Hochschild Lattices, Shuffle Lattices and the FHM-Correspondence

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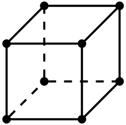
TU Dresden

March 02, 2021

AG Diskrete Mathematik, TU Wien

The Hypercube (dimension $d = 3$)

$\mathsf{Cube}(3)$



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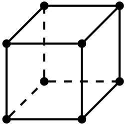
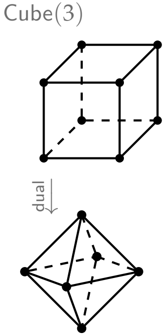


face numbers:

$f_{-1} = 1$
$f_0 = 8$
$f_1 = 12$
$f_2 = 6$
$f_3 = 1$

The Hypercube (dimension $d = 3$)



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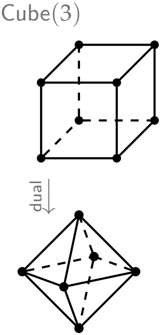

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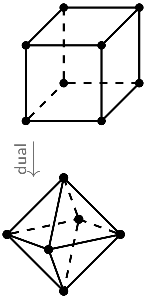

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$$f(x) \overset{\text{def}}{=} \sum_{i=0}^{d} f_{i-1} x^{d-i}$$

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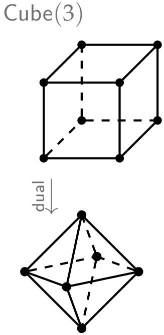

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$$f(x) = x^3+6x^2+12x+8$$

The Hypercube (dimension $d = 3$)

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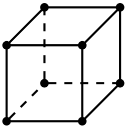

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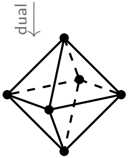

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$$f(x) = x^3 + 6x^2 + 12x + 8$$

$$h(x) \stackrel{\text{def}}{=} f(x-1)$$

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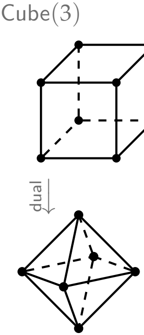

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$$f(x) = x^3 + 6x^2 + 12x + 8$$

$$h(x) = x^3 + 3x^2 + 3x + 1$$

The Hypercube (dimension $d = 3$)

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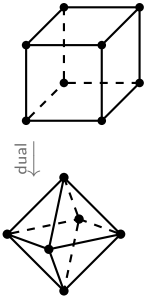

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$$\tilde{f}(x) \overset{\text{def}}{=} x^d f\left(\frac{1}{x}\right)$$

$$h(x) = x^3 + 3x^2 + 3x + 1$$

The Hypercube (dimension $d = 3$)

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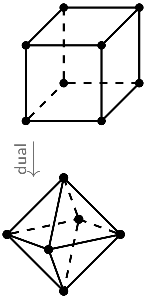

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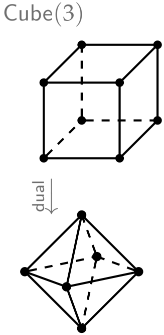
$$\tilde{f}(x) = 1+6x+12x^2+8x^3$$

$$h(x) = x^3+3x^2+3x+1$$

The Hypercube (dimension $d = 3$)



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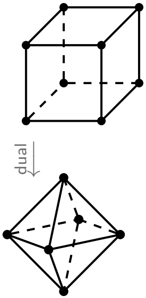
Cube(3)



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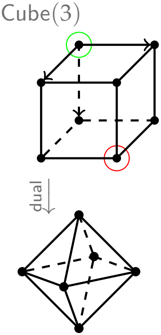

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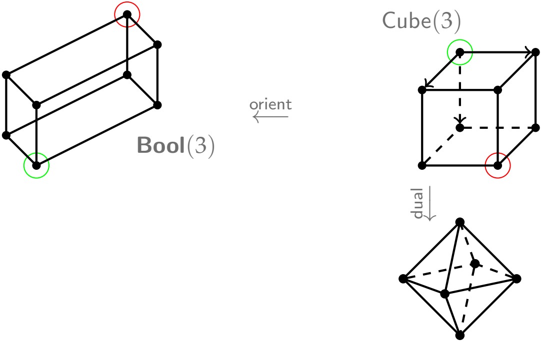

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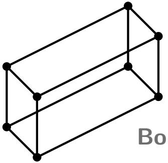
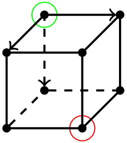


Bool(3)

$\xleftarrow{\text{orient}}$

Cube(3)



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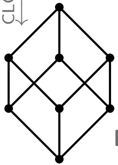
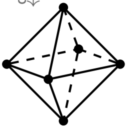
$\downarrow$ CLO



Bool(3)

$\downarrow$ dual



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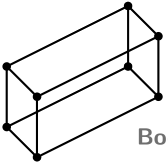
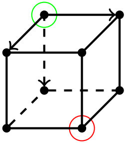


Bool(3)

orient $\longleftarrow$

Cube(3)



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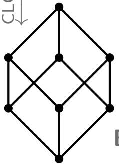
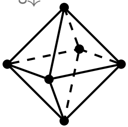
CLO $\downarrow$



