Hochschild Lattices and Shuffle Lattices

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Outline

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The Hochschild Lattice

- **triword**: an integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1, 2\}$ $\rightsquigarrow \mathsf{Tri}(n)$
  - $u_1 \neq 2$
  - $u_i = 0$ implies $u_j \neq 1$ for all $j > i$

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$$(0,0,0),\quad (0,0,2),\quad (0,2,0),\quad (0,2,2),\quad (1,0,0),\quad (1,0,2),$$
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Lemma (C. Combe, 2020)

For $n > 0$, the cardinality of $\mathsf{Tri}(n)$ is $2^{n-2}(n+3)$.

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$1, 2, 5, 12, 28, 64, 144, 320, 704, \ldots$ (A045623 in OEIS)

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  $\mathbf{Hoch}(n) \overset{\mathsf{def}}{=} \big(\mathsf{Tri}(n), \leq_{\mathsf{comp}}\big)$

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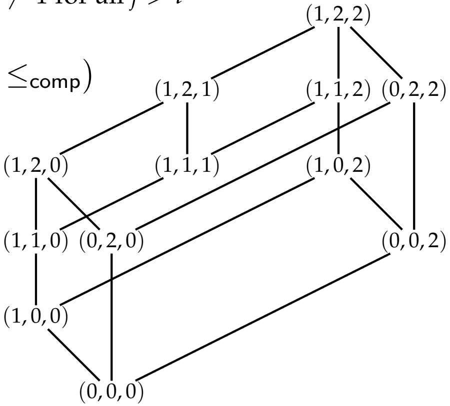

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Theorem (C. Combe, 2020)

For $n > 0$, $\mathbf{Hoch}(n)$ is a lattice.

The Hochschild Lattice

- $\mathbf{L} = (L, \leq)$ .. lattice
- **semidistributive**:
  - $p \vee q = p \vee r \quad \text{implies} \quad (p \vee q) \wedge (p \vee r) = p \vee (q \wedge r)$
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- two statistics:

$$f_0 \colon \mathsf{Tri}(n) \to \{1, 2, \ldots, n+1\}$$
$$\mathfrak{u} \mapsto \begin{cases} n+1, & \text{if } 0 \notin \mathfrak{u} \\ \min\{i \mid u_i = 0\}, & \text{otherwise} \end{cases}$$

$$l_1 \colon \mathsf{Tri}(n) \to \{0, 1, \ldots, n\}$$
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$$\begin{aligned} f_0\colon \mathsf{Tri}(n) &\to \{1,2,\ldots,n{+}1\}\\ \mathfrak{u} &\mapsto \begin{cases} n+1, & \text{if } 0 \notin \mathfrak{u}\\ \min\{i \mid u_i = 0\}, & \text{otherwise}\end{cases}\\ l_1\colon \mathsf{Tri}(n) &\to \{0,1,\ldots,n\}\\ \mathfrak{u} &\mapsto \begin{cases} 0, & \text{if } 1 \notin \mathfrak{u}\\ \max\{i \mid u_i = 1\}, & \text{otherwise}\end{cases}\end{aligned}$$

- by definition, $l_1(\mathfrak{u}) < f_0(\mathfrak{u})$

The Hochschild Lattice

- **edge**: $(\mathfrak{u}, \mathfrak{v})$ such that $\mathfrak{u} < \mathfrak{v}$ without $\mathfrak{u} < \mathfrak{u}' < \mathfrak{v}$ $\rightsquigarrow \mathcal{E}\bigl(\mathsf{Hoch}(n)\bigr)$

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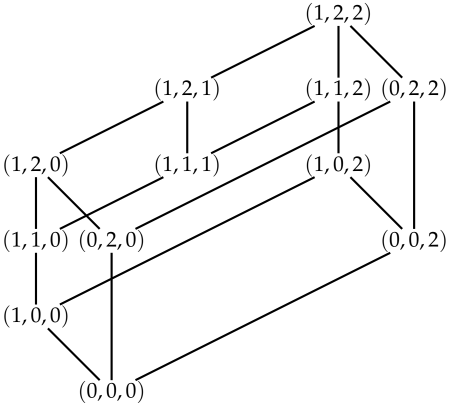
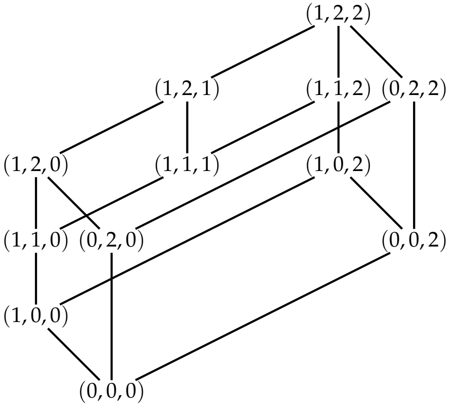
- if $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}\bigl(\mathsf{Hoch}(n)\bigr)$, then $u_i < v_i$ for a unique $i \in [n]$



The Hochschild Lattice

Perspectivity Irreducibility

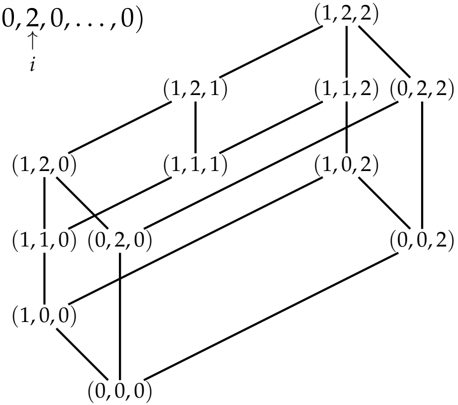
- join-irreducible triwords:



The Hochschild Lattice

Perspectivity Irreducibility

- join-irreducible triwords:
  - $\mathfrak{a}^{(i)} \overset{\text{def}}{=} (\underbrace{1, 1, \ldots, 1}_{i}, 0, 0, \ldots, 0)$
  - $\mathfrak{b}^{(i)} \overset{\text{def}}{=} (0, 0, \ldots, 0, \underset{\overset{\uparrow}{i}}{2}, 0, \ldots, 0)$



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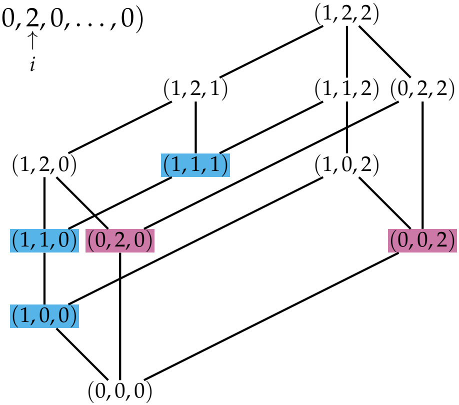

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- $\lambda(\mathfrak{u},\mathfrak{v}) \overset{\text{def}}{=} \begin{cases} \mathfrak{a}^{(i)}, & \text{if } u_i = 0 \text{ and } v_i = 1 \\ \mathfrak{b}^{(i)}, & \text{if } u_i < 2 \text{ and } v_i = 2 \end{cases}$

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Proposition (, 2020)

For $\mathfrak{u} \in \mathsf{Tri}(n)$,

$$\mathsf{Can}(\mathfrak{u}) = \big\{\lambda(\mathfrak{u}',\mathfrak{u}) \mid (\mathfrak{u}',\mathfrak{u}) \in \mathcal{E}\big(\mathbf{Hoch}(n)\big)\big\}.$$

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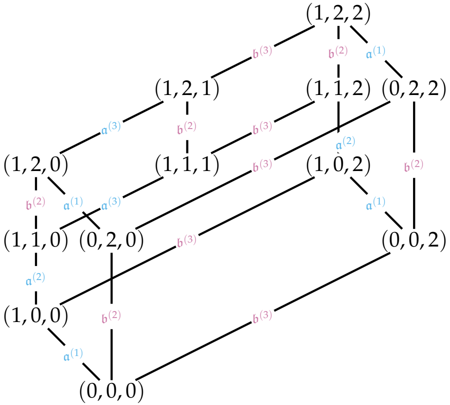

