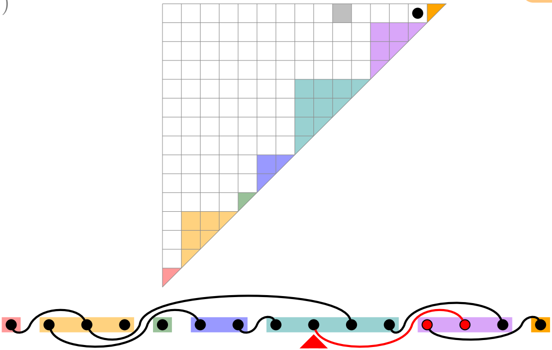

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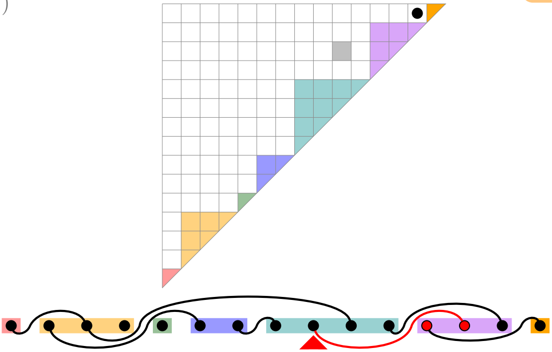

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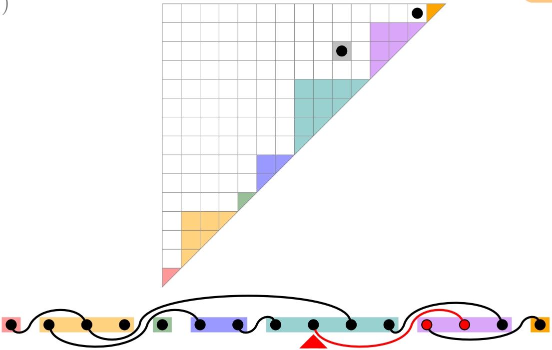

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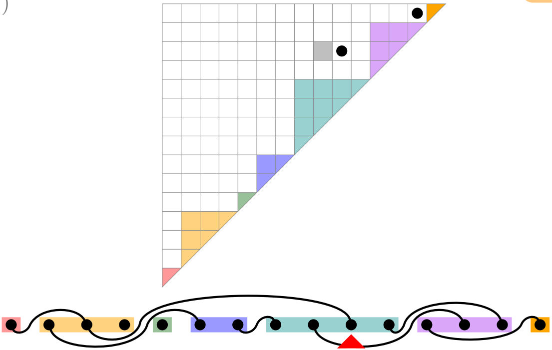

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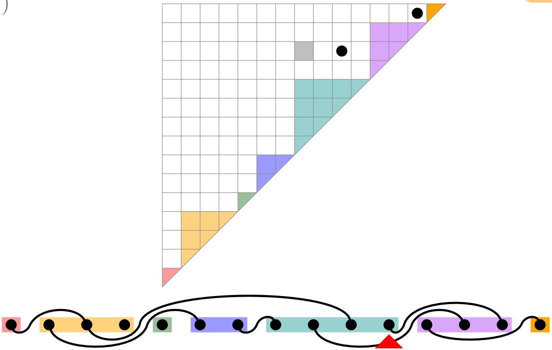

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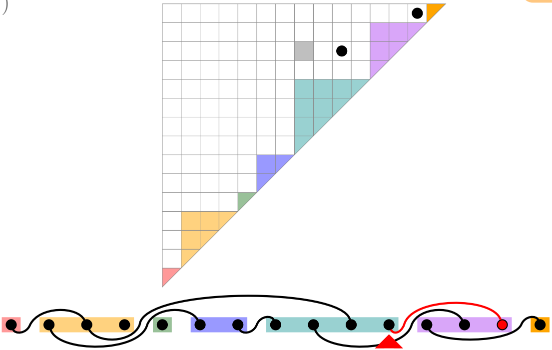

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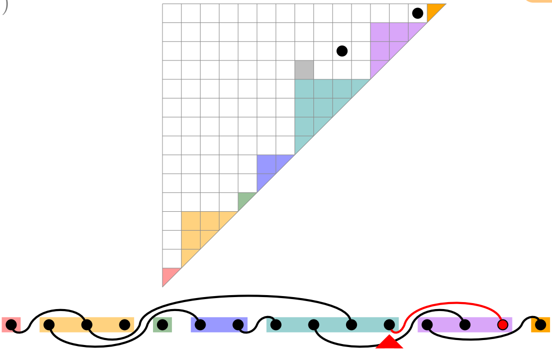

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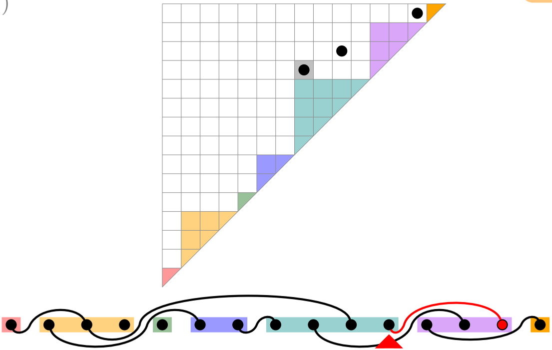

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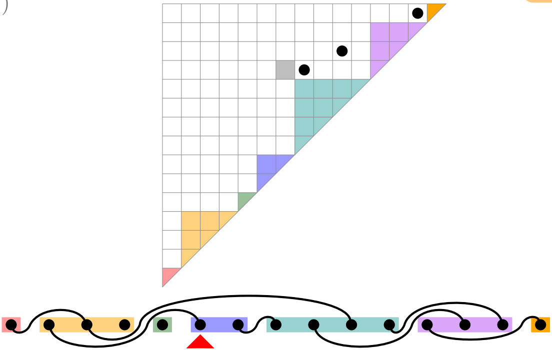

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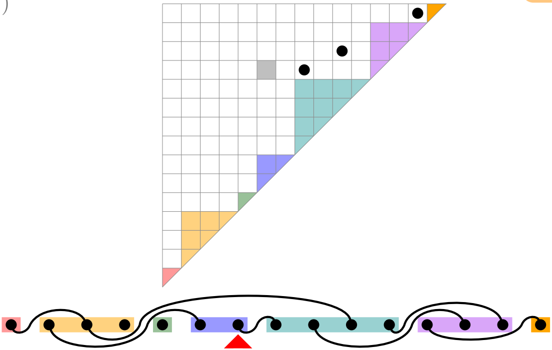

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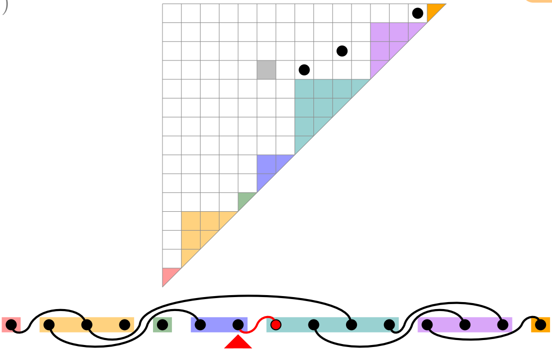

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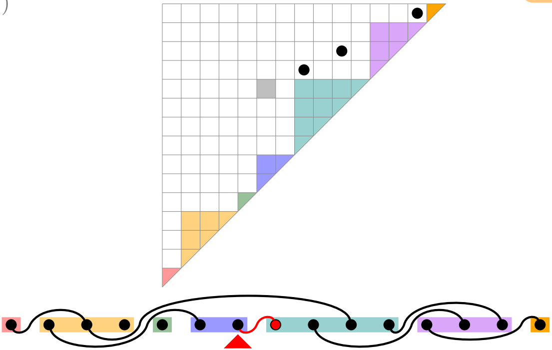

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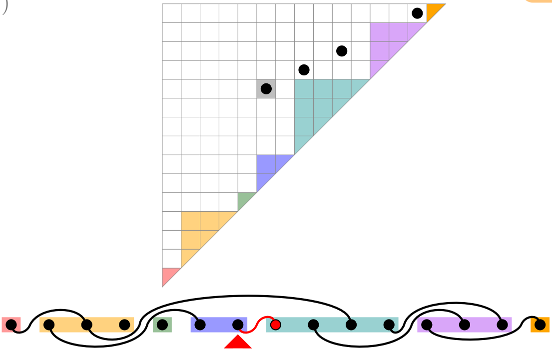

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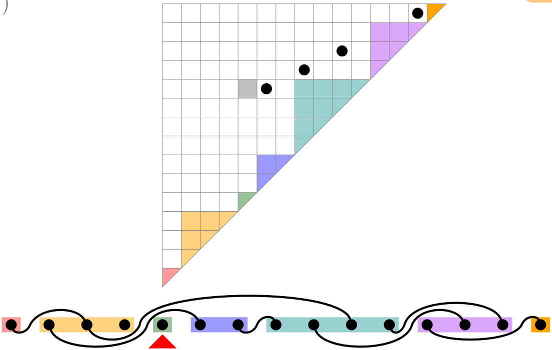

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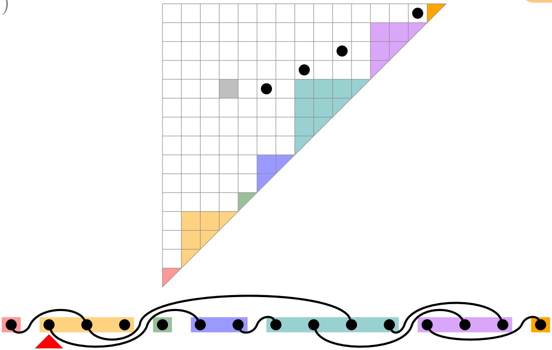

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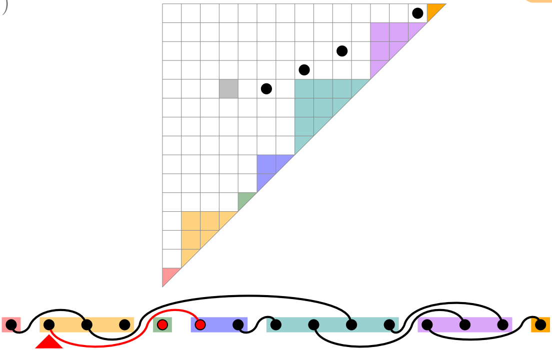

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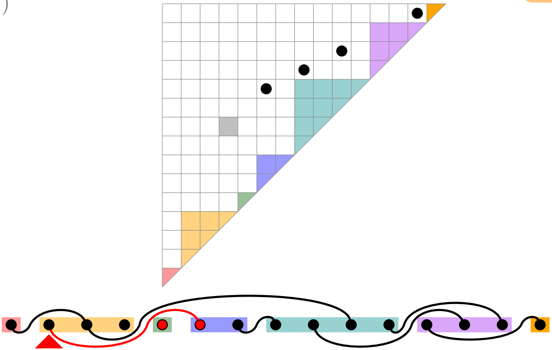

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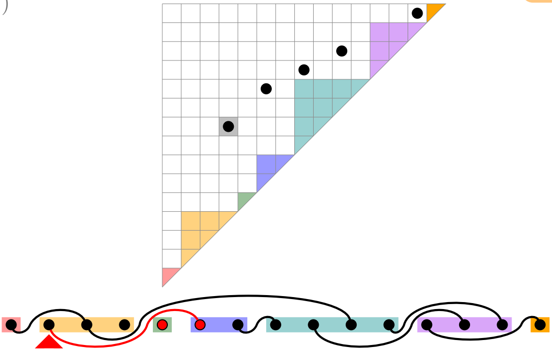

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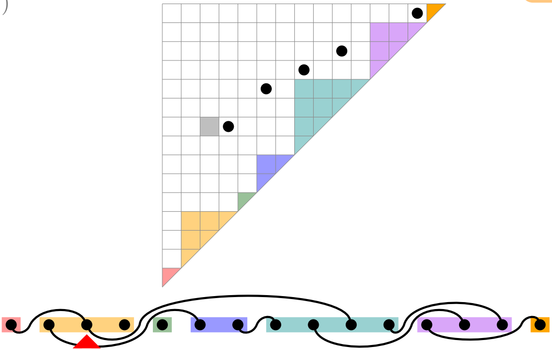

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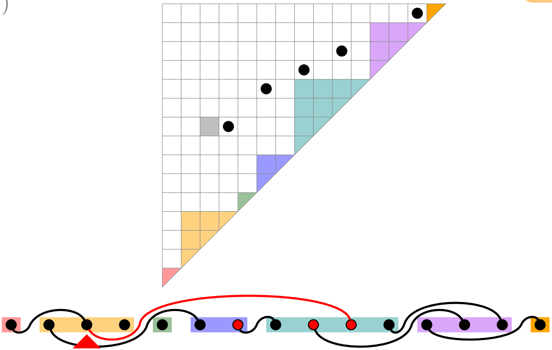

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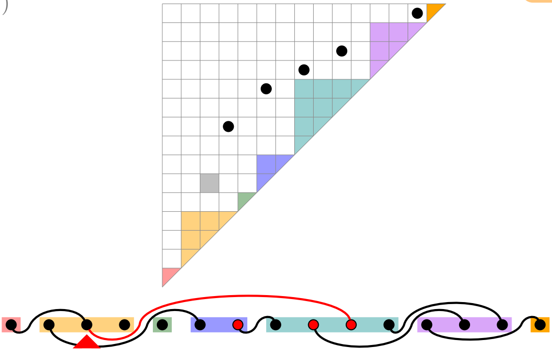