Bool(3)

dual $\downarrow$



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$$c(x) \overset{\text{def}}{=} \sum_a x^{d-\mathsf{rk}(a)}(x+1)^{\mathsf{rk}(a)}$$

$$r(x) \overset{\text{def}}{=} \sum_a x^{\mathsf{rk}(a)}$$

$$\tilde{f}(x) = (2x+1)^3$$

$$h(x) = (x+1)^3$$

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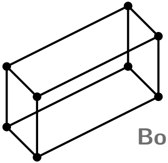

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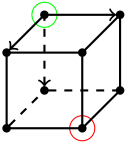

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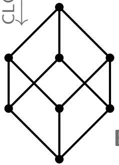

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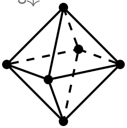

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$c(x) = x^3+3x^2(x+1)+3x(x+1)^2+(x+1)^3$

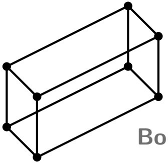
$r(x) = 1+3x+3x^2+x^3$

$\tilde{f}(x) = (2x+1)^3$

$h(x) = (x+1)^3$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Hypercube (dimension $d = 3$)



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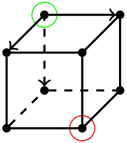
$\xleftarrow{\text{orient}}$

Cube(3)



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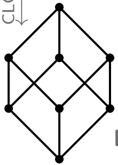

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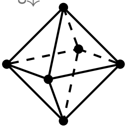

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$\tilde{f}(x) = (2x+1)^3$

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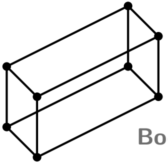
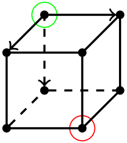


Bool(3)

orient $\longleftarrow$

Cube(3)



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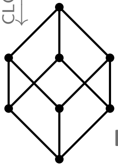

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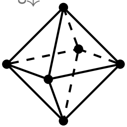

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$c(x) = (2x+1)^3$

$r(x) = (x+1)^3$

$$c(x) = x^3 r\left(\frac{x+1}{x}\right)$$

$\tilde{f}(x) = (2x+1)^3$

$h(x) = (x+1)^3$

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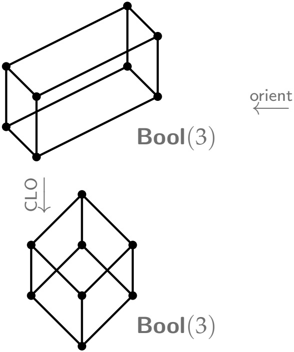
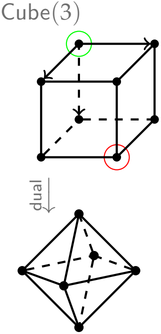
The Hypercube (dimension d)





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$$c(x) = x^d r\left(\frac{x+1}{x}\right)$$

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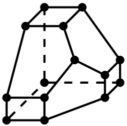
The Associahedron (dimension $d = 3$)

$\mathsf{Asso}(3)$



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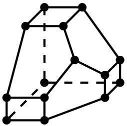

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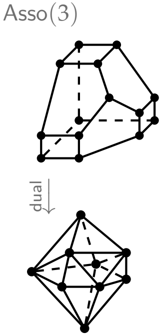

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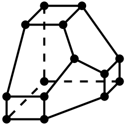

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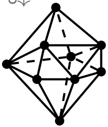
dual ↓



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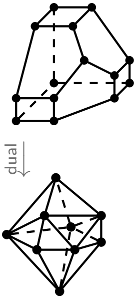
$f_3 = 1$

The Associahedron (dimension $d = 3$)

$\mathsf{Asso}(3)$



face numbers:

$f_{-1} = 1$

$f_0 = 14$

$f_1 = 21$

$f_2 = 9$

$f_3 = 1$

face numbers:

$f_{-1} = 1$

$f_0 = 9$

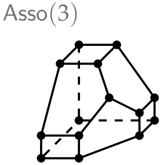
$f_1 = 21$

$f_2 = 14$

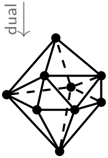
$f_3 = 1$

$$\tilde{f}(x) = 1+9x+21x^2+14x^3$$

The Associahedron (dimension $d = 3$)



face numbers:
$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

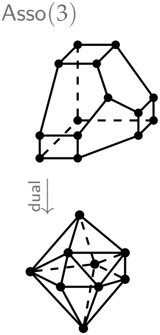


face numbers:
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$$\tilde{f}(x) = 1+9x+21x^2+14x^3$$

$$h(x) = x^3+6x^2+6x+1$$

The Associahedron (dimension $d = 3$)



face numbers:
$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

face numbers:
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

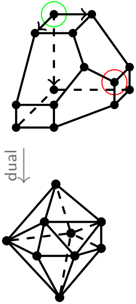
$$\tilde{f}(x) = 1+9x+21x^2+14x^3$$

$$h(x) = x^3+6x^2+6x+1$$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Associahedron (dimension $d = 3$)

Asso(3)



face numbers:

$f_{-1} = 1$

$f_0 = 14$

$f_1 = 21$

$f_2 = 9$

$f_3 = 1$

face numbers:

$f_{-1} = 1$

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$$\tilde{f}(x) = 1+9x+21x^2+14x^3$$