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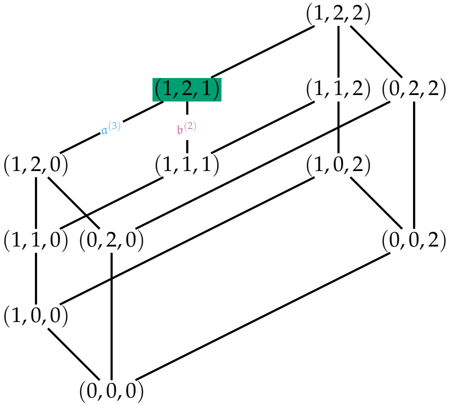

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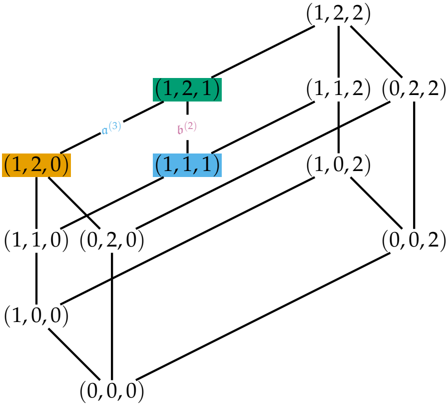

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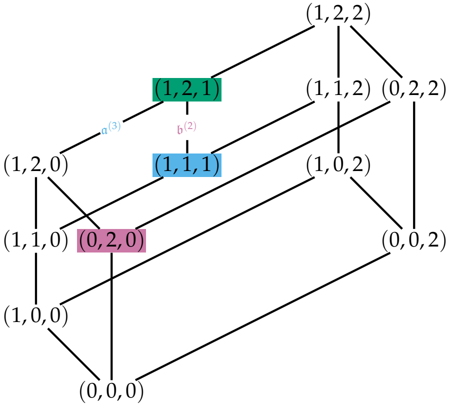

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Proposition (, 2020)

For $\mathfrak{u} \in \mathsf{Tri}(n)$, we have

$$\mathsf{Can}(\mathfrak{u}) = \left\{\mathfrak{a}^{(i)} \mid i = l_1(\mathfrak{u}) \textit{ if } l_1(\mathfrak{u}) > 0\right\} \uplus \left\{\mathfrak{b}^{(i)} \mid u_i = 2\right\}.$$

Outline

Shuffle Lattices

- $\mathbf{a} = a_1a_2\cdots a_r$, $\mathbf{b} = b_1b_2\cdots b_s$
- **(word) shuffle**: word using letters a_i or b_i whose restriction to the a_i's and b_i's preserves order

  $\rightsquigarrow \mathsf{Shuf}(r,s)$

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$b_2a_1b_1b_3 \notin \mathsf{Shuf}(2,3)$

Shuffle Lattices

- $\mathbf{u}, \mathbf{v} \in \mathsf{Shuf}(r, s)$
- $\mathbf{u} \leq_{\mathsf{shuf}} \mathbf{v}$ if $\mathbf{v}$ is obtained from $\mathbf{u}$ by deleting a_i's or adding b_i's without changing order of letters

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$$a_1a_2 \leq_{\mathsf{shuf}} b_1b_2b_3$$

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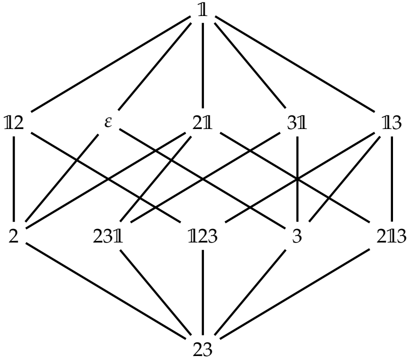
$\mathbf{Shuf}(2,1)$



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Theorem (C. Greene, 1988)

For $r, s \geq 0$, the poset $\mathbf{Shuf}(r, s) \overset{\text{def}}{=} \big(\mathsf{Shuf}(r, s), \leq_{\mathsf{shuf}}\big)$ is a supersolvable lattice.

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Proposition (C. Greene, 1988)

For $r,s \geq 0$, we have $\bigl|\mathsf{Shuf}(r,s)\bigr| = 2^{r+s}\sum_{j\geq 0}\binom{r}{j}\binom{s}{j}\left(\frac{1}{4}\right)^j.$

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Corollary

For $n > 0$, we have $\bigl|\mathsf{Shuf}(n-1,1)\bigr| = 2^{n-2}(n+3).$

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$\mathbf{a} = 23\cdots n, \mathbf{b} = \mathbb{1}$

A Bijection

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$$\mathfrak{u} = (1, 1, 1, 2, 2, 2, 1, 0, 0, 2) \in \mathsf{Tri}(10)$$

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$$\tau(\mathfrak{u}) = 23789$$

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- $\mathbf{w} \sqcup_i \mathbb{1} \overset{\text{def}}{=} \begin{cases} \mathbf{w}, & \text{if } i = 0\\ \mathbb{1}\mathbf{w}, & \text{if } i > 0, i \notin \mathbf{w}\\ w_1w_2\cdots w_j\mathbb{1}w_{j+1}\cdots w_k, & \text{if } i > 0, w_j = i \end{cases}$

$\mathbf{w} = 23789$

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$$\mathbf{w} \sqcup\!\sqcup_0 \mathbb{1} = 23789$$

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Proposition (, 2020)

For $n > 0$, the map $\sigma\colon \mathsf{Tri}(n) \to \mathsf{Shuf}(n-1,1)$ is a bijection.

Outline

The Core Label Order

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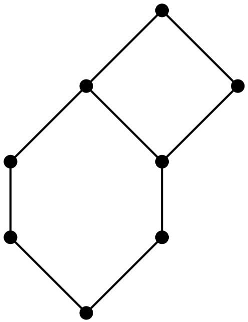

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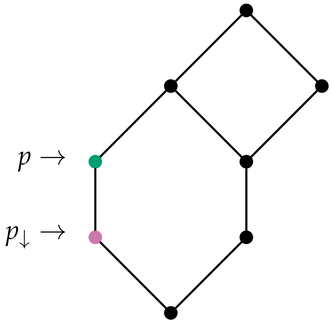

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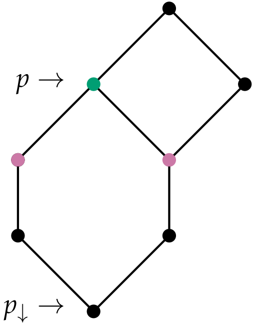

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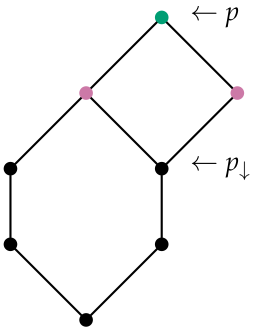

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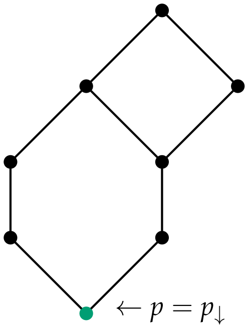

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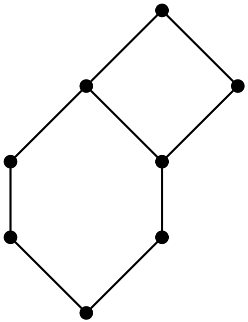

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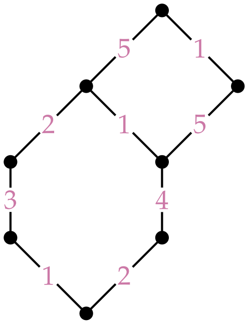

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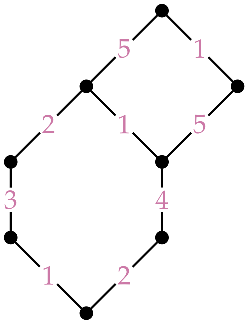