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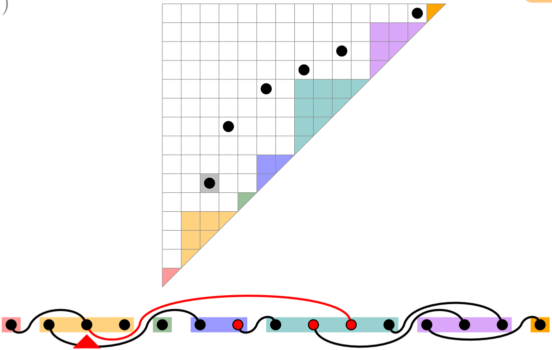

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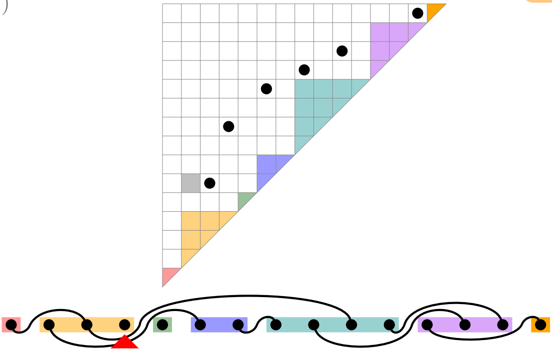

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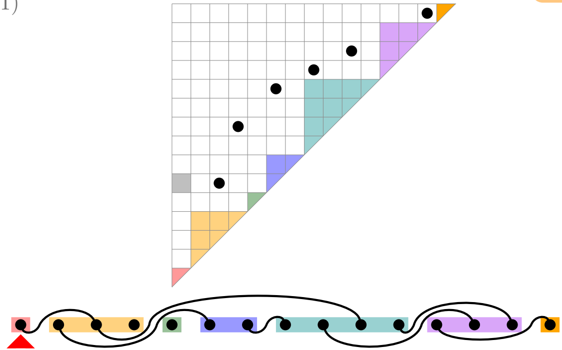

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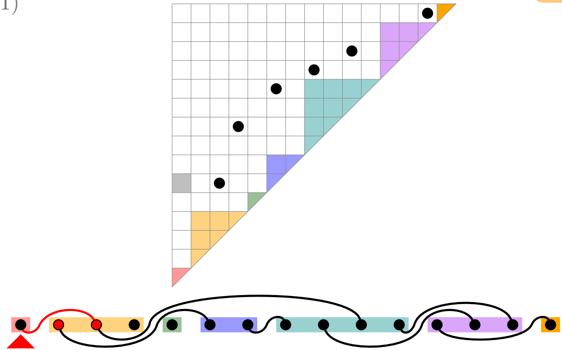

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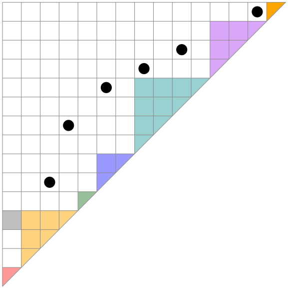

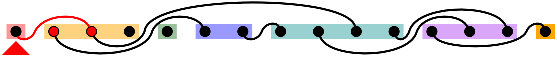

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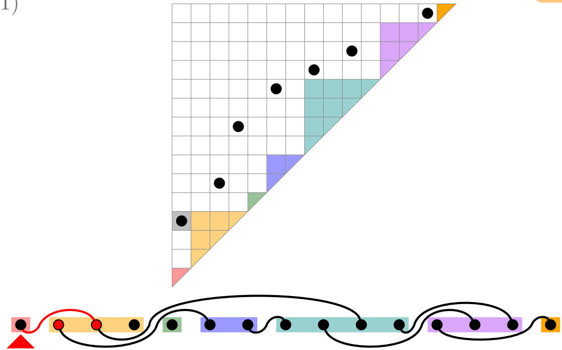

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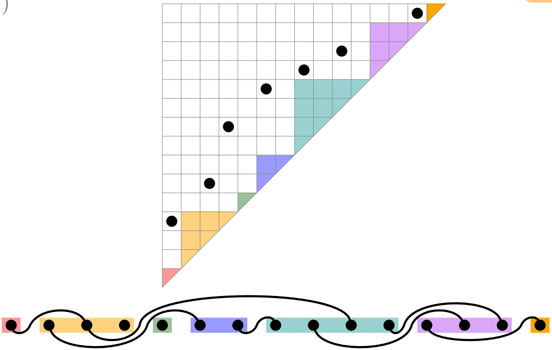

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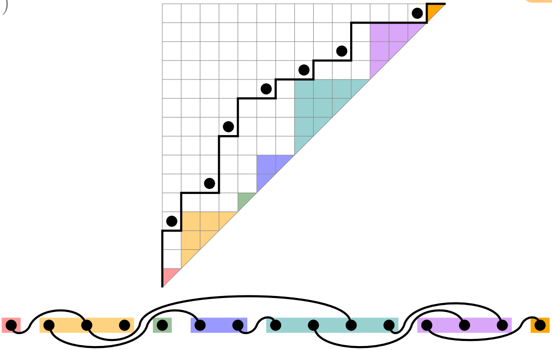

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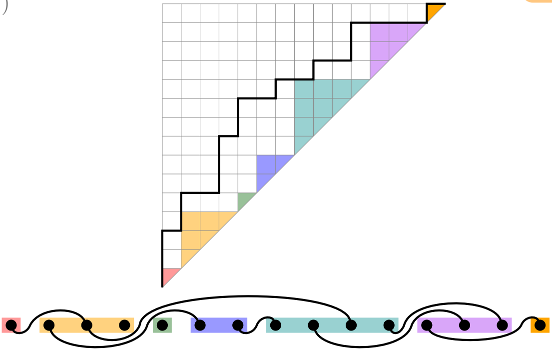

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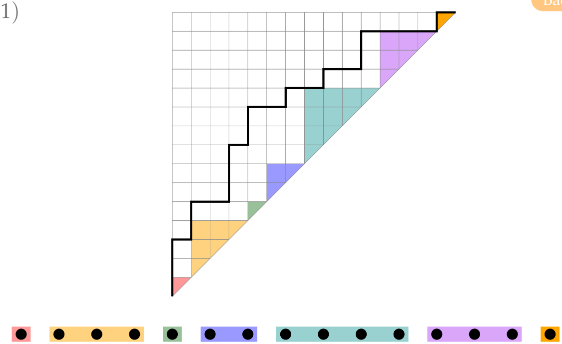

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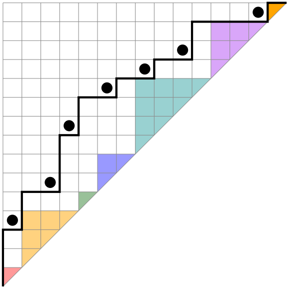

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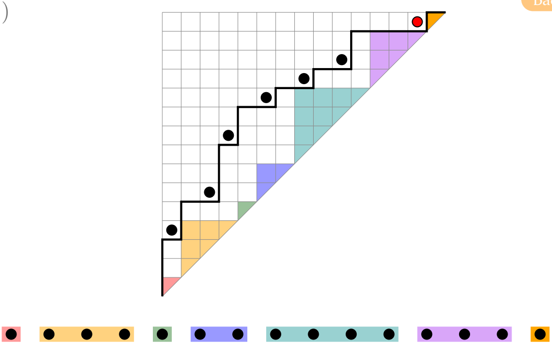

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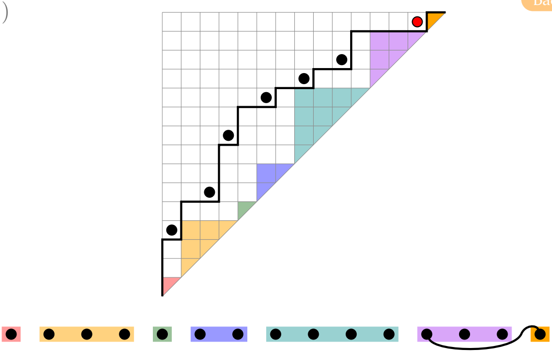

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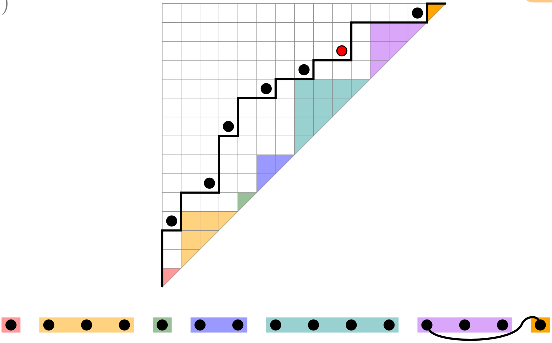

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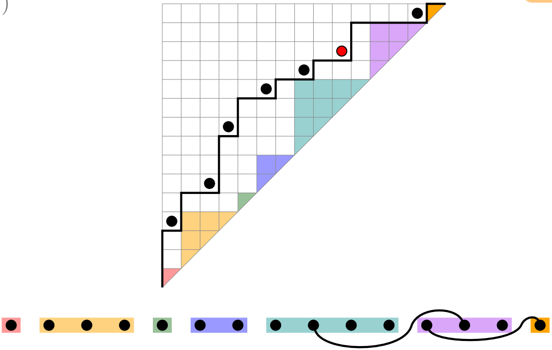

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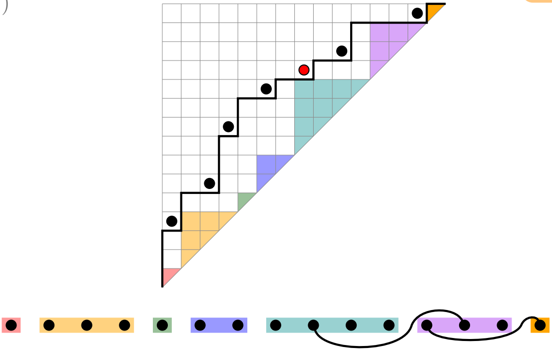

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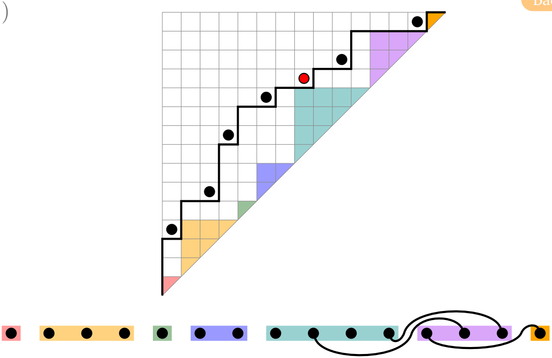

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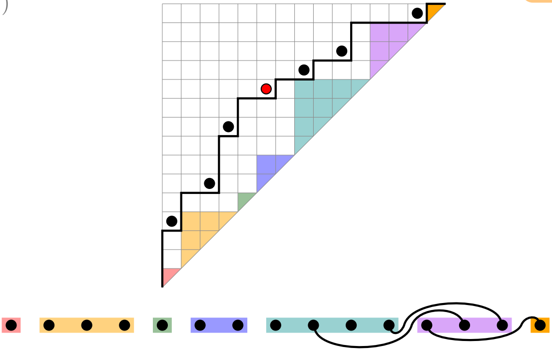

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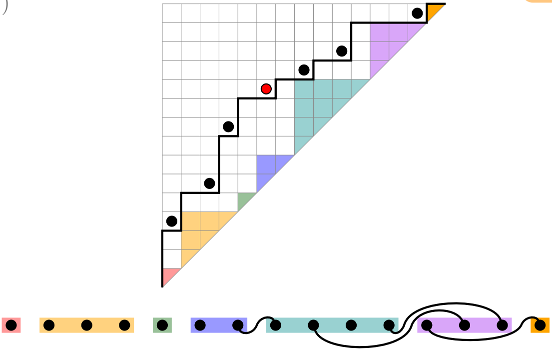

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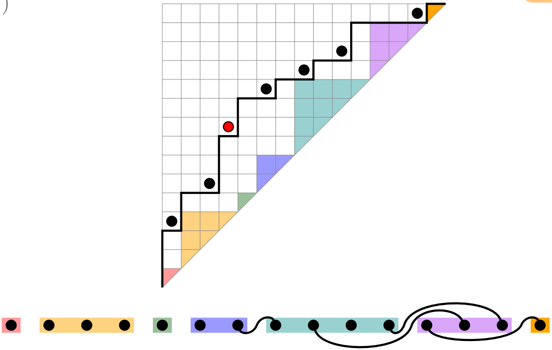

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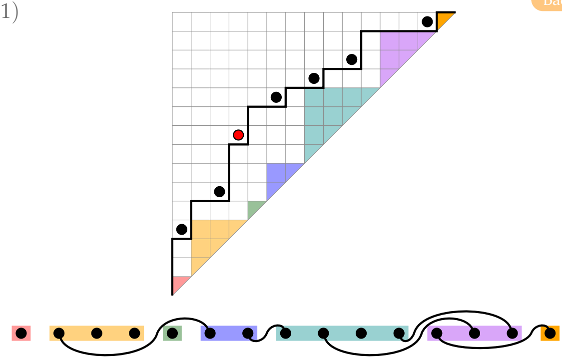