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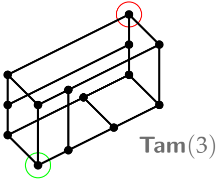
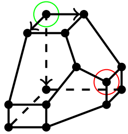
The Associahedron (dimension $d = 3$)



orient ⟵

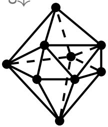
Asso(3)



face numbers:

$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

dual ↓



face numbers:

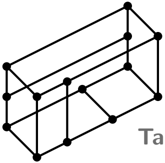
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$$\tilde{f}(x) = 1+9x+21x^2+14x^3$$

$$h(x) = x^3+6x^2+6x+1$$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

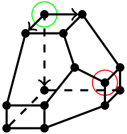
The Associahedron (dimension $d = 3$)



Tam(3)

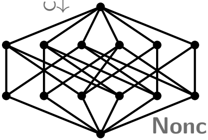
orient $\longleftarrow$

Asso(3)



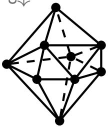
face numbers:
$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

CLO $\downarrow$



Nonc(3)

dual $\downarrow$



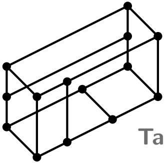
face numbers:
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$\tilde{f}(x) = 1+9x+21x^2+14x^3$

$h(x) = x^3+6x^2+6x+1$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

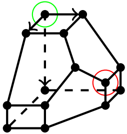
The Associahedron (dimension $d = 3$)



Tam(3)

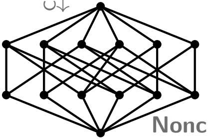
orient $\longleftarrow$

Asso(3)



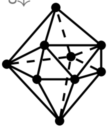
face numbers:
$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

CLO $\downarrow$



Nonc(3)

dual $\downarrow$



face numbers:
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$c(x) = x^3+6x^2(x+1)+6x(x+1)^2+(x+1)^3$

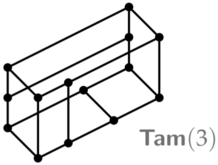
$r(x) = 1+6x+6x^2+x^3$

$\tilde{f}(x) = 1+9x+21x^2+14x^3$

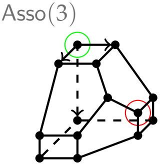
$h(x) = x^3+6x^2+6x+1$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Associahedron (dimension $d = 3$)



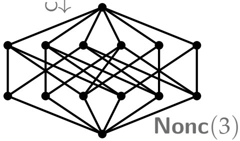
orient $\longleftarrow$



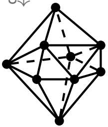
face numbers:

$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

CLO $\downarrow$



dual $\downarrow$



face numbers:

$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$c(x) = 1+9x+21x^2+14x^3$

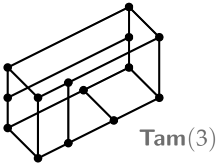
$r(x) = 1+6x+6x^2+x^3$

$\tilde{f}(x) = 1+9x+21x^2+14x^3$

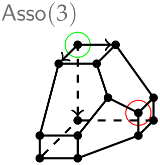
$h(x) = x^3+6x^2+6x+1$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Associahedron (dimension $d = 3$)

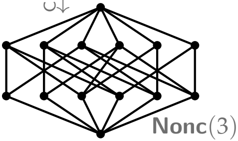


orient $\longleftarrow$

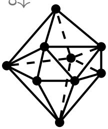


face numbers:
$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

CLO $\downarrow$



dual $\downarrow$



face numbers:
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$c(x) = 1+9x+21x^2+14x^3$

$r(x) = 1+6x+6x^2+x^3$

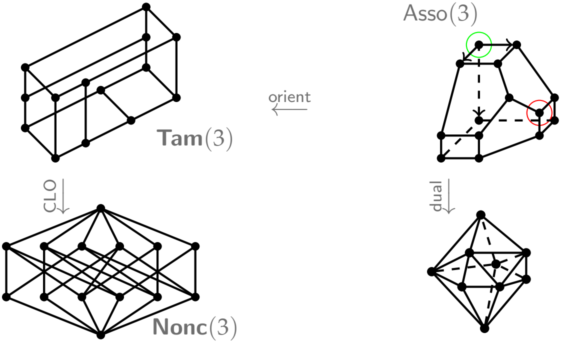
$$c(x) = x^3 r\left(\frac{x+1}{x}\right)$$

$\tilde{f}(x) = 1+9x+21x^2+14x^3$

$h(x) = x^3+6x^2+6x+1$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Associahedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 14$
$f_1 = 21$
$f_2 = 9$
$f_3 = 1$

face numbers:

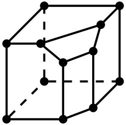
$f_{-1} = 1$
$f_0 = 9$
$f_1 = 21$
$f_2 = 14$
$f_3 = 1$

$$c(x) = x^d r\left(\frac{x+1}{x}\right)$$

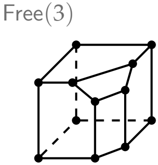
$$\tilde{f}(x) = x^d h\left(\frac{x+1}{x}\right)$$

The Freehedron (dimension $d = 3$)