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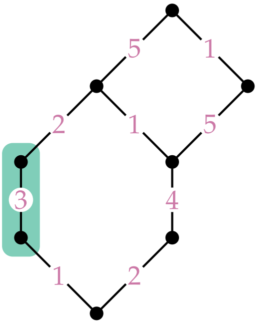
$\{3\} \rightarrow$



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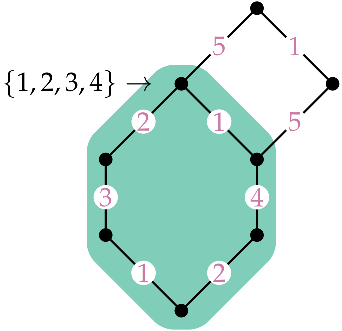

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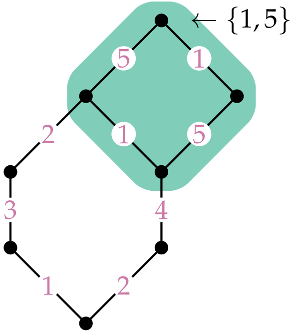

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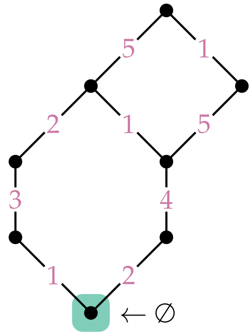

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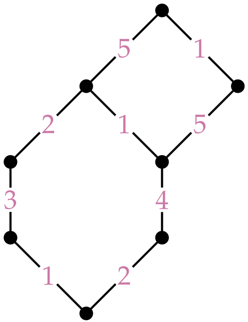

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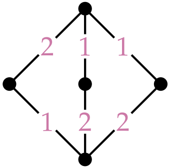

not a core labeling

The Core Label Order

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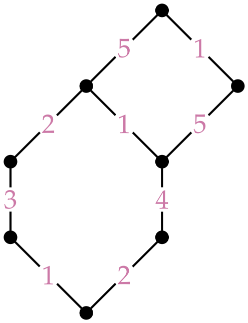

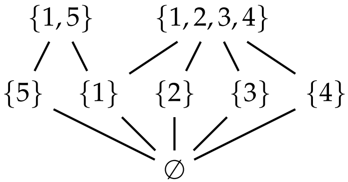

The Hochschild Lattice

- $\lambda(\mathfrak{u},\mathfrak{v}) \overset{\text{def}}{=} \begin{cases} \mathfrak{a}^{(i)}, & \text{if } u_i = 0 \text{ and } v_i = 1\\ \mathfrak{b}^{(i)}, & \text{if } u_i < 2 \text{ and } v_i = 2\end{cases}$

Proposition (⚙, 2020)

The labeling λ is a core labeling of $\mathbf{Hoch}(n)$.

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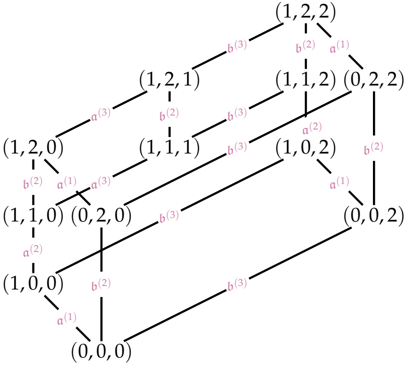
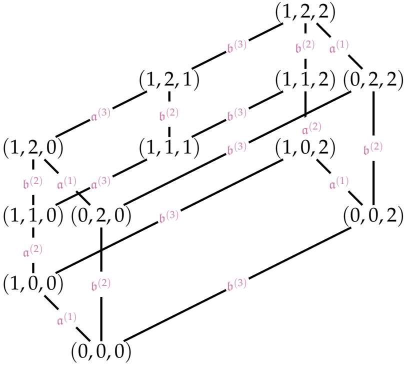


The Hochschild Lattice

Proposition (, 2020)

The core label set of $\mathfrak{u} \in \mathsf{Tri}(n)$ *is*

$$\Psi(\mathfrak{u}) = \left\{\mathfrak{a}^{(i)} \mid 0 < l_1(\mathfrak{u}) \leq i < f_0(\mathfrak{u})\right\} \uplus \left\{\mathfrak{b}^{(i)} \mid u_i = 2\right\}.$$



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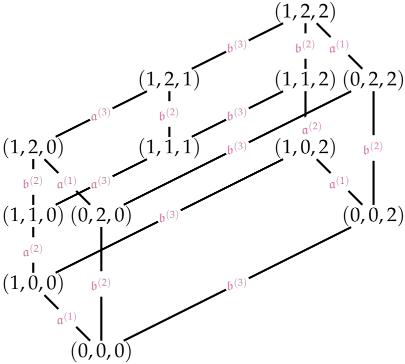

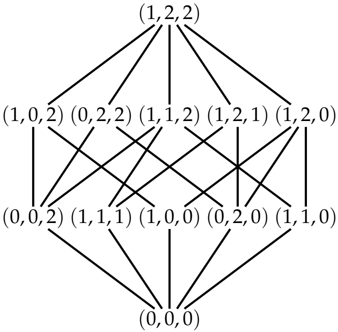

The Hochschild Lattice

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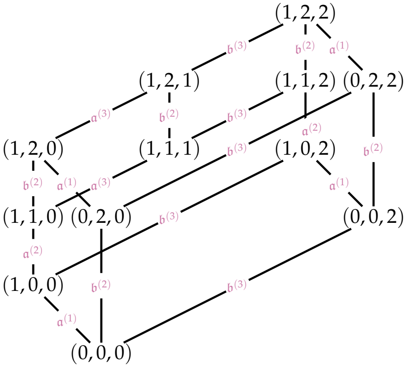
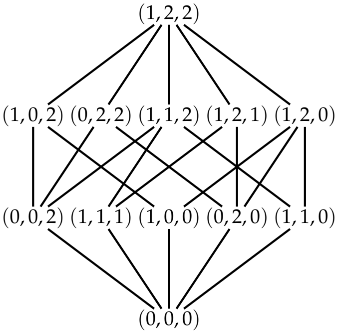
For $n > 0$, the map σ extends to an isomorphism from $\mathbf{CLO}\big(\mathbf{Hoch}(n)\big)$ *to* $\mathbf{Shuf}(n-1,1)$.





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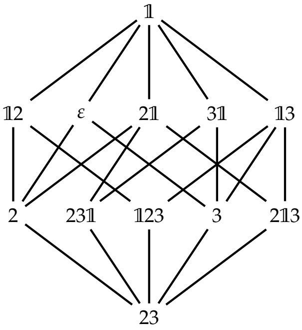

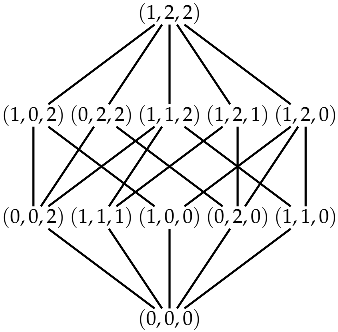

Outline

Enumeration

- $\mathbf{w} \in \mathsf{Shuf}(n-1,1)$
- $a(\mathbf{w})$ denotes the number of a_i's contained in $\mathbf{w}$

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Proposition (C. Greene, 1988)

Let $\mathbf{w} \in \mathsf{Shuf}(n-1,1)$. *The rank of* $\mathbf{w}$ *in* $\mathbf{Shuf}(n-1,1)$ *is*

$$n-1-a(\mathbf{w}) + \begin{cases} 1, & \textit{if } \mathbf{w} \textit{ contains } \mathbb{1}, \\ 0, & \textit{otherwise.} \end{cases}$$

Enumeration

Corollary (, 2020)