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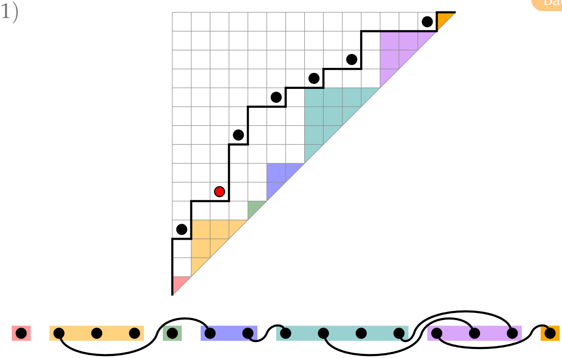

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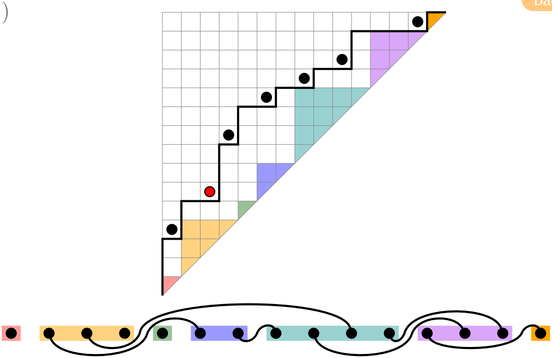

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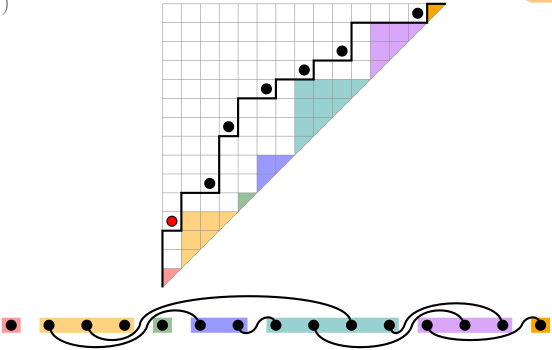

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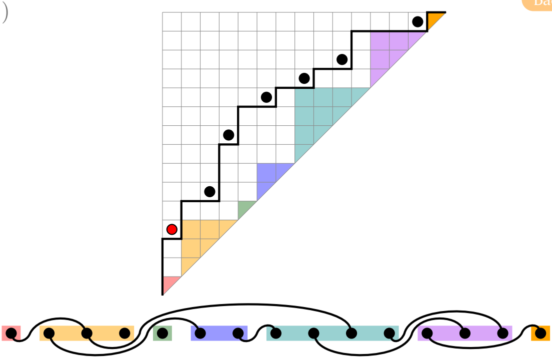

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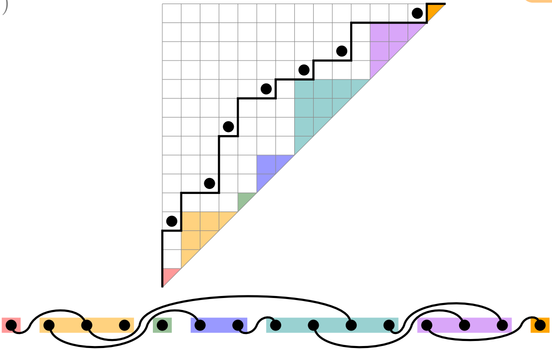

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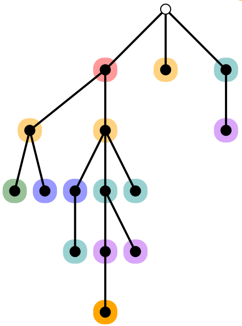

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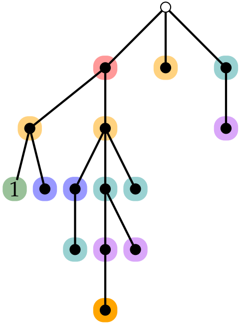

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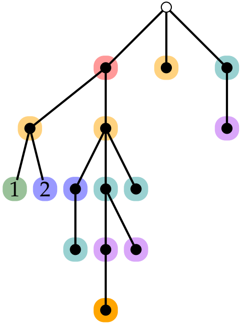

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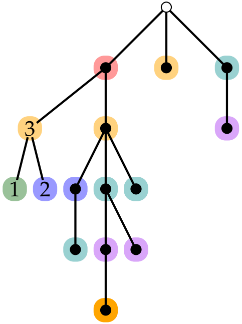

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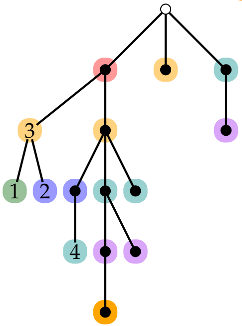

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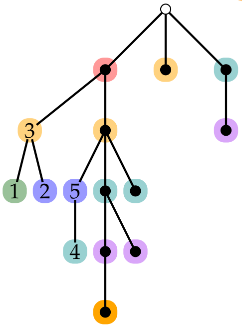

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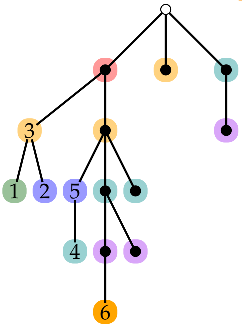

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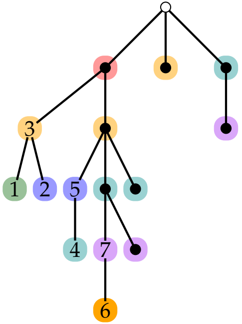

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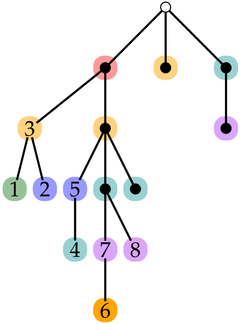

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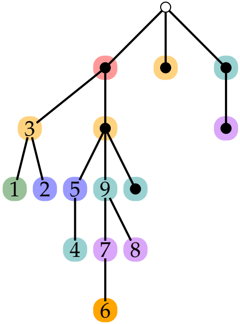

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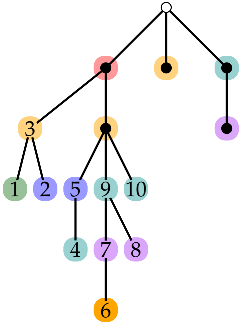

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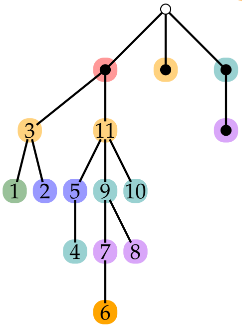

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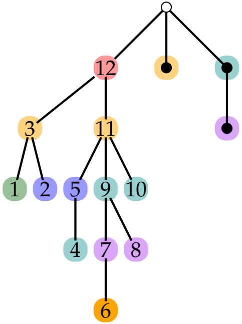

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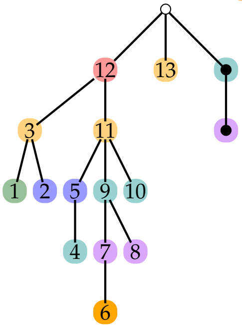

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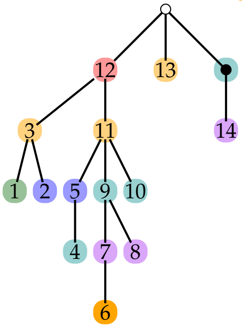

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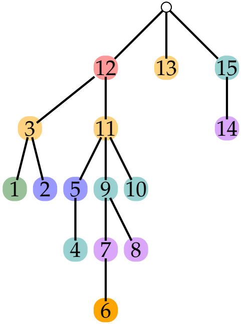

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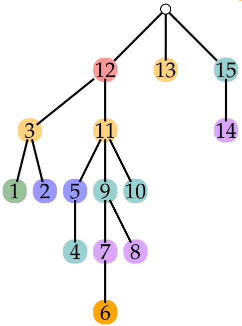

12

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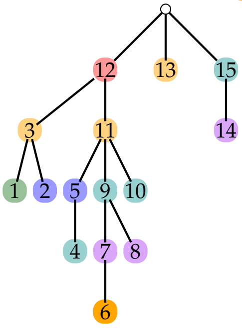

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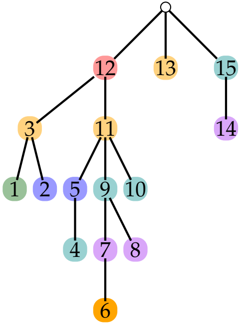

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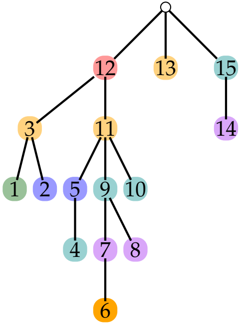

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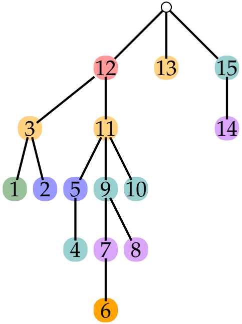

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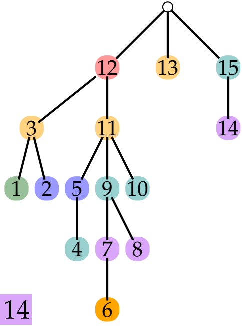

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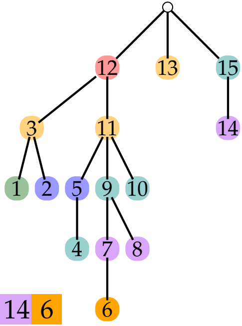

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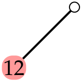

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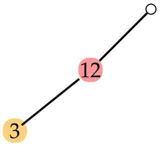

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Parabolic Cataland

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Bijections in Parabolic Cataland

Chapoton Triangles in Parabolic Cataland

The Ballot Case

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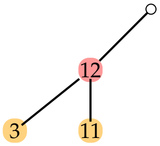

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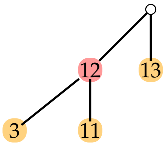

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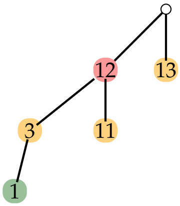

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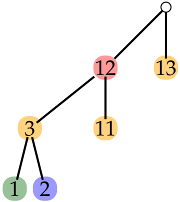

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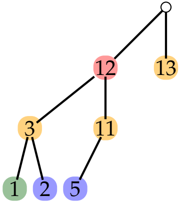

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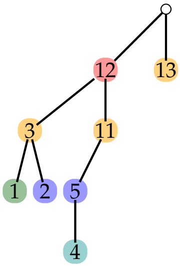

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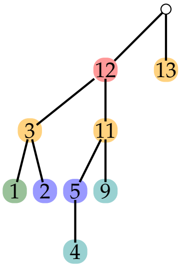

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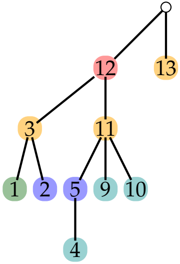

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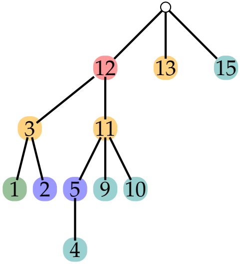

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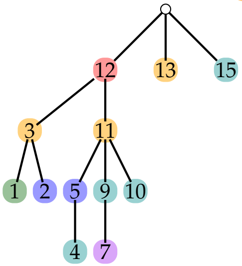

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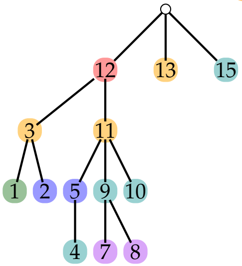

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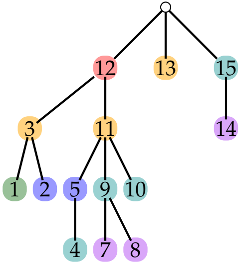

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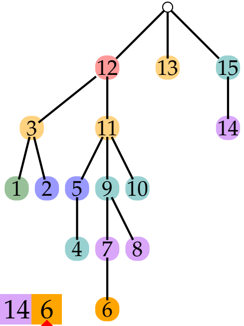

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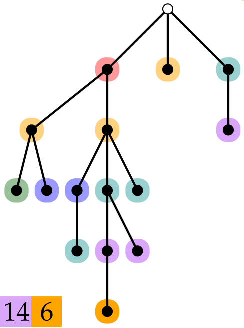

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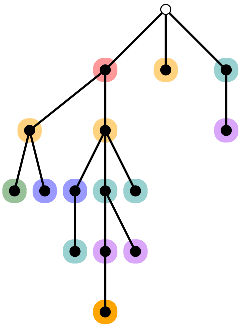

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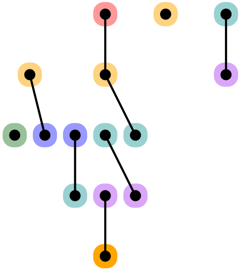

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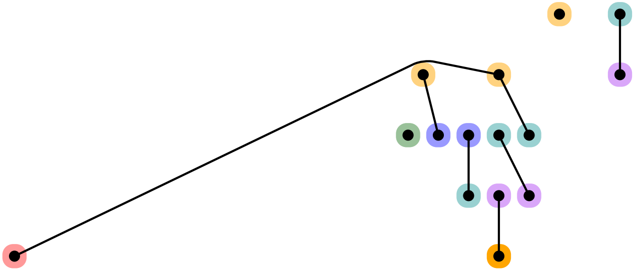