$\mathsf{Free}(3)$



The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$

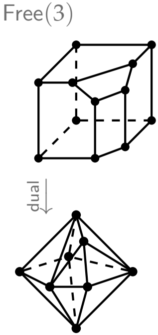
$f_0 = 12$

$f_1 = 18$

$f_2 = 8$

$f_3 = 1$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$

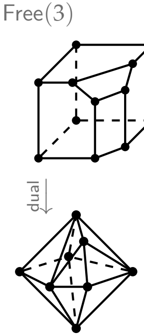
$f_0 = 12$

$f_1 = 18$

$f_2 = 8$

$f_3 = 1$

The Freehedron (dimension $d = 3$)



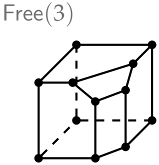
face numbers:

- $f_{-1} = 1$
- $f_0 = 12$
- $f_1 = 18$
- $f_2 = 8$
- $f_3 = 1$

face numbers:

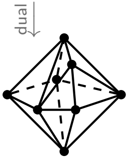
- $f_{-1} = 1$
- $f_0 = 8$
- $f_1 = 18$
- $f_2 = 12$
- $f_3 = 1$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$



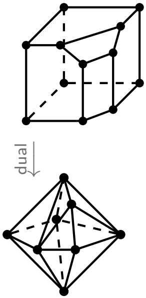
face numbers:

$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$\tilde{f}(x) = 1+8x+18x^2+12x^3$

The Freehedron (dimension $d = 3$)

$\mathsf{Free}(3)$



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

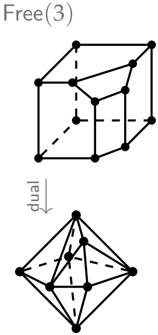
face numbers:

$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$$\tilde{f}(x) = 1+8x+18x^2+12x^3$$

$$h(x) = x^3+5x^2+5x+1$$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

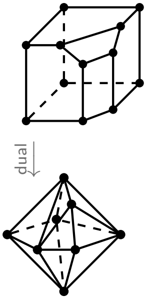
face numbers:

$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$$\tilde{f}(x) = (2x+1)\big(6(x^2+x)+1\big)$$

$$h(x) = (x+1)(x^2+4x+1)$$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

face numbers:

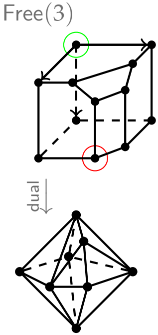
$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$$\tilde{f}(x) = (2x+1)\big(6(x^2+x)+1\big)$$

$$h(x) = (x+1)(x^2+4x+1)$$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

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$f_{-1} = 1$
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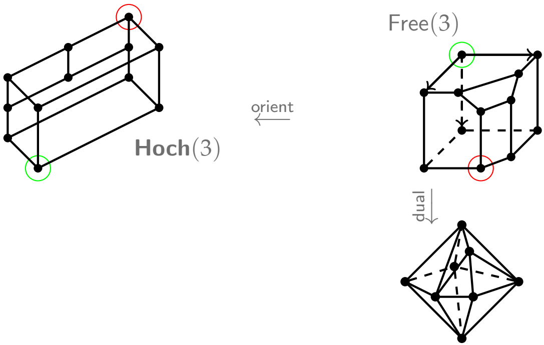
The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

face numbers:

$f_{-1} = 1$
$f_0 = 8$
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$$\tilde{f}(x) = (2x+1)\big(6(x^2+x)+1\big)$$

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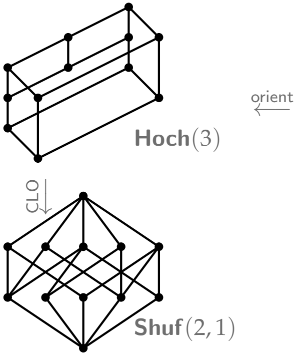
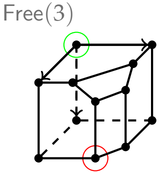
$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

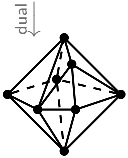
The Freehedron (dimension $d = 3$)





face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$



face numbers:

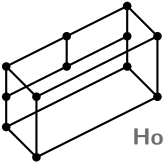
$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$$\tilde{f}(x) = (2x+1)\big(6(x^2+x)+1\big)$$

$$h(x) = (x+1)(x^2+4x+1)$$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

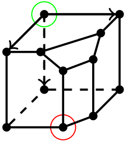
The Freehedron (dimension $d = 3$)



Hoch(3)

orient $\longleftarrow$

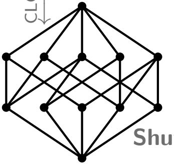
Free(3)



face numbers:

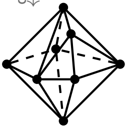
$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
$f_2 = 8$
$f_3 = 1$

CLO $\downarrow$



Shuf(2, 1)

dual $\downarrow$



face numbers:

$f_{-1} = 1$
$f_0 = 8$
$f_1 = 18$
$f_2 = 12$
$f_3 = 1$

$c(x) = x^3 + 5x^2(x+1) + 5x(x+1)^2 + (x+1)^3$

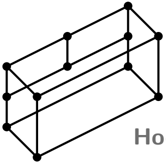
$r(x) = 1 + 5x + 5x^2 + x^3$

$\tilde{f}(x) = (2x+1)(6(x^2+x)+1)$

$h(x) = (x+1)(x^2+4x+1)$

$$\tilde{f}(x) = x^3 h\left(\frac{x+1}{x}\right)$$

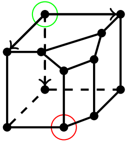
The Freehedron (dimension $d = 3$)



Hoch(3)

orient $\longleftarrow$

Free(3)



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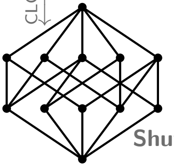

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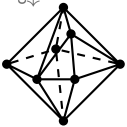

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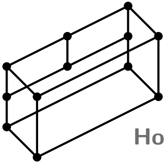

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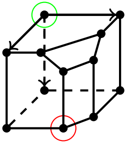

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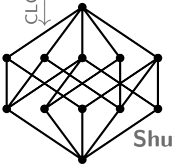

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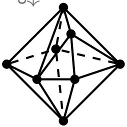

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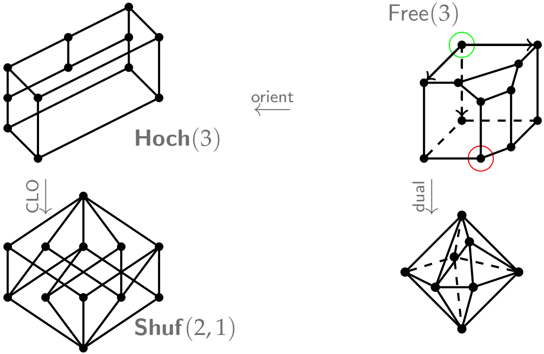

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Outline

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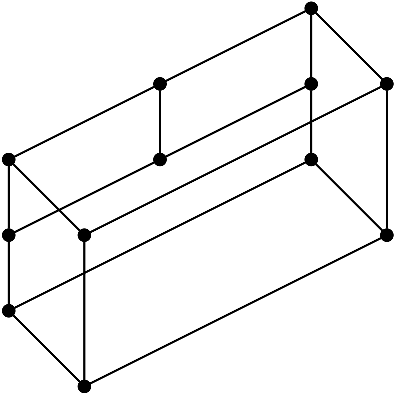
The Hochschild Lattice

Tamari



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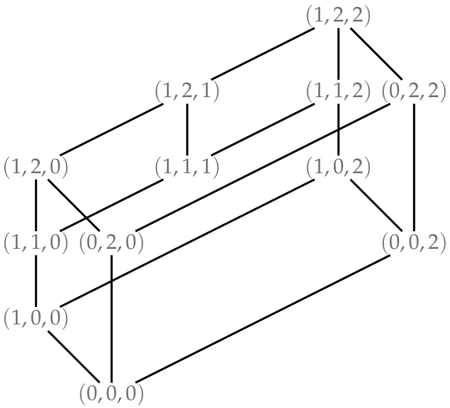

The Hochschild Lattice

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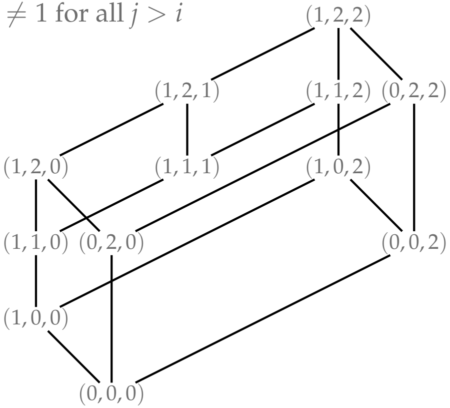
- **triword**: an integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1, 2\}$ $\rightsquigarrow \mathsf{Tri}(n)$
  - $u_1 \neq 2$
  - $u_i = 0$ implies $u_j \neq 1$ for all $j > i$



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Theorem (C. Combe, 2020)

For $n > 0$, the componentwise order on $\mathsf{Tri}(n)$ realizes the Hochschild lattice of order n.

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$1, 2, 5, 12, 28, 64, 144, 320, 704, \ldots$ (A045623 in OEIS)

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For $n > 0$, the Hochschild lattice $\mathbf{Hoch}(n)$ is semidistributive.

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$$f_0 \colon \mathsf{Tri}(n) \to \{1, 2, \ldots, n{+}1\}$$
$$\mathfrak{u} \mapsto \begin{cases} n+1, & \text{if } 0 \notin \mathfrak{u} \\ \min\{i \mid u_i = 0\}, & \text{otherwise} \end{cases}$$
$$l_1 \colon \mathsf{Tri}(n) \to \{0, 1, \ldots, n\}$$
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- by definition, $l_1(\mathfrak{u}) < f_0(\mathfrak{u})$