Let $\mathfrak{u} \in \mathsf{Tri}(n)$. *The rank of* $\mathfrak{u}$ *in* $\mathbf{CLO}\big(\mathbf{Hoch}(n)\big)$ *is*

$$\big|\{i \mid u_i = 2\}\big| + \begin{cases} 1, & \text{if } l_1(\mathfrak{u}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Enumeration

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The number of $\mathfrak{u} \in \mathsf{Tri}(n)$ *having rank* i *in* $\mathbf{CLO}\bigl(\mathbf{Hoch}(n)\bigr)$ *is*

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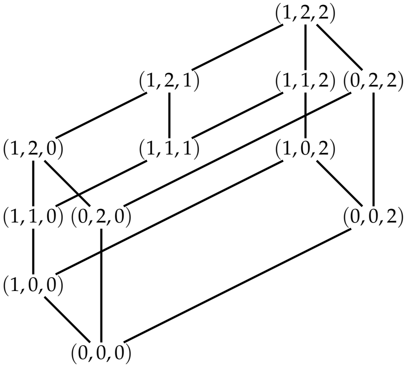
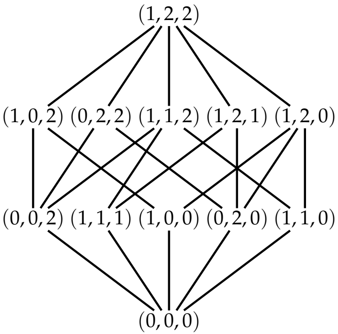
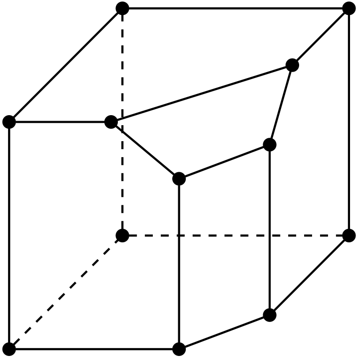




Facial Intervals in **Hoch**(n)

- **Hoch**(n) arises from an orientation of the 1-skeleton of a (simple) polytope



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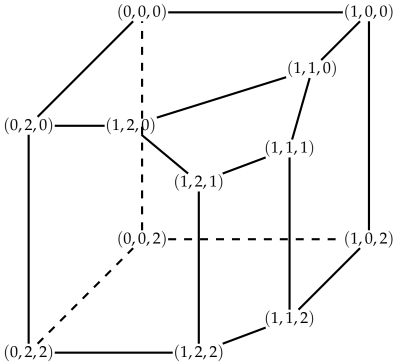

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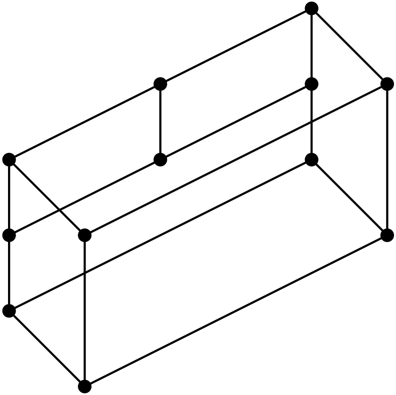

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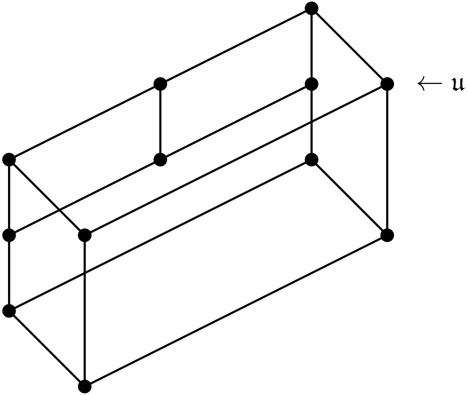

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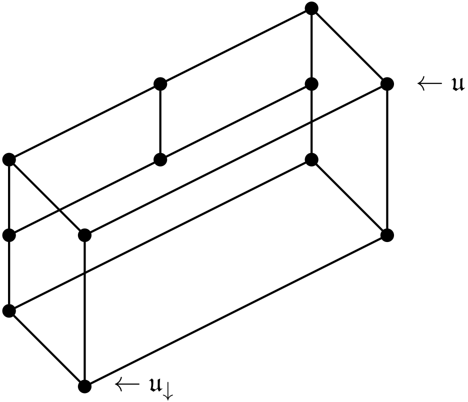

Facial Intervals in **Hoch**(n)

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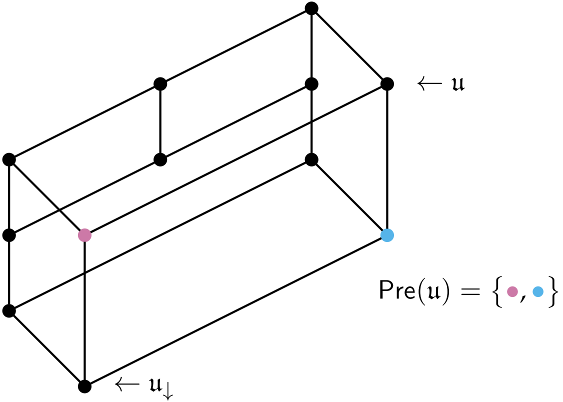

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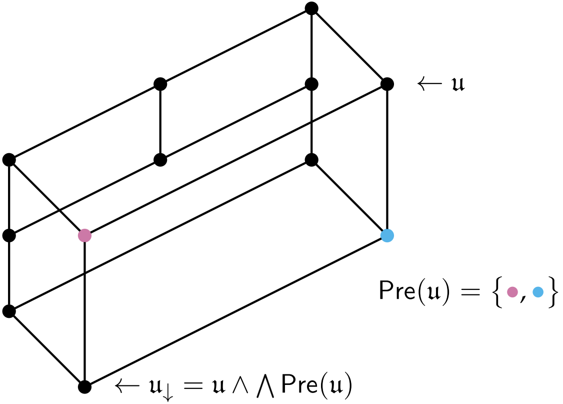

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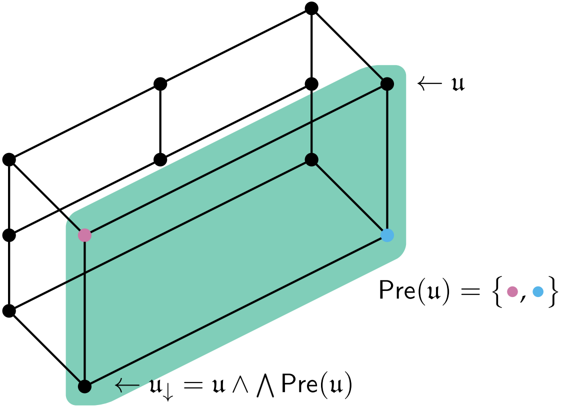

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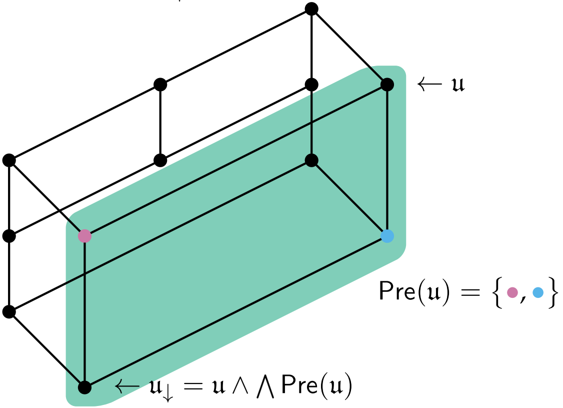

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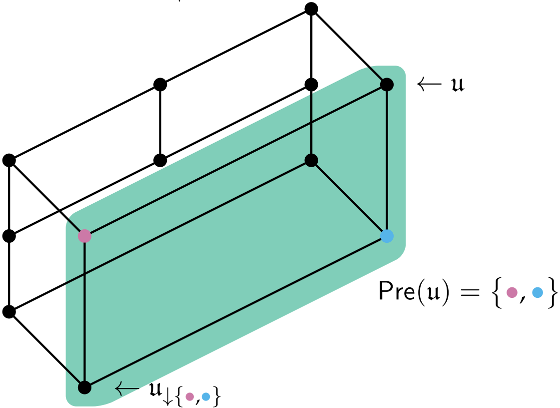