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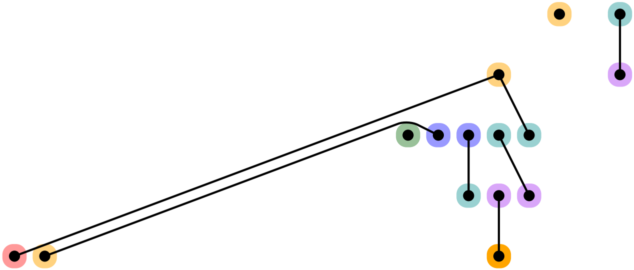

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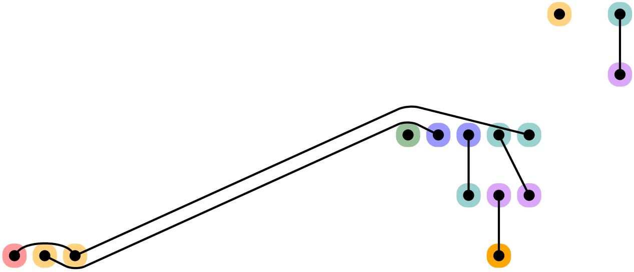

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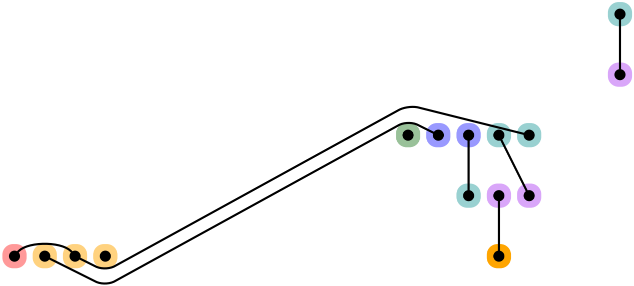

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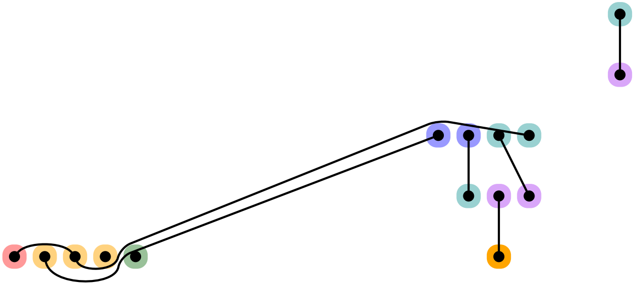

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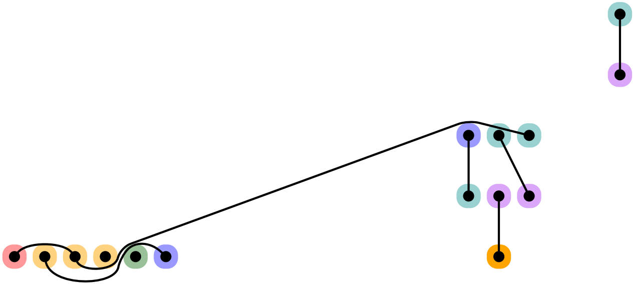

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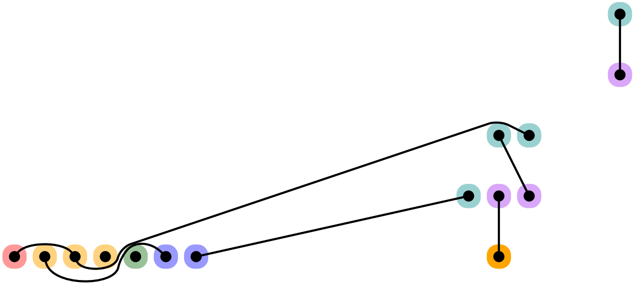

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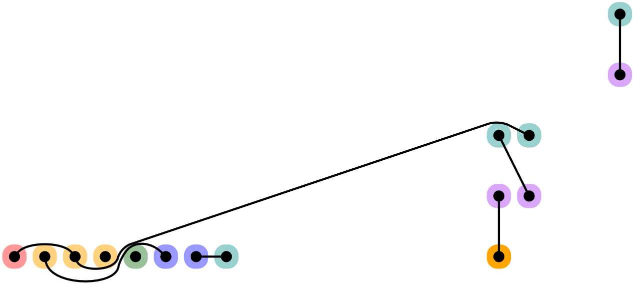

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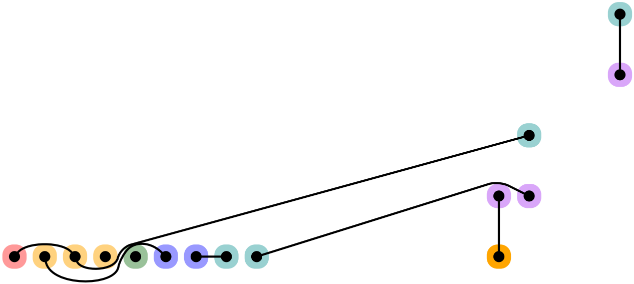

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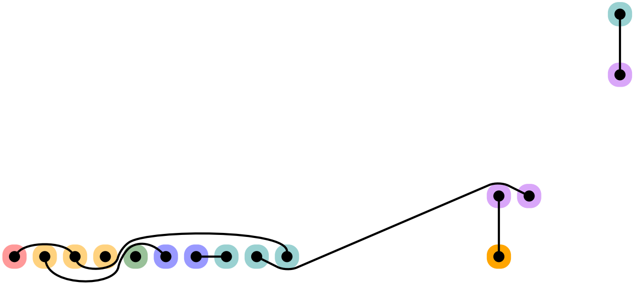

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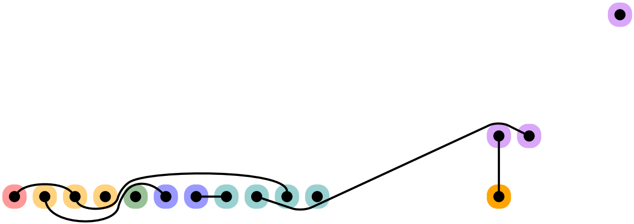

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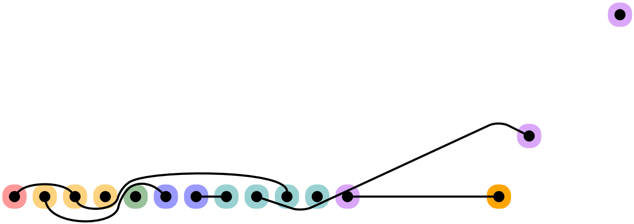

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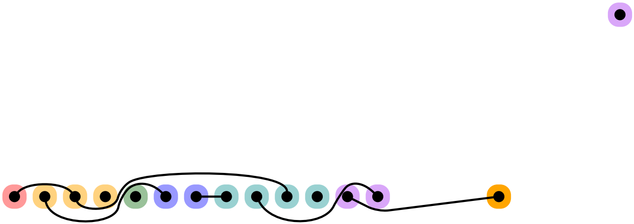

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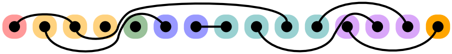

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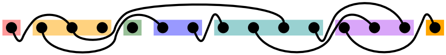

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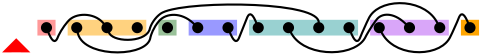

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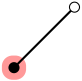

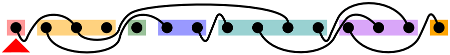

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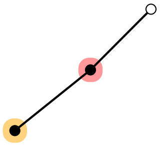

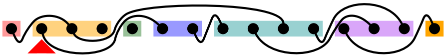

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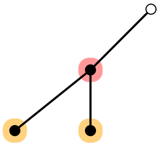

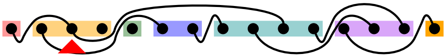

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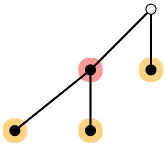

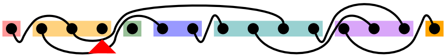

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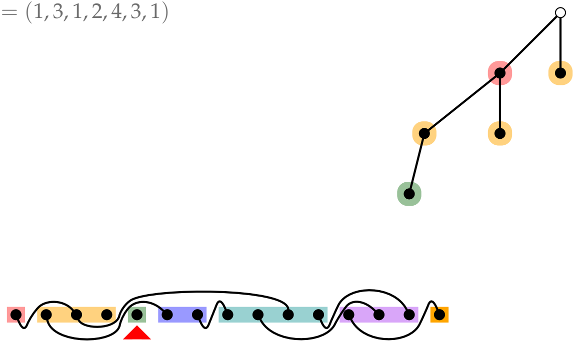

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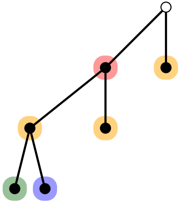

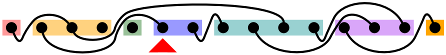

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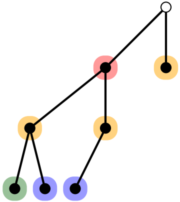

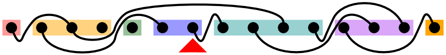

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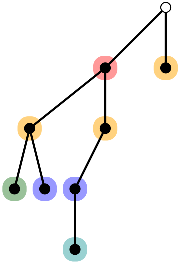

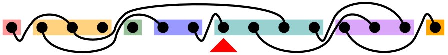

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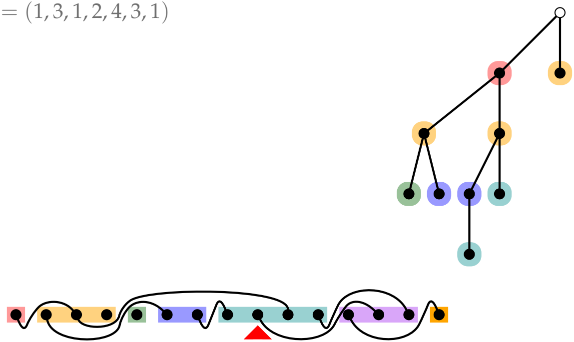

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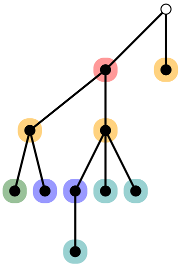

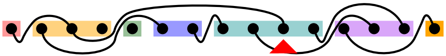

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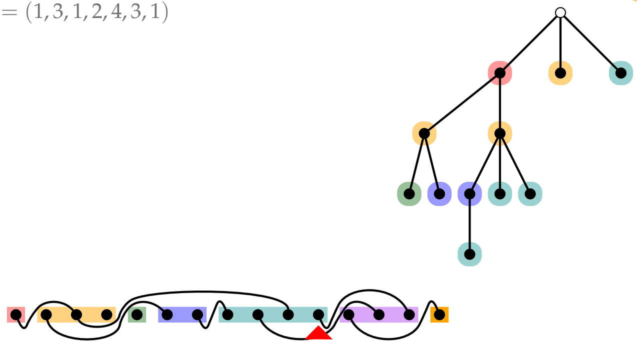

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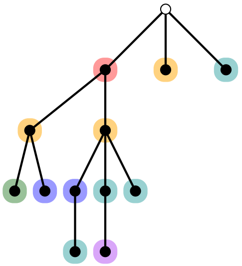

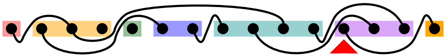

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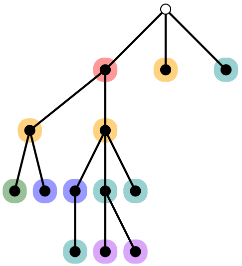

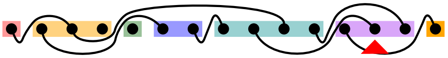

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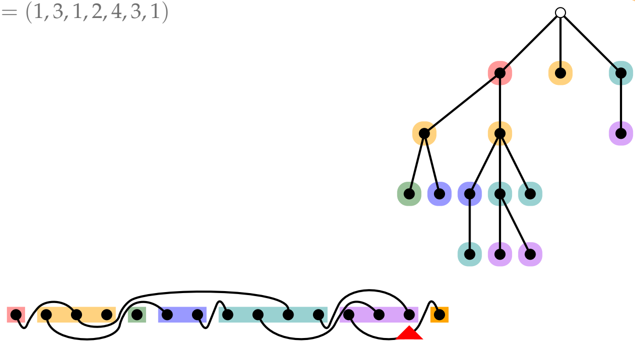

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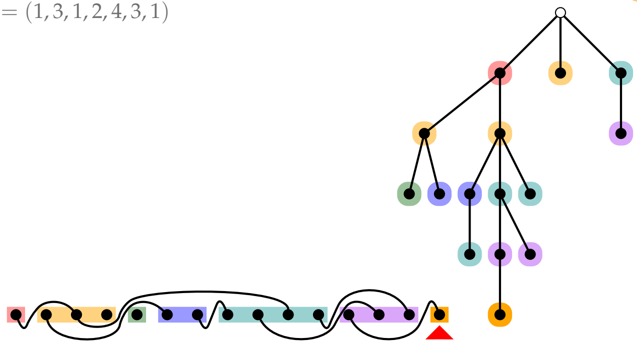

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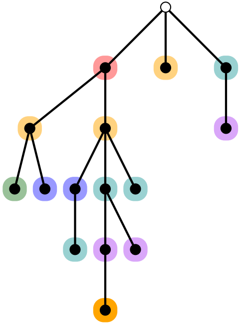

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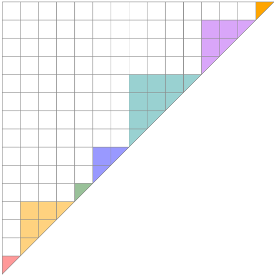