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- **edge**: $(\mathfrak{u}, \mathfrak{v})$ such that $\mathfrak{u} < \mathfrak{v}$ without $\mathfrak{u} < \mathfrak{u}' < \mathfrak{v}$
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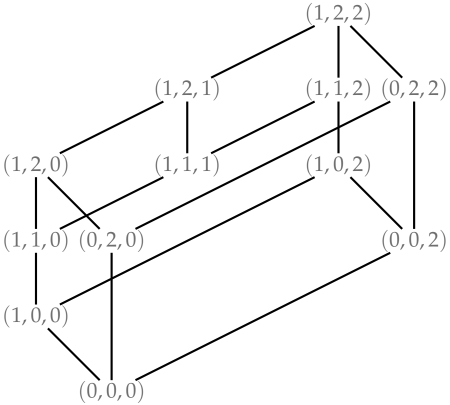
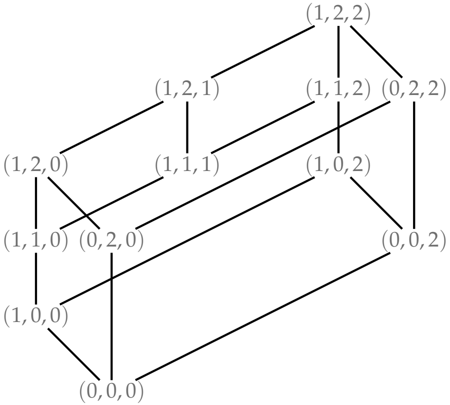
- if $(\mathfrak{u},\mathfrak{v}) \in \mathcal{E}\bigl(\mathsf{\mathbf{Hoch}}(n)\bigr)$, then $u_i < v_i$ for a unique $i \in [n]$



The Hochschild Lattice

Perspectivity Irreducibility

- join-irreducible triwords:



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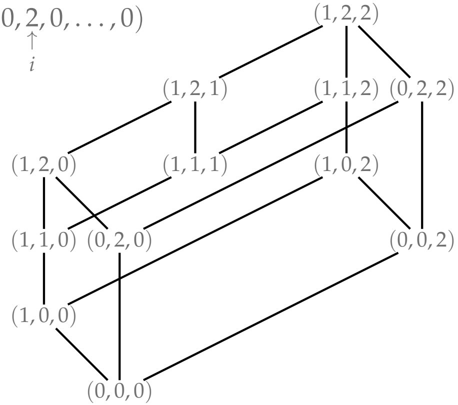

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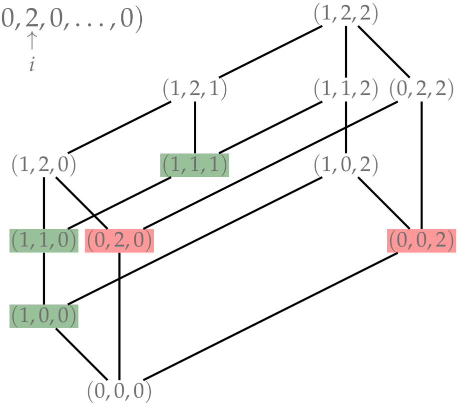

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- $\lambda(\mathfrak{u},\mathfrak{v}) \overset{\text{def}}{=} \begin{cases}\mathfrak{a}^{(i)}, & \text{if } u_i = 0 \text{ and } v_i = 1\\ \mathfrak{b}^{(i)}, & \text{if } u_i < 2 \text{ and } v_i = 2\end{cases}$

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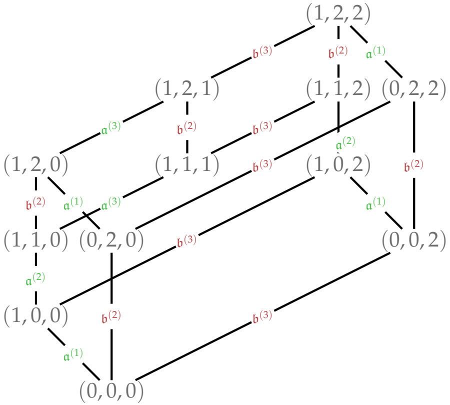

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Proposition (, 2020)

For $\mathfrak{u} \in \mathsf{Tri}(n)$, we have

$$\mathsf{Can}(\mathfrak{u}) = \left\{\mathfrak{a}^{(i)} \mid i = l_1(\mathfrak{u}) \text{ if } l_1(\mathfrak{u}) > 0\right\} \uplus \left\{\mathfrak{b}^{(i)} \mid u_i = 2\right\}.$$

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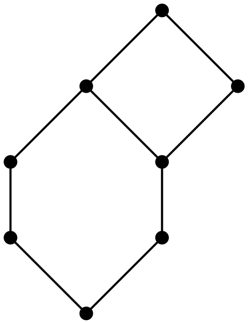

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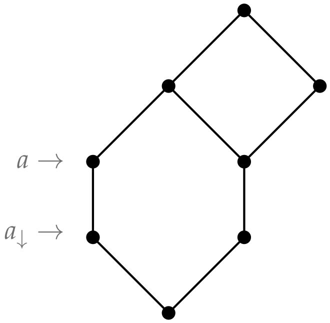

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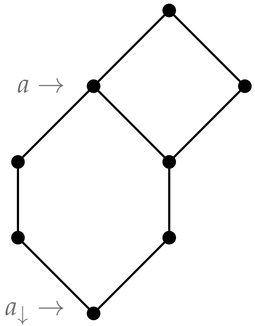

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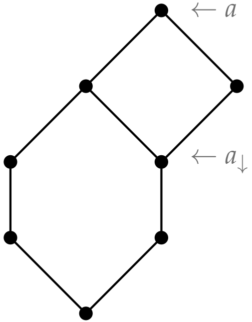