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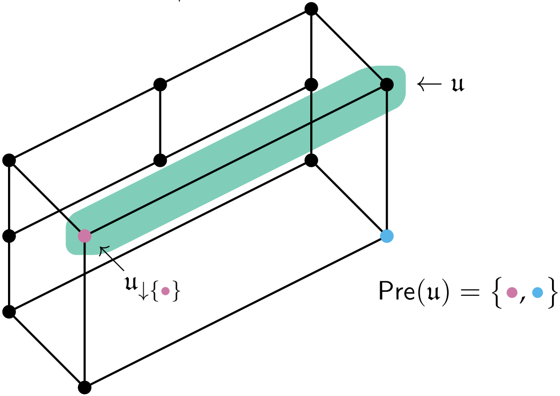

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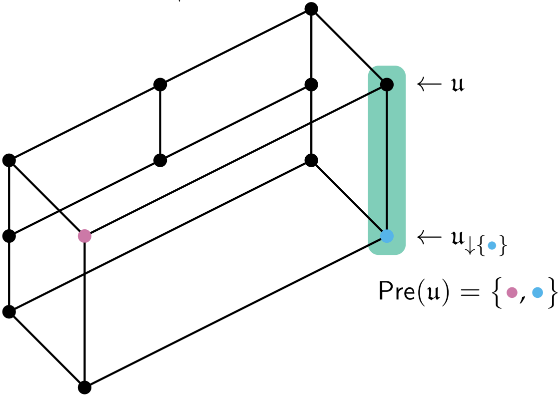

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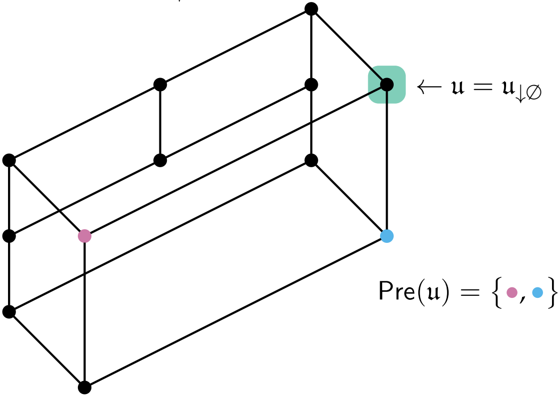
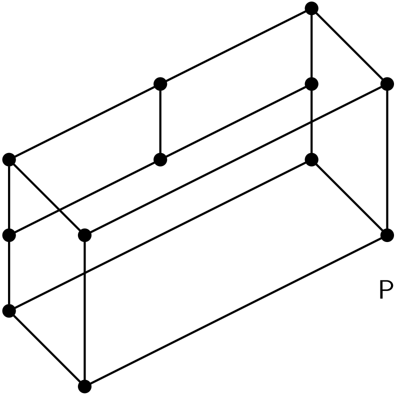


Facial Intervals in $\mathbf{Hoch}(n)$

- $\mathsf{CP}\big(\mathbf{Hoch}(n)\big) \stackrel{\mathsf{def}}{=} \big\{\langle\mathfrak{u}, P\rangle \mid \mathfrak{u} \in \mathsf{Tri}(n), P \subseteq \mathsf{Pre}(\mathfrak{u})\big\}$



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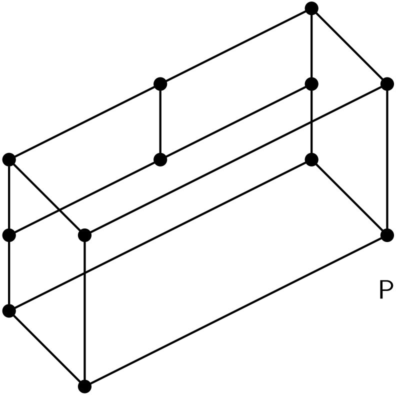

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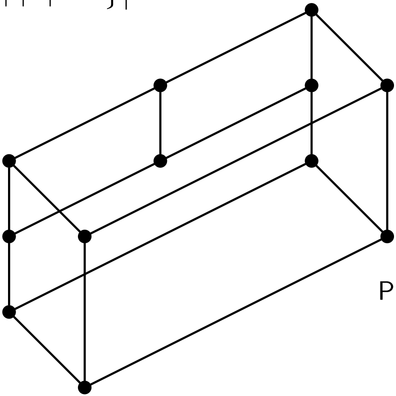
- $\dim\langle\mathfrak{u},P\rangle \stackrel{\text{def}}{=} \lvert P\rvert$
- $f_i \stackrel{\text{def}}{=} \Big\lvert\big\{\langle\mathfrak{u},P\rangle \mid \lvert P\rvert = i\big\}\Big\rvert$



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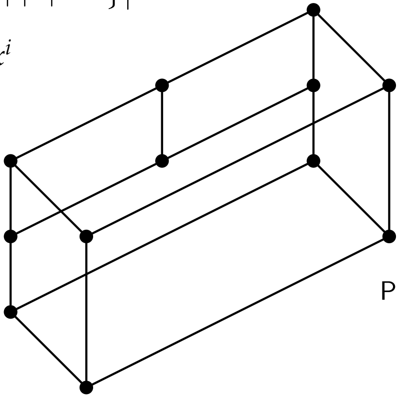

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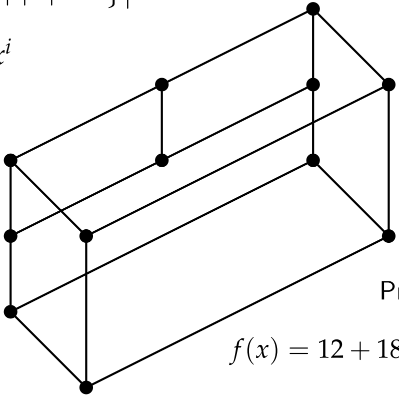

$\mathsf{Pre}(\mathfrak{u}) = \{\bullet, \bullet\}$

$f(x) = 12 + 18x + 8x^2 + x^3$

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Proposition (, 2020)

For $n>0$ and $0\leq i\leq n$, we have

$$f_i = \binom{n}{i}2^{n-i-2}\frac{n(n+3)-i(i-1)}{n}.$$

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- $f(x) \overset{\text{def}}{=} \sum_{i=0}^{n} f_i x^i$
- $h(x) \overset{\text{def}}{=} f(x-1)$

Corollary (, 2020)

For $n>0$, we have

$$f(x) = (x+2)^{n-2}\Big(x^2 + (n+3)x + n + 3\Big),$$
$$h(x) = (x+1)^{n-2}\Big(x^2 + (n+1)x + 1\Big).$$

The f-Polynomial of $\mathsf{Hoch}(n)$

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Corollary (, 2020)

The number of $\mathfrak{u} \in \mathsf{Tri}(n)$ *with* $|\mathsf{Can}(\mathfrak{u})| = i$ *is*

$$\binom{n-1}{i} + \binom{n-1}{i-1} + (n-1)\binom{n-2}{i-1}.$$

The f-Polynomial of $\mathsf{Hoch}(n)$

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Refined Face Enumeration

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice; $\hat{0}$ .. least element
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Proposition (, 2020)

For $n > 0$, we have $\mathcal{A}\bigl(\mathsf{Hoch}(n)\bigr) = \bigl\{\mathfrak{a}^{(1)}, \mathfrak{b}^{(2)}, \ldots, \mathfrak{b}^{(n)}\bigr\}$.