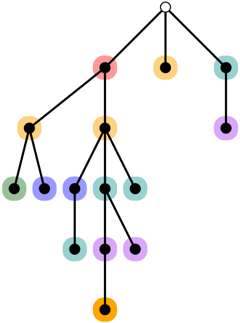

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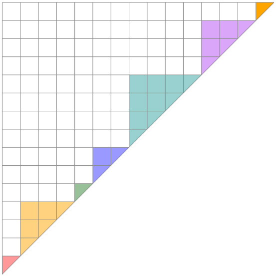

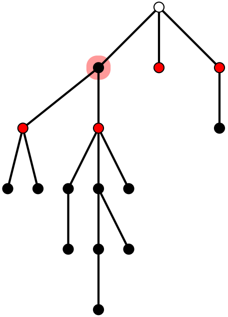

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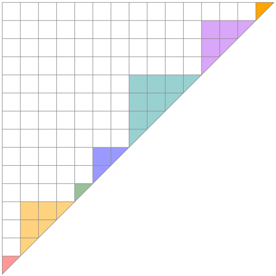

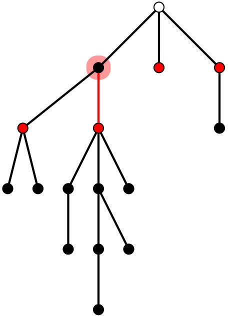

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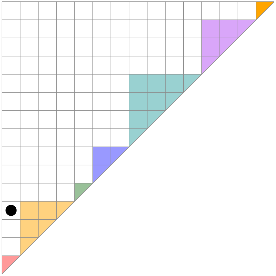

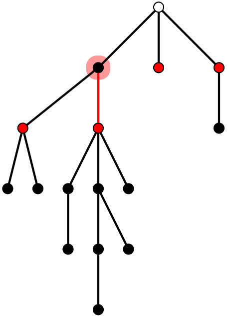

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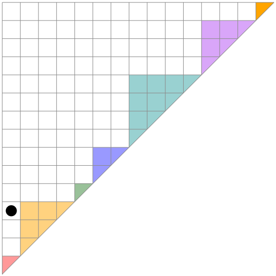

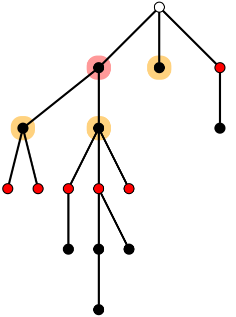

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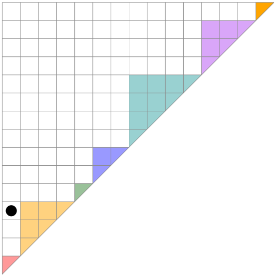

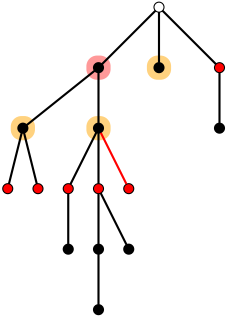

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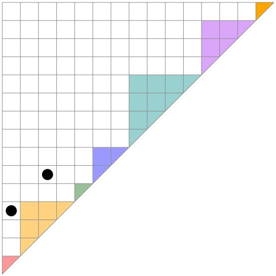

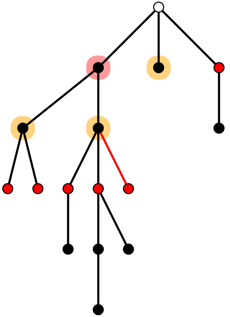

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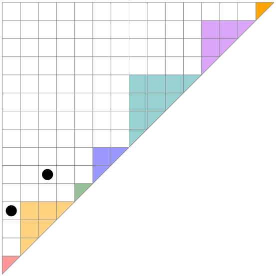

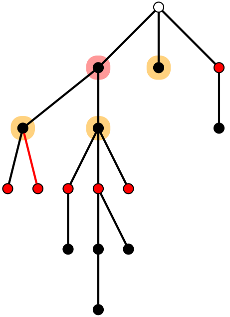

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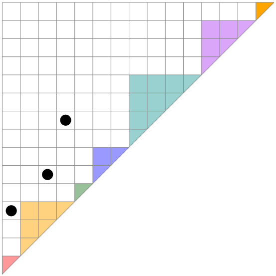

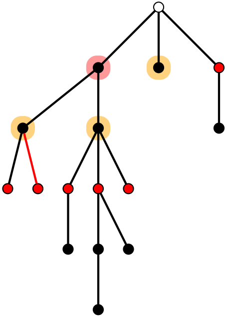

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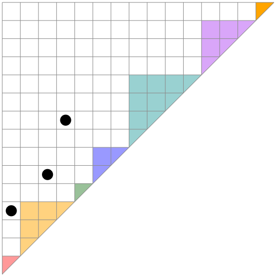

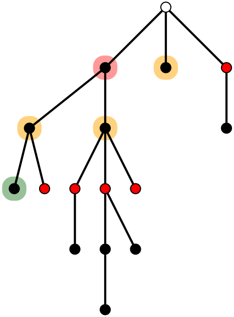

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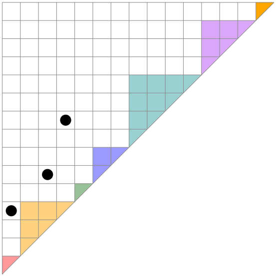

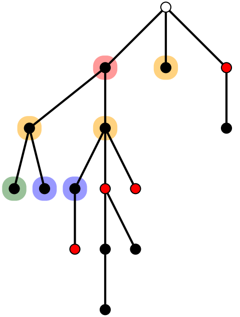

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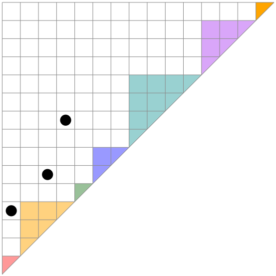

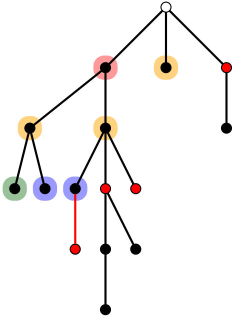

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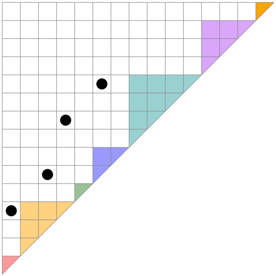

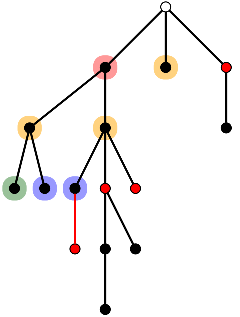

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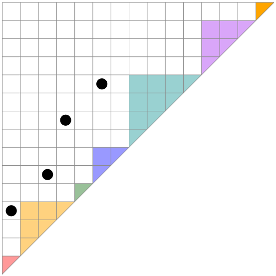

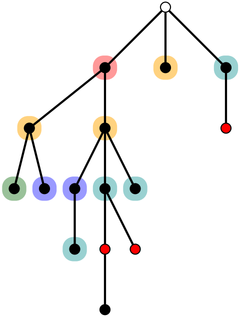

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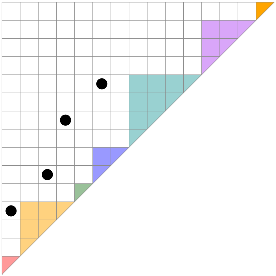

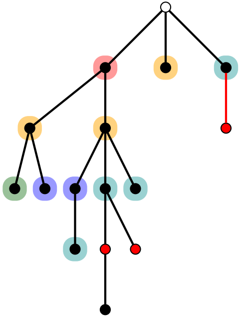

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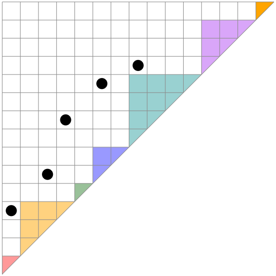

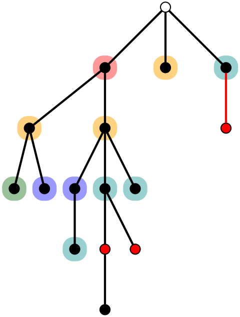

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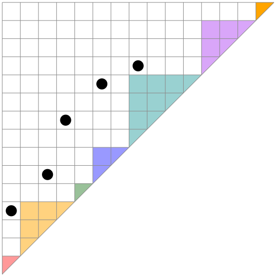

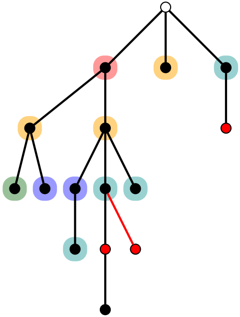

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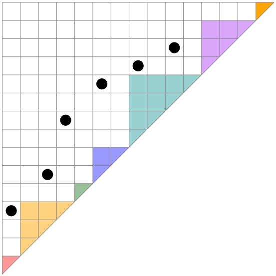

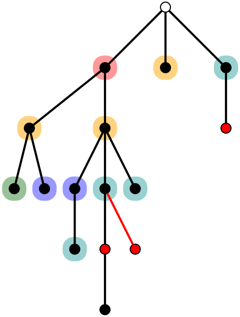

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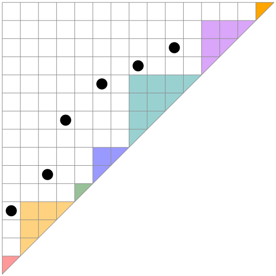

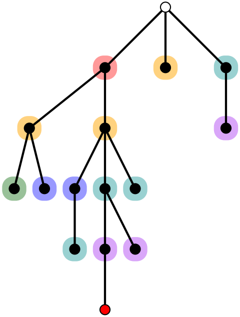

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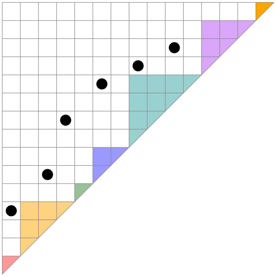

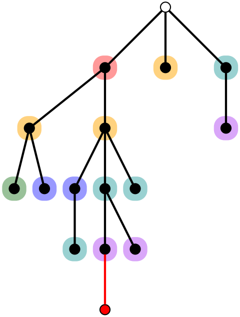

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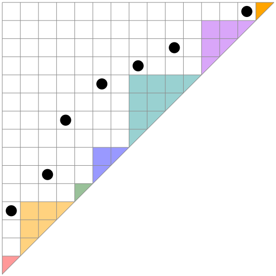

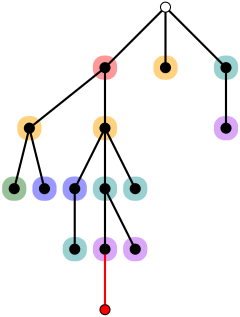

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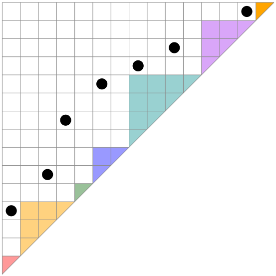

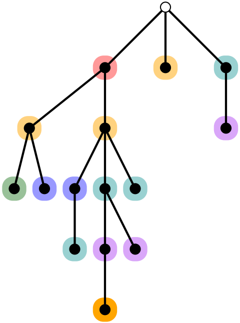

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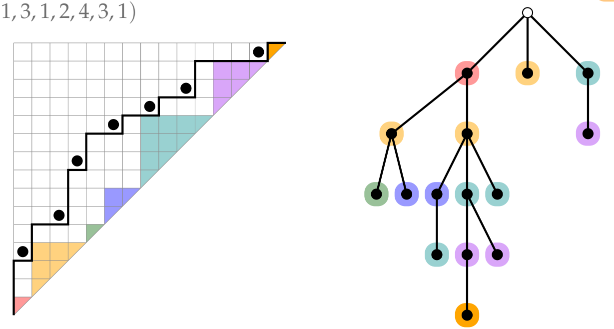

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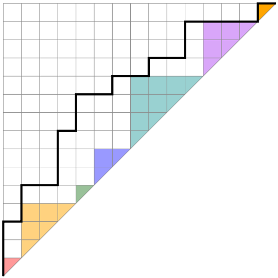

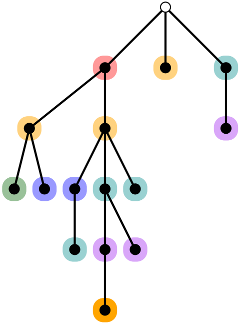

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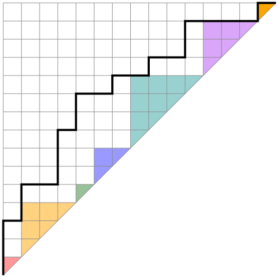

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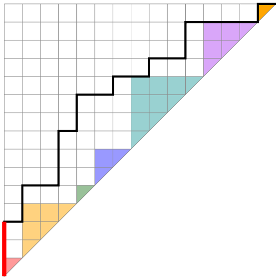

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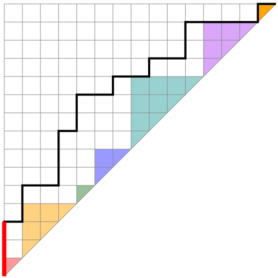

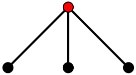

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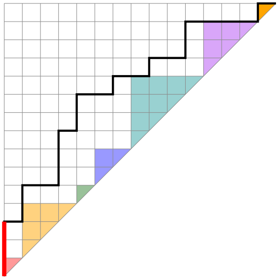