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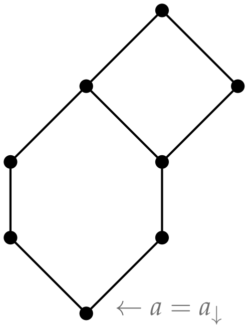

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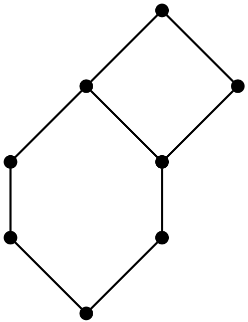

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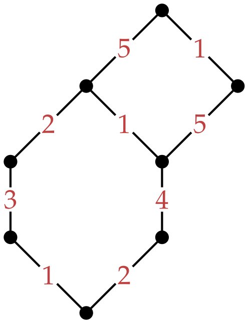

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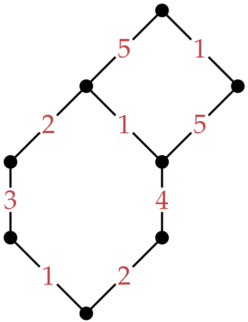

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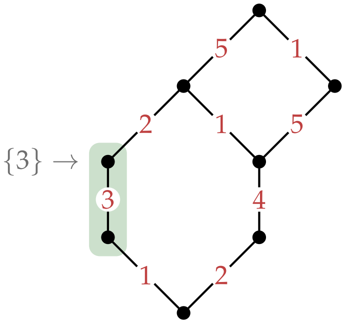

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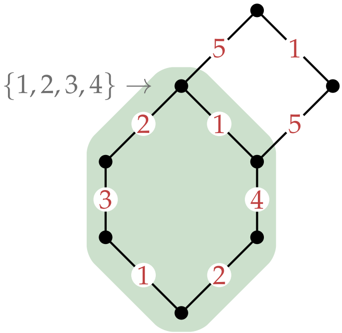

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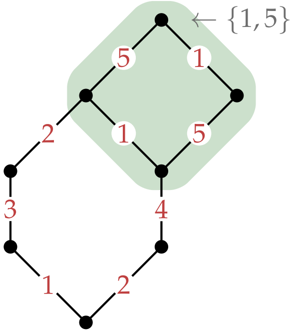

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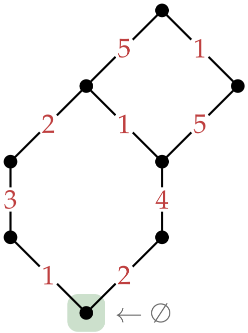

The Core Label Order

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice, $a \in L$, λ .. edge labeling
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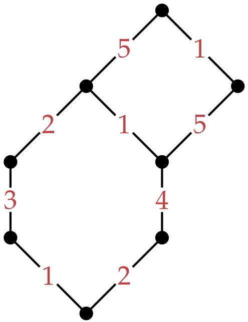

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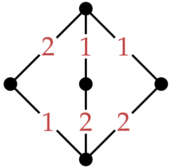

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Proposition (, 2020)

The labeling λ is a core labeling of $\mathsf{Hoch}(n)$.

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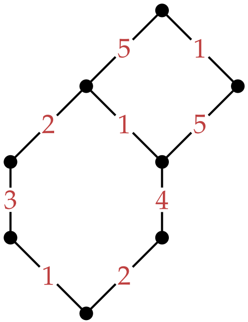

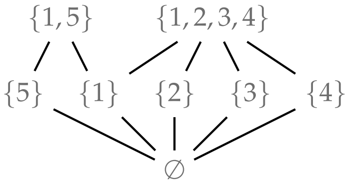

The Hochschild Lattice

Proposition (🏠, 2020)

The core label set of $\mathfrak{u} \in \mathsf{Tri}(n)$ *is*

$$\Psi(\mathfrak{u}) = \left\{\mathfrak{a}^{(i)} \mid 0 < l_1(\mathfrak{u}) \leq i < f_0(\mathfrak{u})\right\} \uplus \left\{\mathfrak{b}^{(i)} \mid u_i = 2\right\}.$$

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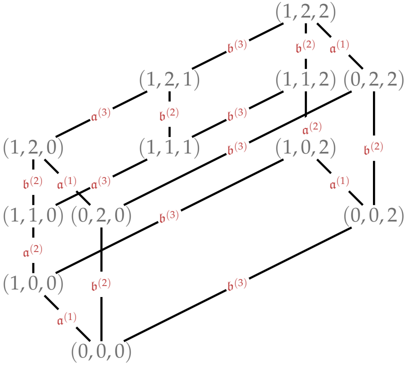

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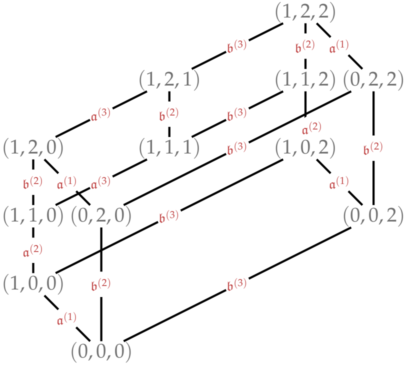