Refined Face Enumeration

- $\mathsf{pos}(\mathfrak{u}) \overset{\mathsf{def}}{=} \bigl|\mathsf{Can}(\mathfrak{u}) \setminus \mathcal{A}\bigl(\mathbf{Hoch}(n)\bigr)\bigr|$
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Proposition (🌬, 2020)

For $n>0$, we have

$$F_{\mathbf{Hoch}(n)}(x,y) = (x+y+1)^{n-2}\bigl(nx^2+2xy+(n+1)x+(y+1)^2\bigr).$$

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Proposition (, 2020)

For $n > 0$, we have

$$F_{\mathbf{Hoch}(n)}(x,x) = (2x+1)^{n-2}\bigl((n+3)x^2+(n+3)x+1\bigr).$$

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- $x^n F_{\mathbf{Hoch}(n)}\left(\frac{1}{x}, \frac{1}{x}\right) = \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} (x+1)^{|\mathsf{Can}(\mathfrak{u})|}$

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For $n > 0$, we have

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For $n > 0$, we have

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- $F_{\mathsf{Hoch}(n)}(x,y) \overset{\mathsf{def}}{=} \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{n-|\mathsf{Can}(\mathfrak{u})|}(x+1)^{\mathsf{pos}(\mathfrak{u})}(y+1)^{\mathsf{neg}(\mathfrak{u})}$
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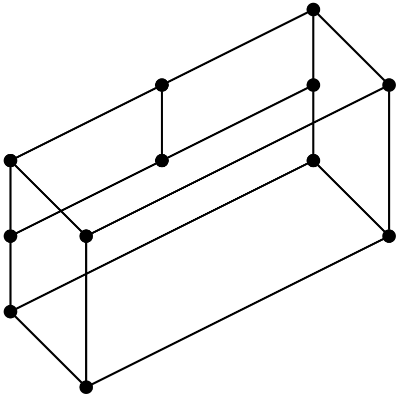
Corollary (, 2020)

For $n > 0$, we have

$$F_{\mathsf{Hoch}(n)}(x,y) = x^n H_{\mathsf{Hoch}(n)}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

Refined Face Enumeration

$n = 3$



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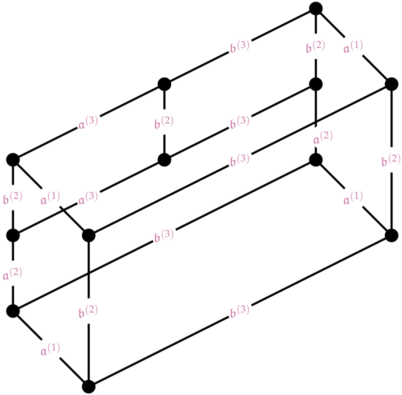

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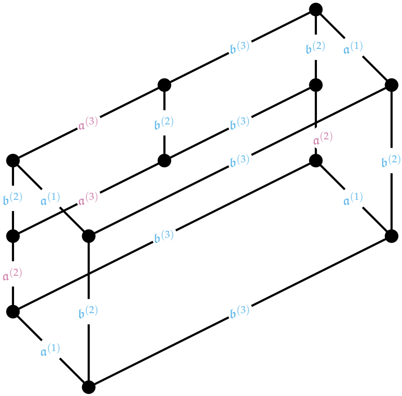

Refined Face Enumeration

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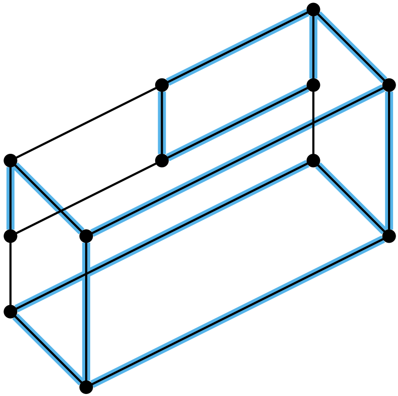

Refined Face Enumeration

- $F_{\mathsf{Hoch}(n)}(x,y) \overset{\text{def}}{=} \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{n-|\mathsf{Can}(\mathfrak{u})|}(x+1)^{\mathsf{pos}(\mathfrak{u})}(y+1)^{\mathsf{neg}(\mathfrak{u})}$

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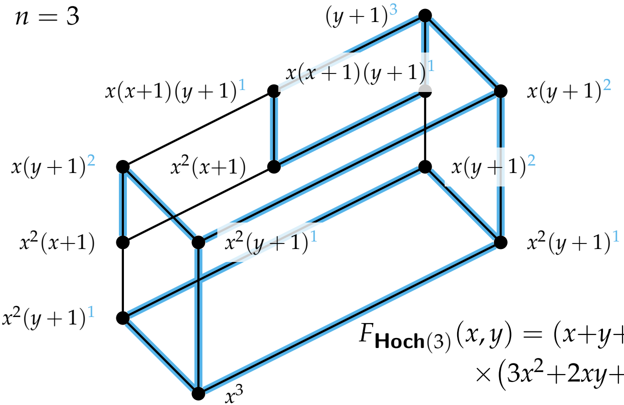

$$F_{\mathsf{Hoch}(3)}(x,y) = (x+y+1) \times \big(3x^2+2xy+4x+(y+1)^2\big)$$

Refined Face Enumeration

- $H_{\mathsf{Hoch}(n)}(x,y) \overset{\text{def}}{=} \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{|\mathsf{Can}(\mathfrak{u})|} y^{\mathsf{neg}(\mathfrak{u})}$

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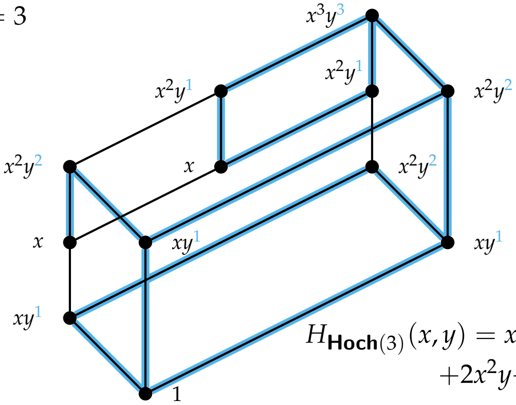
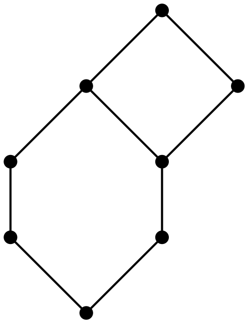


$$H_{\mathsf{Hoch}(3)}(x,y) = x^3y^3+3x^2y^2+2x^2y+3xy+2x+1$$

Möbius Polynomials

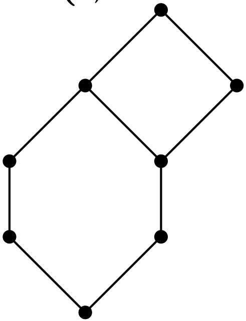
- $\mathbf{P} = (P, \leq)$ .. (finite) poset



Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. (finite) poset
- **Möbius function**:

$$\mu_{\mathbf{P}}(p,q) \stackrel{\mathsf{def}}{=} \begin{cases} 1, & \text{if } p = q \\ -\sum_{p \leq r < q} \mu_{\mathbf{P}}(p,r), & \text{if } p < q \\ 0, & \text{otherwise} \end{cases}$$



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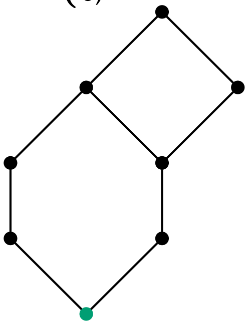

$\mu(\bullet, \bullet) = 1$

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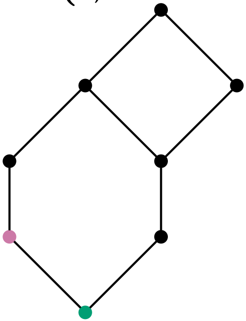

$\mu(\bullet, \bullet) = -1$

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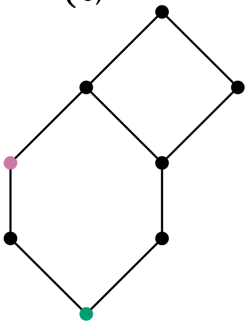

$\mu(\bullet, \bullet) = 0$

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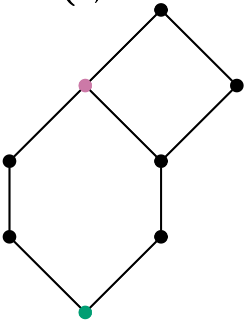

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Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. graded (finite) poset with bounds $\hat{0}$ and $\hat{1}$
- (reverse) **characteristic polynomial**:

$$\chi_{\mathbf{P}}(x) \stackrel{\mathsf{def}}{=} \sum_{p \in P} \mu_{\mathbf{P}}(\hat{0}, p) x^{\mathsf{rk}(p)}$$

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Lemma

- $M_{\mathbf{P}}(x, y) = \sum_{p \in P} (xy)^{\mathsf{rk}(p)} \chi_{[p, \hat{1}]}(y)$.
- $\chi_{\mathbf{P}}(x) = M_{\mathbf{P}}(0, x)$.