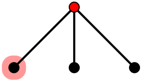

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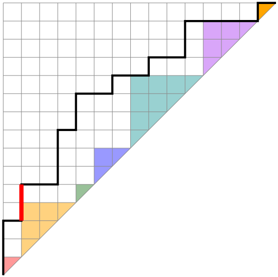

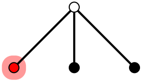

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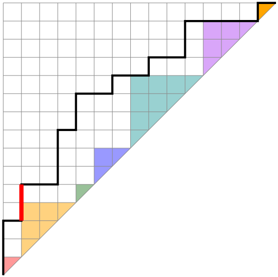

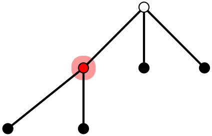

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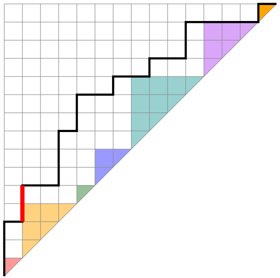

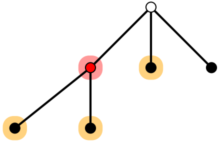

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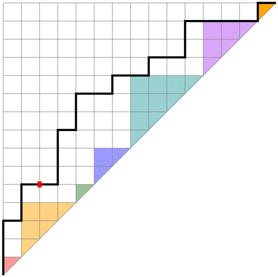

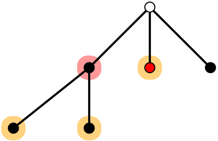

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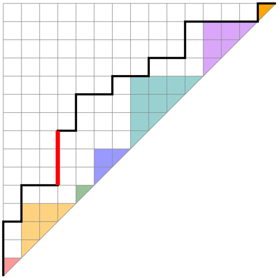

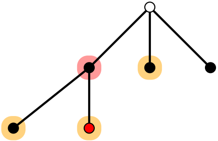

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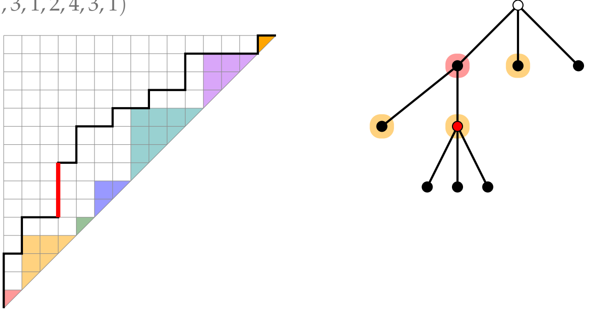

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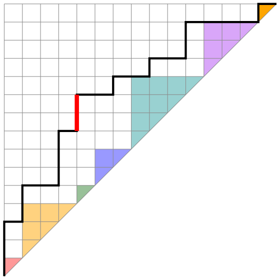

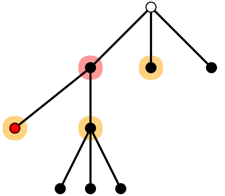

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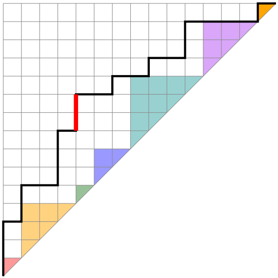

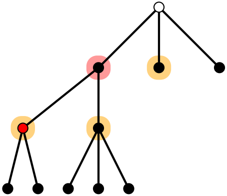

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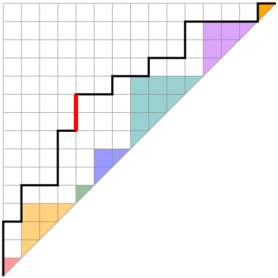

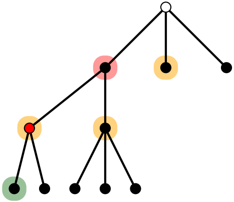

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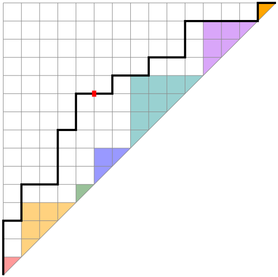

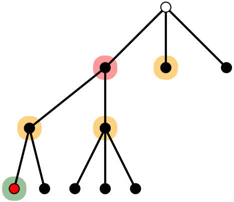

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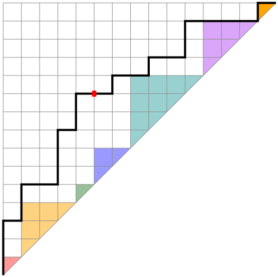

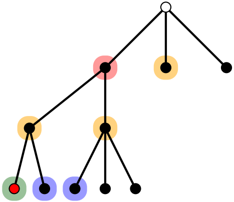

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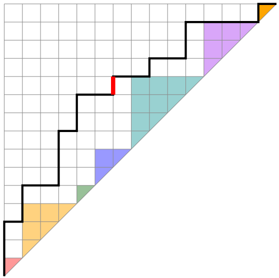

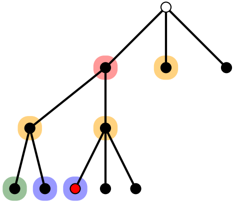

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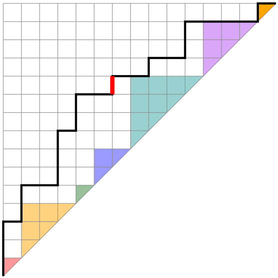

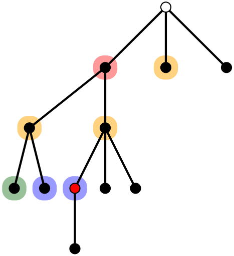

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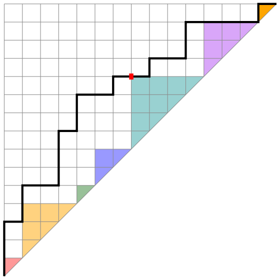

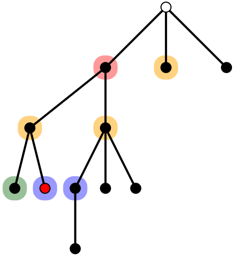

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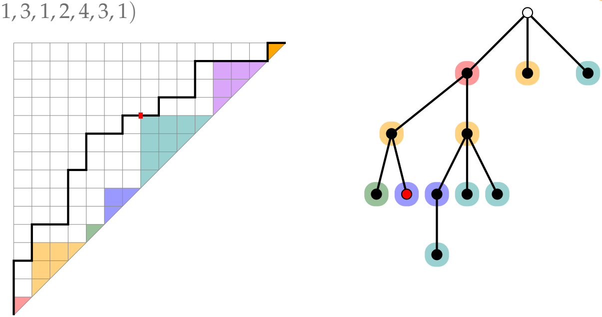

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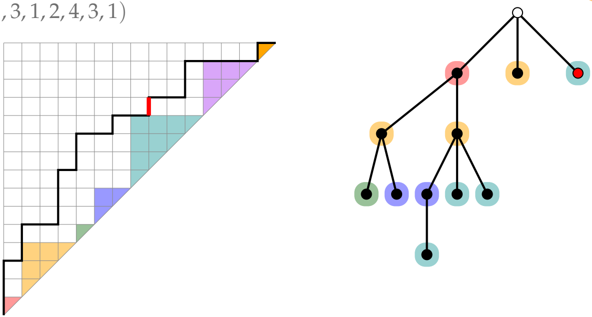

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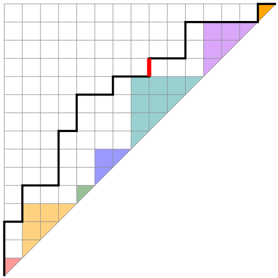

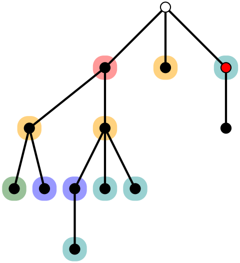

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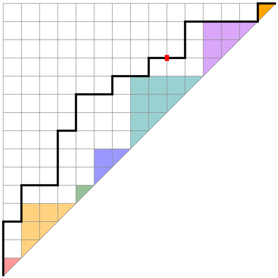

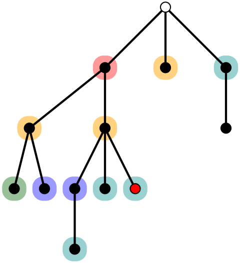

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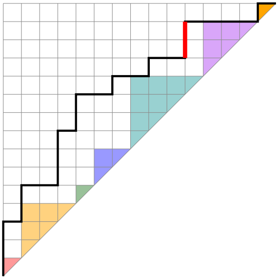

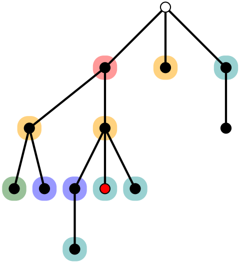

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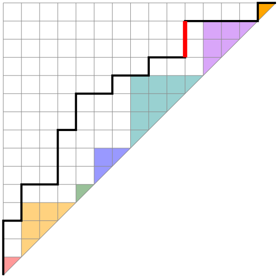

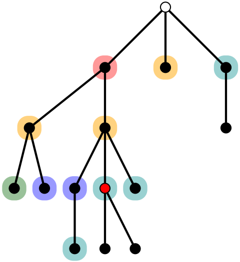

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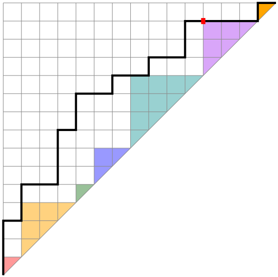

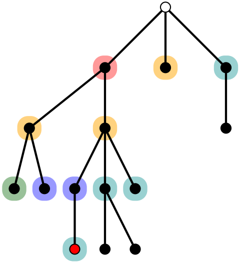

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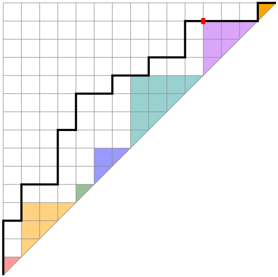

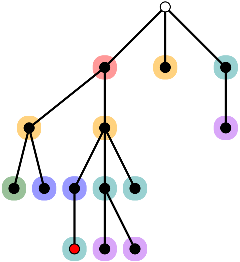

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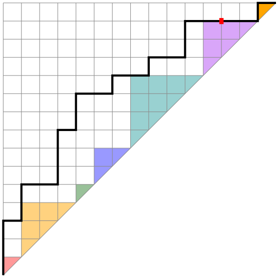

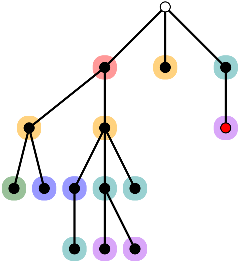

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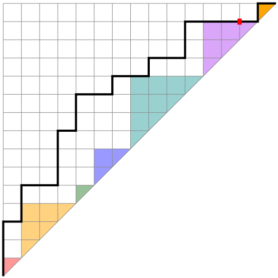

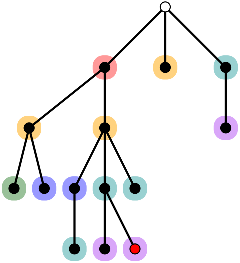

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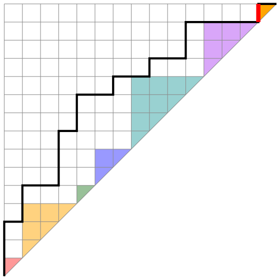

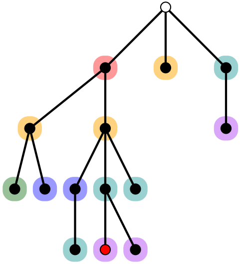

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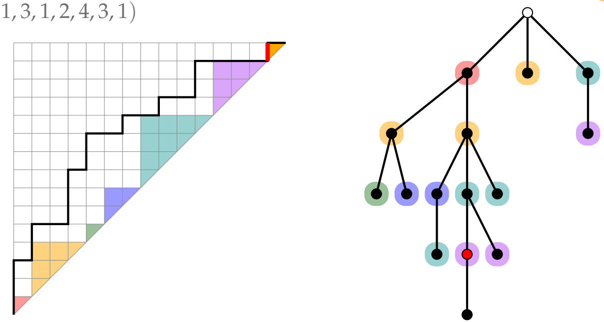

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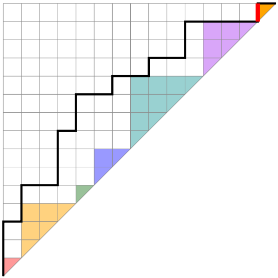

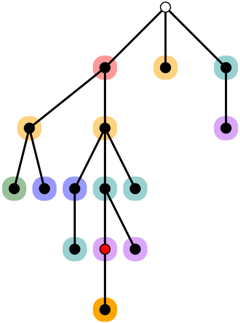

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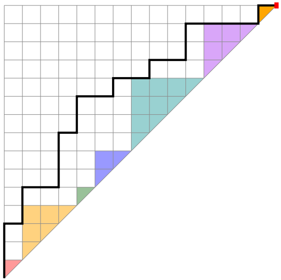