The FHM-Correspondence for $\mathsf{Hoch}(n)$

- $\mathfrak{t} \overset{\text{def}}{=} (1,2,2,\ldots,2)$ .. top element of $\mathsf{CLO}\bigl(\mathsf{Hoch}(n)\bigr)$
- if $\bigl|\mathsf{Can}(\mathfrak{u})\bigr| = i$, then

$$[\mathfrak{u},\mathfrak{t}]_{\mathsf{CLO}\bigl(\mathsf{Hoch}(n)\bigr)} \cong \begin{cases}\mathsf{CLO}\bigl(\mathsf{Hoch}(n-i)\bigr), & \text{if } l_1(\mathfrak{u}) = 0\\ \mathsf{Bool}(n-i), & \text{otherwise}\end{cases}$$

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Proposition (C. Greene, 1988)

For $n > 0$, we have

$$\chi_{\mathsf{Bool}(n)}(x) = (1-x)^n,$$

$$\chi_{\mathsf{Shuf}(n-1,1)}(x) = (1-x)^{n-1}(1-nx).$$

The FHM-Correspondence for $\mathbf{Hoch}(n)$

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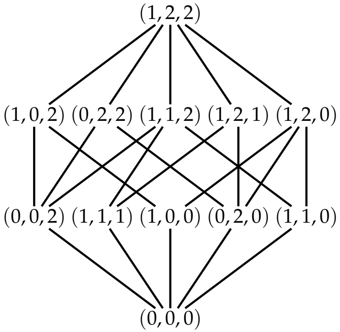
Proposition (, 2020)

For $n > 0$, we have

$$M_{\mathbf{CLO}\left(\mathbf{Hoch}(n)\right)}(x,y) = (xy-y+1)^{n-2} \times \Bigl((n+1)\bigl((x-1)y-xy^2\bigr)+(n+x^2)y^2+1\Bigr).$$

The FHM-Correspondence for $\mathsf{Hoch}(n)$



$$M_{\mathsf{CLO}(\mathsf{Hoch}(3))}(x,y) = x^3y^3 - 5x^2y^3 + 5x^2y^2 + 7xy^3 - 12xy^2 - 3y^3 + 5xy + 7y^2 - 5y + 1$$

The FHM-Correspondence for $\mathsf{Hoch}(n)$

Theorem (, 2020)

For $n > 0$, we have

$$M_{\mathsf{CLO}(\mathsf{Hoch}(n))}(x,y) = (xy-1)^n F_{\mathsf{Hoch}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$$
$$= (1-y)^n H_{\mathsf{Hoch}(n)}\left(\frac{y(x-1)}{1-y}, \frac{x}{x-1}\right).$$

The FHM-Correspondence for $\mathsf{Hoch}(n)$

- prototypical relation observed by F. Chapoton (2004/2006) connecting Tamari lattices, noncrossing partition lattices and cluster complexes

Theorem (, 2020)

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$$= (1-y)^n H_{\mathsf{Hoch}(n)}\left(\frac{y(x-1)}{1-y}, \frac{x}{x-1}\right).$$

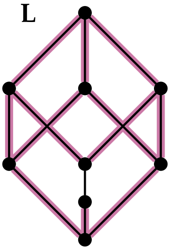
Open Questions

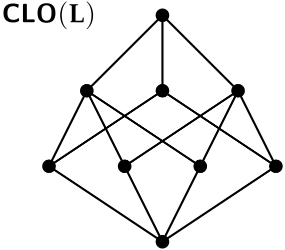
Tamari

- what is the relation between $\chi_{\mathsf{CLO}(\mathsf{Hoch}(n))}(x), f(x)$ and $h(x)$?
- what is the geometric nature of $M_{\mathsf{CLO}(\mathsf{Hoch}(n))}(x,y)$?
- can we characterize lattices satisfying the FHM-correspondence?

Thank You.

Abstract Examples





$$F(x,y) = (x+y+1)^3 + x^2(x+1)$$

$$H(x,y) = (xy+1)^3 + x$$

$$M(x,y) = (xy-y+1)^3 + (x-1)y(y-1)^2$$

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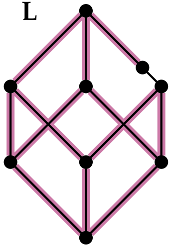

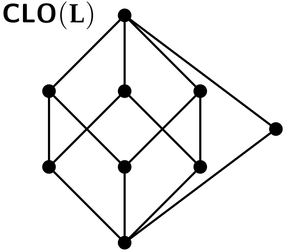

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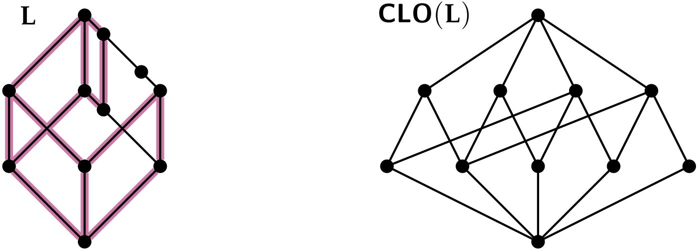
Abstract Examples



$$F(x,y) = (x+y+1)^3 + x(x+1)(2x+y+1)$$

$$H(x,y) = (xy+1)^3 + x^2y + 2x$$

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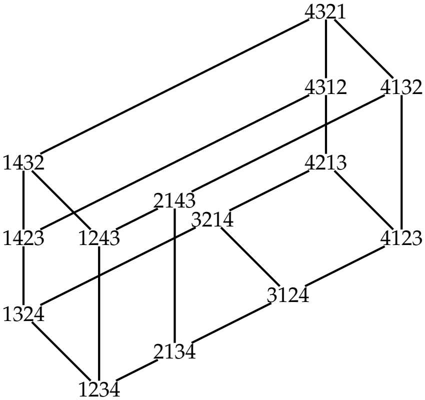
The Tamari Lattice

Questions

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$



The Tamari Lattice

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- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Theorem (A. Björner & M. Wachs, 1997)

For $n > 0$, the weak order on $\mathfrak{S}_n(231)$ realizes the Tamari lattice of order $n-1$.

The Tamari Lattice

Questions

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Lemma (D. Knuth, 1968)

For $n > 0$, the cardinality of $\mathfrak{S}_n(231)$ is $\frac{1}{n+1}\binom{2n}{n}$.

The Tamari Lattice

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- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Lemma (D. Knuth, 1968)

For $n > 0$, the cardinality of $\mathfrak{S}_n(231)$ is $\frac{1}{n+1}\binom{2n}{n}$.

$1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$ (A000108 in OEIS)

The Tamari Lattice

Questions

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Theorem (A. Urquhart, 1978)

For $n > 0$, the Tamari lattice $\textsf{Tam}(n)$ is semidistributive.

A Bijection

Questions

- $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n(231)$
- $\mathsf{nc}(w)$ is the noncrossing partition whose bumps are the descents of w

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Proposition (P. Biane, 1997)

For $n > 0$, the map $\mathsf{nc}\colon \mathfrak{S}_n(231) \to \mathsf{Nonc}(n)$ is a bijection.

A Bijection

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Theorem (N. Reading, 2011)

For $n > 0$, the map nc *extends to an isomorphism from* $\mathbf{CLO}\bigl(\mathbf{Tam}(n)\bigr)$ *to* $\mathbf{Nonc}(n)$.

A Bijection

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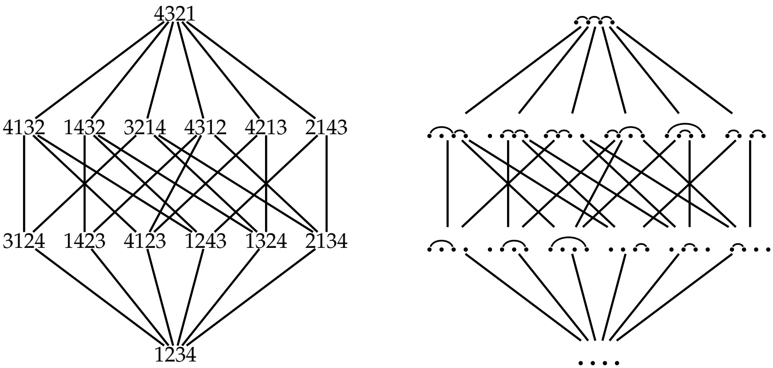
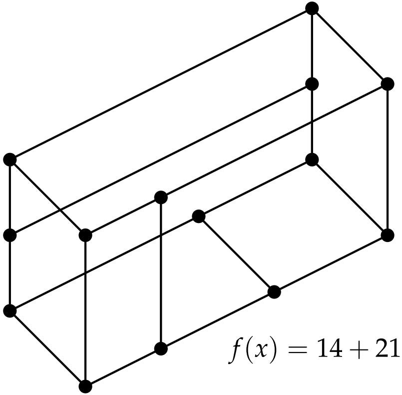


Facial Intervals in $\mathbf{Tam}(n)$

Questions



$$f(x) = 14 + 21x + 9x^2 + x^3$$

Facial Intervals in $\textbf{Tam}(n)$

Questions

Proposition (C. Lee, 1989)

For $n > 0$ and $0 \leq i \leq n$, we have

$$f_i = \frac{1}{n+1-i}\binom{n}{i}\binom{2n+2-i}{n-i}.$$

Facial Intervals in $\mathsf{Tam}(n)$

Questions

Corollary

For $n > 0$, we have

$$f(x) = \sum_{i=0}^{n} \frac{1}{n+1-i} \binom{n}{i} \binom{2n+2-i}{n-i} x^i,$$

$$h(x) = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} \binom{n+1}{i} x^i.$$

Perspectivity

- **L** .. (finite) lattice

Perspectivity

Hochschild

- $\mathbf{L}$ .. (finite) lattice
- **edge**: (p,q) such that $p < q$ and no $p < r < q$ $\rightsquigarrow \mathcal{E}(\mathbf{L})$
- **perspective**: $(p,q) \,\overline{\overline{\wedge}}\, (p',q')$ such that $q \wedge p' = p$ and $q \vee p' = q'$ (or $q' \wedge p = p'$ and $q' \vee p = q$)

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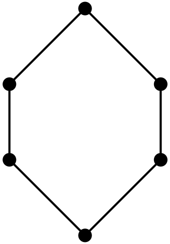

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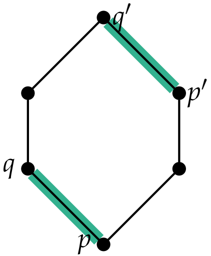

perspective

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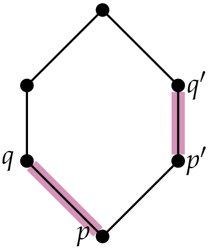

not perspective

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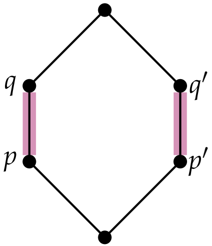

not perspective

Irreducibility

- $\mathbf{L}$ .. (finite) lattice
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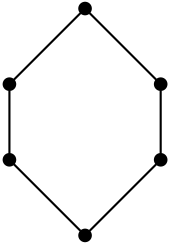

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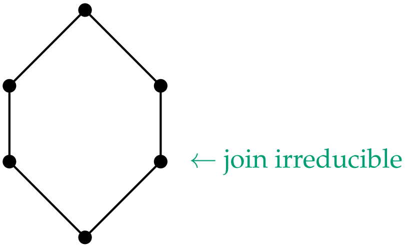

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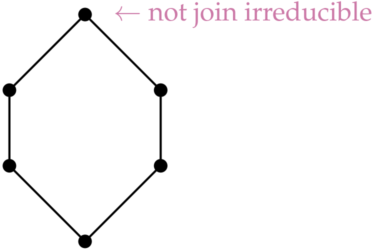

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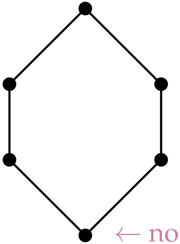

$\leftarrow$ not join irreducible

Irreducibility

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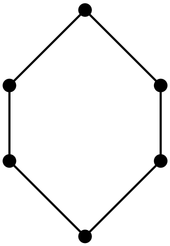
- **join irreducible**: $j = p \vee q$ implies $j \in \{p, q\}$ $\rightsquigarrow \mathcal{J}(\mathbf{L})$
  $\rightsquigarrow$ there exists a unique edge (j_*, j)
- **edge determined**: for all $(p, q) \in \mathcal{E}(\mathbf{L})$ exists a unique $j \in \mathcal{J}(\mathbf{L})$ such that $(p, q) \,\overline{\overline{\wedge}}\, (j_*, j)$



Irreducibility

Hochschild

- **L** .. (finite) lattice
- **join irreducible**: $j = p \vee q$ implies $j \in \{p, q\}$ $\rightsquigarrow \mathcal{J}(\mathbf{L})$
  $\rightsquigarrow$ there exists a unique edge (j_*, j)
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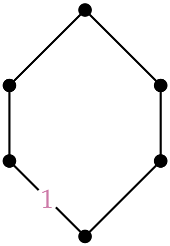

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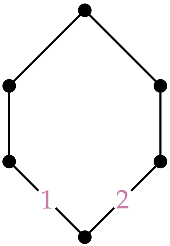

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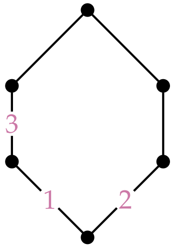

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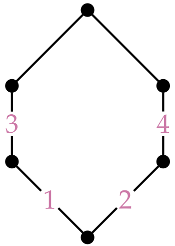

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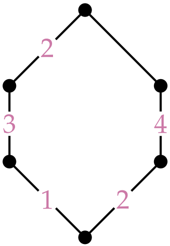

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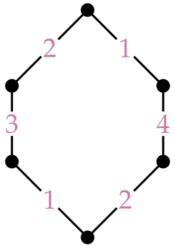

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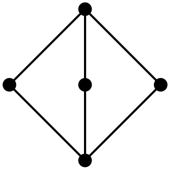

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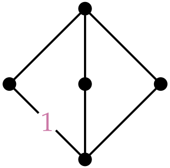

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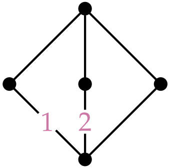

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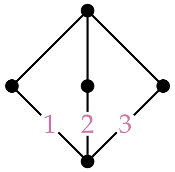

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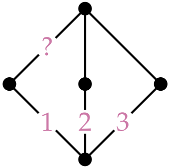

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Proposition

Every semidistributive lattice is edge determined.

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- **perspectivity labeling**: $\lambda\colon \mathcal{E}(\mathbf{L}) \to \mathcal{J}(\mathbf{L}), (p, q) \mapsto j$ such that $(p, q) \overline{\overline{\wedge}} (j_*, j)$

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