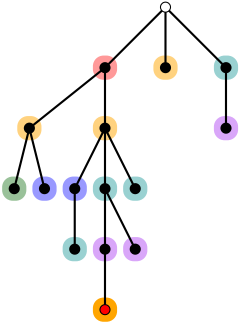

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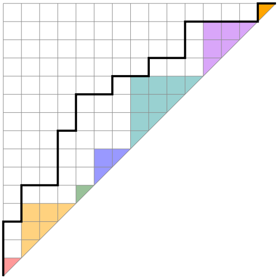
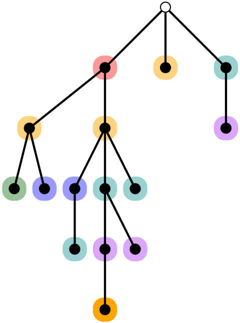


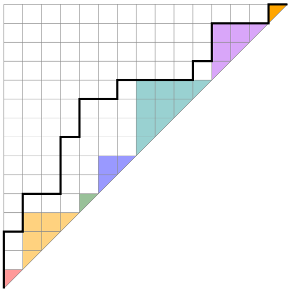


Outline

Statistics on Dyck Paths

- $\mu \in \mathcal{D}_\alpha$

$\alpha = (1,3,1,2,4,3,1)$



Statistics on Dyck Paths

- $\mu \in \mathcal{D}_\alpha$
- **peak**: coordinate preceded by N and followed by E

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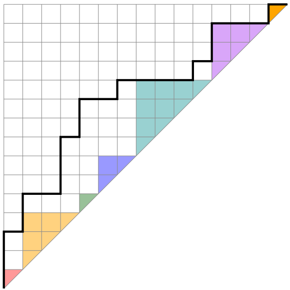

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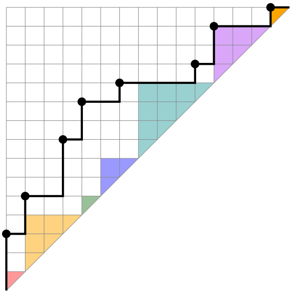

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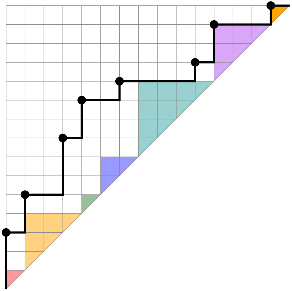
$\alpha = (1,3,1,2,4,3,1)$

peak $= 8$



Statistics on Dyck Paths

- $\mu \in \mathcal{D}_\alpha$
- **peak**: coordinate preceded by N and followed by E
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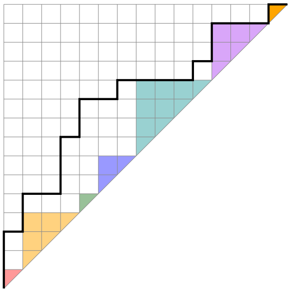
$\text{peak} = 8$



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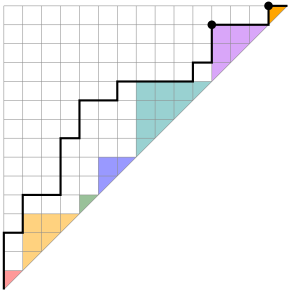

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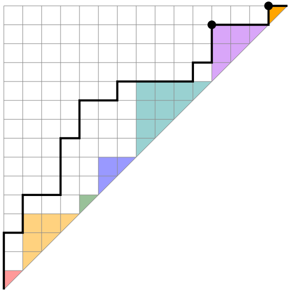
$$\text{peak} = 8$$
$$\text{bouncepeak} = 2$$



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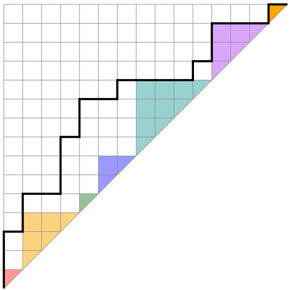

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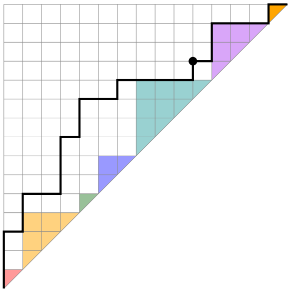

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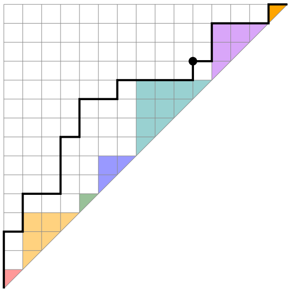

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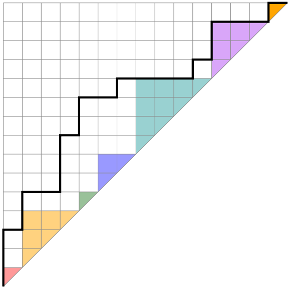
- ***H*-triangle**:

$$H_\alpha(s,t) \stackrel{\text{def}}{=} \sum_{\mu \in \mathcal{D}_\alpha} s^{\text{peak}(\mu) - \text{bouncepeak}(\mu)} t^{\text{basepeak}(\mu)}$$

$\alpha = (1,3,1,2,4,3,1)$

$$\begin{aligned} \text{peak} &= 8 \\ \text{bouncepeak} &= 2 \\ \text{basepeak} &= 1 \end{aligned}$$



The Parabolic H-Triangle

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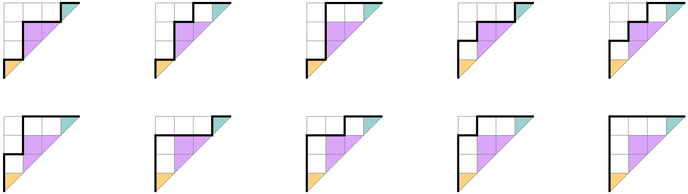

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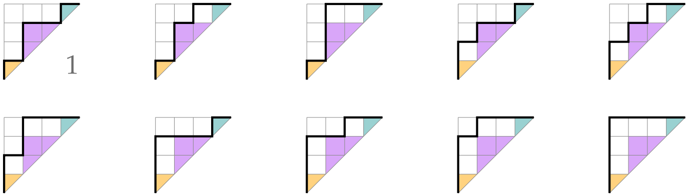

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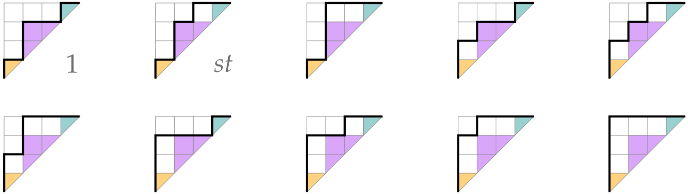

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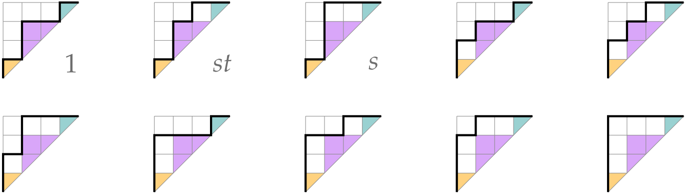

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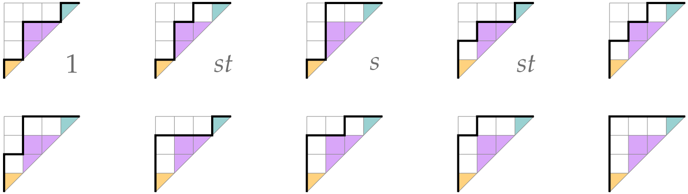

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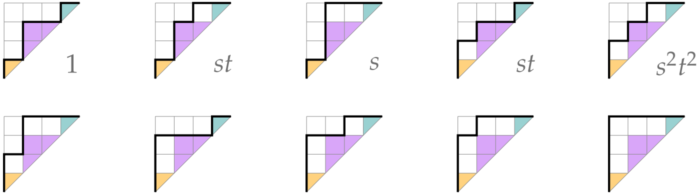

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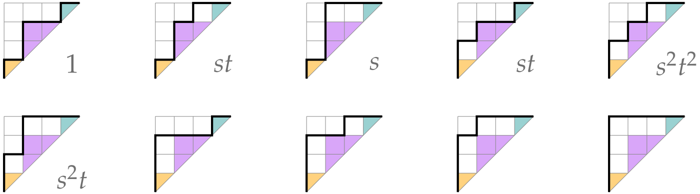

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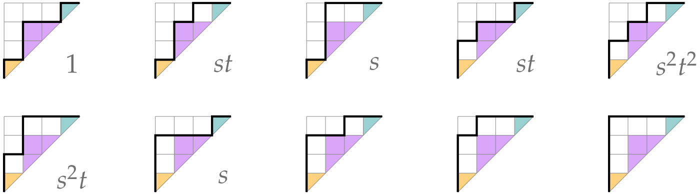

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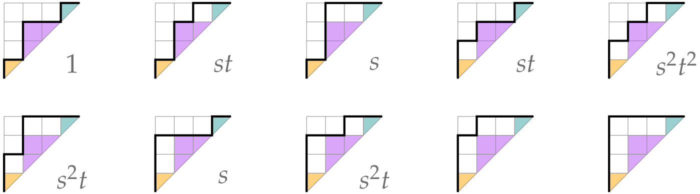

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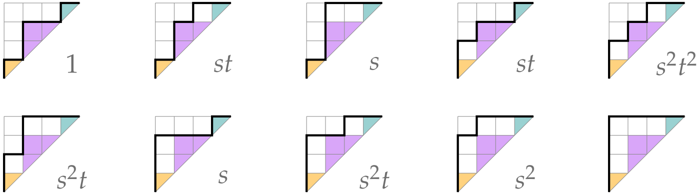

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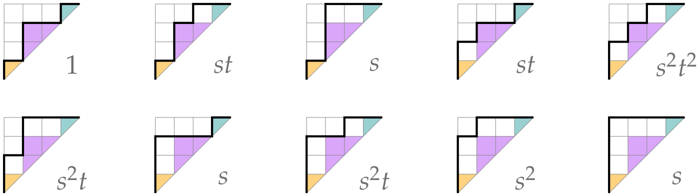

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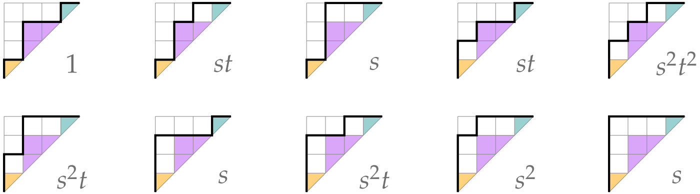
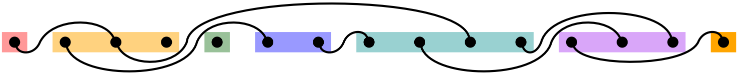


- $\mathbf{P} \in NC_\alpha$

$\alpha = (1,3,1,2,4,3,1)$



Statistics on Noncrossing Partitions

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- **bump**: number of bumps of $\mathbf{P}$

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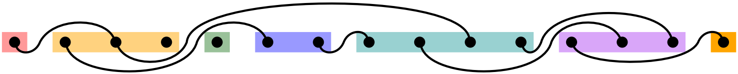

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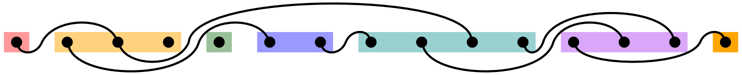

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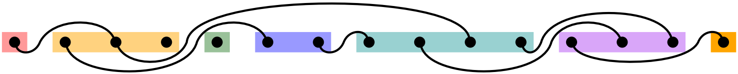

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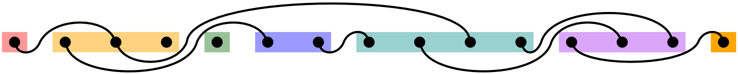

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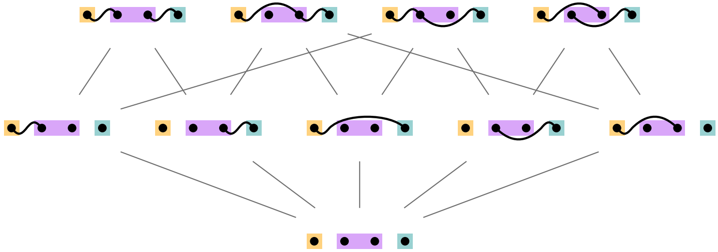

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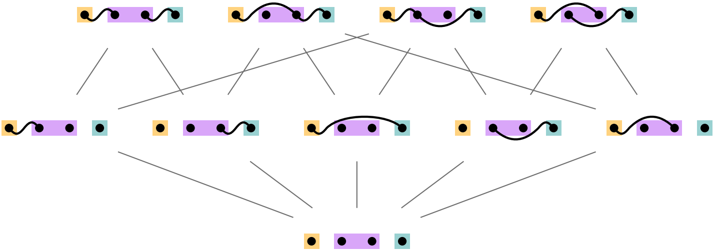
$$M_{(1,2,1)}(s,t) = 1$$



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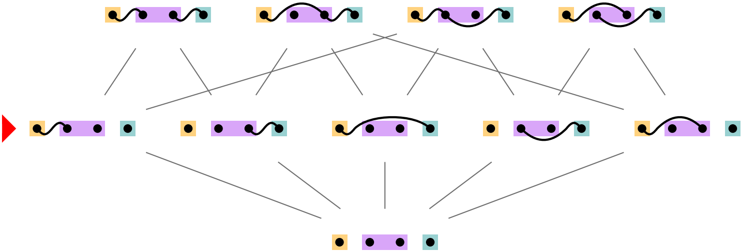

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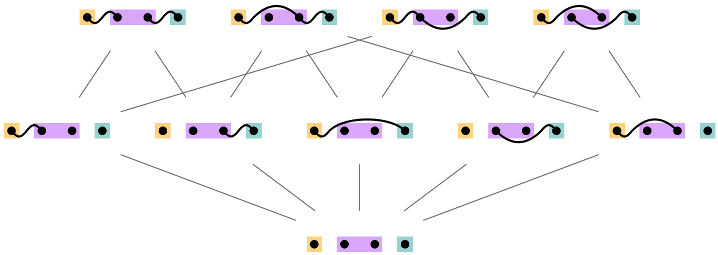
$$M_{(1,2,1)}(s, t) = 1 + 5st + 4s^2t^2$$



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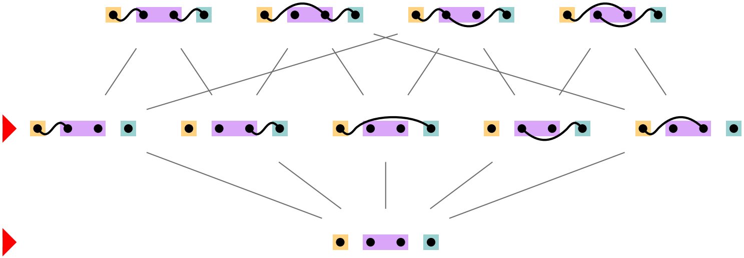
$$M_{(1,2,1)}(s,t) = 1 + 5st + 4s^2t^2 - 5s$$



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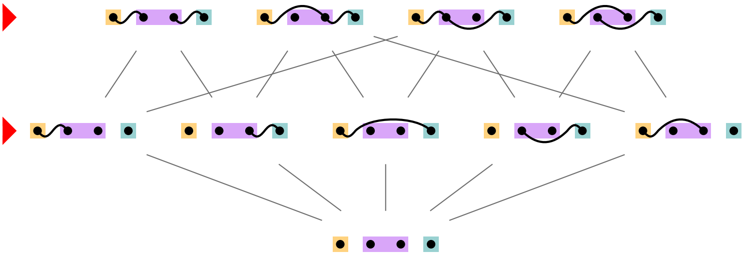

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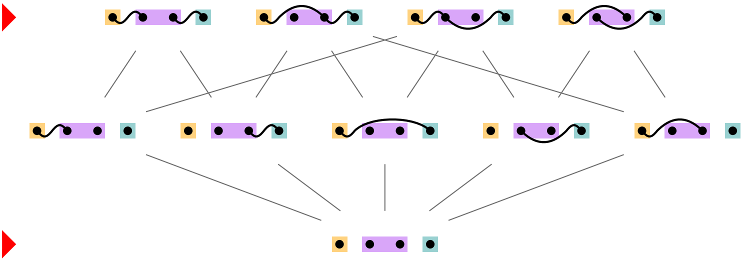

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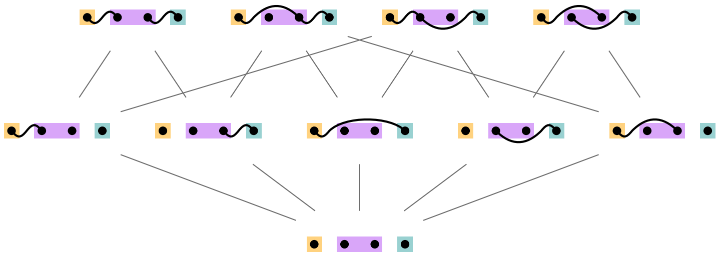

An Enumerative Connection

- ***H*-triangle**:
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Conjecture (; 2018)

The following equation holds if and only if α has r parts, where either the first or the last may exceed 1*:*
$$H_\alpha(s,t) = \bigl(s(t-1)+1\bigr)^{r-1} M_\alpha\left(\frac{s(t-1)}{\bigl(s(t-1)+1\bigr)}, \frac{t}{t-1}\right).$$

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The following rational function is a polynomial with positive integer coefficients if and only if α has r parts, of which at most one exceeds 1:

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Question

Which family of combinatorial objects realizes F_α? What are the statistics?

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$$F_{(2,2)}(s,t) = \frac{5s^2+st+6s+1}{s}$$

Outline

The Ballot Case

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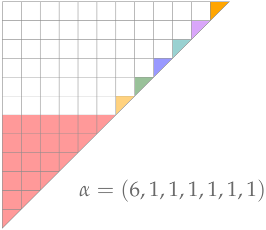

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Theorem (; 2018)

For $n > 0$ and $1 \leq t \leq n$, the common cardinality of the sets $\mathfrak{S}_{\alpha_{(n;t)}}(231)$, $NC_{\alpha_{(n;t)}}$, $\mathcal{D}_{\alpha_{(n;t)}}$, and $\mathbb{T}_{\alpha_{(n;t)}}$ is

$$\mathrm{Cat}\big(\alpha_{(n;t)}\big) \overset{\text{def}}{=} \frac{t+1}{n+1}\binom{2n-t}{n-t}.$$

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Theorem (; 2018)

For $n>0$ and $1\leq t\leq n$, the number of noncrossing $\alpha_{(n;t)}$-partitions with exactly k bumps is

$$\binom{n}{k}\binom{n-t}{k}-\binom{n-1}{k-1}\binom{n-t+1}{k+1}.$$

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For $n > 0$ and $1 \leq t \leq n$, we have $\mathrm{CLO}\bigl(\mathcal{T}_{\alpha_{(n;t)}}\bigr) \cong \mathcal{NC}_{\alpha_{(n;t)}}$.

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Theorem (C. Krattenthaler; 2019)

For $n>0$ and $1\leq t\leq n$, the zeta polynomial of $\mathcal{NC}_{\alpha_{(n;t)}}$ is

$$\mathcal{Z}_{\mathcal{NC}_{\alpha_{(n;t)}}}(q)=\frac{t(q-1)+1}{n(q-1)+1}\binom{nq-t}{n-t}.$$

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Theorem (C. Krattenthaler; 2019)

For $n > 0$ and $1 \leq t \leq n$, the number of maximal chains in $\mathcal{NC}_{\alpha_{(n;t)}}$ is tn^{n-t-1}.

Outline

Parabolic Quotients of the Symmetric Group

Back

- $\mathfrak{S}_n$: symmetric group of degree n
- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$: composition of n

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- **parabolic quotient**:
$$\mathfrak{S}_\alpha \overset{\text{def}}{=} \mathfrak{S}_n / (\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_r})$$

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$n = 4$

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

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$\alpha = (1, 2, 1)$

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Parabolic Quotients of the Symmetric Group

Back

- $\mathfrak{S}_n$: symmetric group of degree n
- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$: composition of n
- let $s_i \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_i$
- **α-region**: a set $\{s_i + 1, s_i + 2, \ldots, s_{i+1}\}$
- **parabolic quotient**:

$$\mathfrak{S}_\alpha \overset{\text{def}}{=} \{w \in \mathfrak{S}_n \mid w(k) < w(k+1) \text{ for all } k \notin \{s_1, s_2, \ldots, s_{r-1}\}\}$$

$n = 4$
$\alpha = (1, 2, 1)$

1234	1243	~~1324~~	1342	~~1423~~	~~1432~~
2134	2143	~~2314~~	2341	~~2413~~	~~2431~~
3124	3142	~~3214~~	3241	~~3412~~	~~3421~~
4123	4132	~~4213~~	4231	~~4312~~	~~4321~~

Möbius Function

Back

- $\mathcal{P} = (P, \leq)$ finite poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ given by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0, & \text{otherwise} \end{cases}$$

Möbius Function

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Theorem (G.-C. Rota; 1964)

Let $\mathcal{P} = (P, \leq)$ be a finite poset, and let $f, g\colon P \times P \to \mathbb{Z}$. It holds $f(y) = \sum_{x \leq y} g(x)$ if and only if $g(y) = \sum_{x \leq y} g(x)\mu_{\mathcal{P}}(x,y)$.

Möbius Function

- $\mathcal{P} = (P, \leq)$ finite bounded poset; $\hat{0}, \hat{1}$ least/greatest element

Back

- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ given by

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Theorem (P. Hall; 1936)

Let $\mathcal{P} = (P, \leq)$ be a finite bounded poset. The reduced Euler characteristic of the order complex of $\big(P \setminus \{\hat{0}, \hat{1}\}, \leq\big)$ equals $\mu_{\mathcal{P}}(\hat{0}, \hat{1})$ up to sign.

Interlude: Extremal Lattices

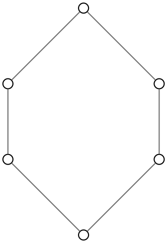
Back

- $\mathcal{L} = (L, \leq)$ finite lattice
- **join irreducible**: $j = x \vee y$ implies $j \in \{x, y\}$ $\rightsquigarrow \mathcal{J}(\mathcal{L})$
- **meet irreducible**: $m = x \wedge y$ implies $m \in \{x, y\}$ $\rightsquigarrow \mathcal{M}(\mathcal{L})$
- **length**: maximal length of a chain $\rightsquigarrow \ell(\mathcal{L})$

Interlude: Extremal Lattices

Back

- $\mathcal{L} = (L, \leq)$ finite lattice
- **extremal**: $\lvert\mathcal{J}(\mathcal{L})\rvert = \ell(\mathcal{L}) = \lvert\mathcal{M}(\mathcal{L})\rvert$



$\lvert\mathcal{J}(\mathcal{L})\rvert = 4$

$\lvert\mathcal{M}(\mathcal{L})\rvert = 4$

$\ell(\mathcal{L}) = 3$

Interlude: Extremal Lattices

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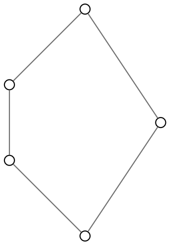

$$\lvert\mathcal{J}(\mathcal{L})\rvert = 3$$

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Interlude: Extremal Lattices

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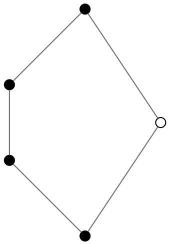
- $\mathcal{L} = (L, \leq)$ finite lattice
- **extremal**: $\lvert\mathcal{J}(\mathcal{L})\rvert = \ell(\mathcal{L}) = \lvert\mathcal{M}(\mathcal{L})\rvert$
- $C : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{\ell(\mathcal{L})}$



Interlude: Extremal Lattices

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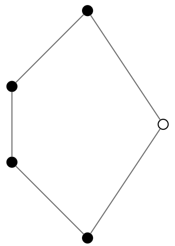
- $C : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{\ell(\mathcal{L})}$
- sort irreducibles such that

$$j_1 \vee j_2 \vee \cdots \vee j_k = x_k = m_{k+1} \wedge m_{k+2} \wedge \cdots \wedge m_{\ell(\mathcal{L})}$$



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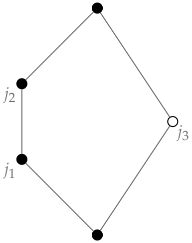

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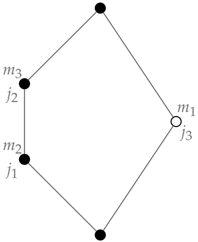

Interlude: Extremal Lattices

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- $\mathcal{L} = (L, \leq)$ finite lattice
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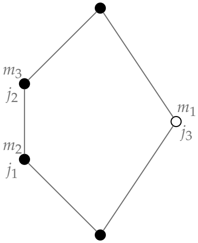

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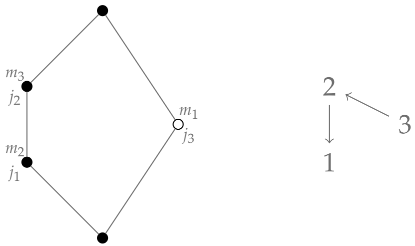

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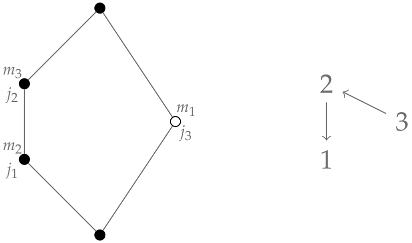

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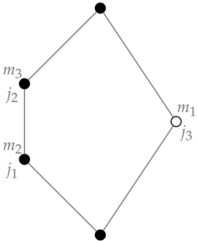

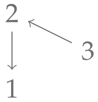

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Theorem (G. Markowsky; 1992)

Every finite extremal lattice is isomorphic to the lattice of maximal orthogonal pairs of its Galois graph.

This is a special case of a formal context.

Interlude: Extremal Lattices

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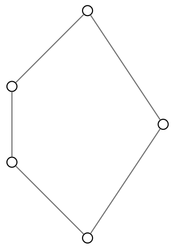

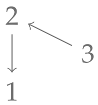

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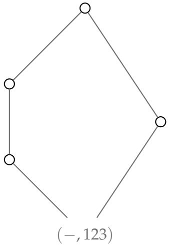

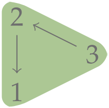

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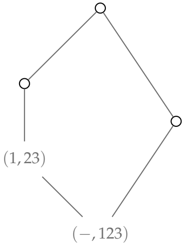

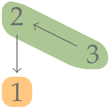

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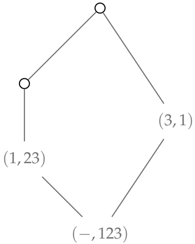

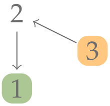

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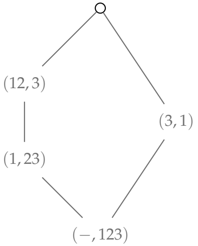

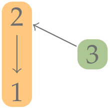

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