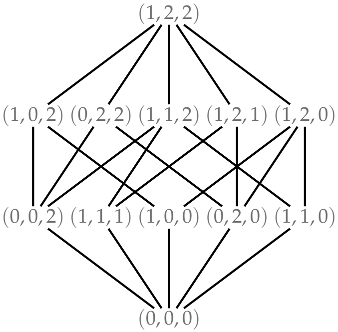

Outline

Shuffle Lattices

- $\mathbf{a} = a_1 a_2 \cdots a_r$, $\mathbf{b} = b_1 b_2 \cdots b_s$
- **(word) shuffle**: word using letters a_i or b_i whose restriction to the a_i's and b_i's preserves order

$\rightsquigarrow \mathsf{Shuf}(r,s)$

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Theorem (C. Greene, 1988)

For $r,s \geq 0$, the poset $\mathbf{Shuf}(r,s) \stackrel{\text{def}}{=} \big(\mathsf{Shuf}(r,s), \preceq\big)$ *is a lattice.*

Shuffle Lattices

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Proposition (C. Greene, 1988)

For $r,s \geq 0$, we have $\left|\mathsf{Shuf}(r,s)\right| = 2^{r+s}\sum_{j\geq 0}\binom{r}{j}\binom{s}{j}\left(\frac{1}{4}\right)^j.$

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Corollary

For $n > 0$, we have $\big|\mathsf{Shuf}(n-1, 1)\big| = 2^{n-2}(n+3)$.

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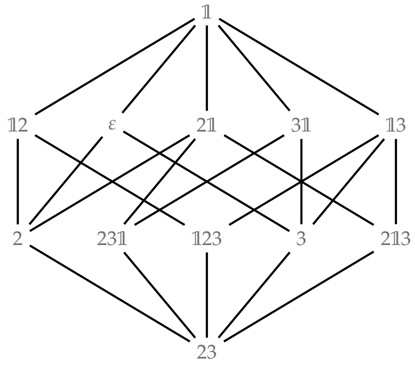
Shuf$(2, 1)$



A Bijection

Tamari

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$$\mathfrak{u} = (1, 1, 1, 2, 2, 2, 1, 0, 0, 2) \in \mathsf{Tri}(10)$$

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Proposition (, 2020)

For $n > 0$, the map $\sigma\colon \mathsf{Tri}(n) \to \mathsf{Shuf}(n-1,1)$ is a bijection.

A Bijection

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- $\sigma(\mathfrak{u}) \overset{\text{def}}{=} \tau(\mathfrak{u}) \sqcup_{l_1(\mathfrak{u})} \mathbb{1}$

Theorem (2020)

For $n > 0$, the map σ extends to an isomorphism from $\mathbf{CLO}(\mathbf{Hoch}(n))$ to $\mathbf{Shuf}(n-1,1)$.

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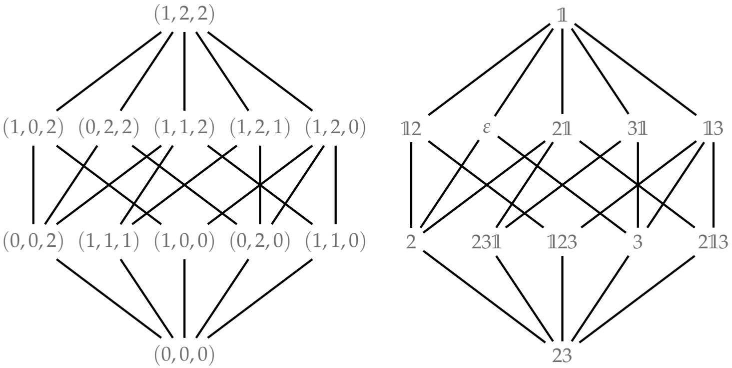

Enumeration

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Proposition (C. Greene, 1988)

Let $\mathbf{w} \in \mathsf{Shuf}(n-1,1)$*. The rank of* $\mathbf{w}$ *in* $\mathbf{Shuf}(n-1,1)$ *is*

$$n - 1 - a(\mathbf{w}) + \begin{cases} 1, & \text{if } \mathbf{w} \text{ contains } \mathbb{1}, \\ 0, & \text{otherwise.} \end{cases}$$

Enumeration

Corollary (, 2020)

Let $\mathfrak{u} \in \mathsf{Tri}(n)$. *The rank of* $\mathfrak{u}$ *in* $\mathbf{CLO}\big(\mathbf{Hoch}(n)\big)$ *is*

$$\Big|\{i \mid u_i = 2\}\Big| + \begin{cases} 1, & \text{if } l_1(\mathfrak{u}) > 0,\\ 0, & \text{otherwise.}\end{cases}$$

Enumeration

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The number of $\mathfrak{u} \in \mathsf{Tri}(n)$ *having rank* i *in* $\mathbf{CLO}\bigl(\mathbf{Hoch}(n)\bigr)$ *is*

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$$\underset{l_1(\mathfrak{u})=0}{\uparrow} \qquad \underset{l_1(\mathfrak{u})=1}{\uparrow} \qquad \underset{l_1(\mathfrak{u})>1}{\uparrow}$$

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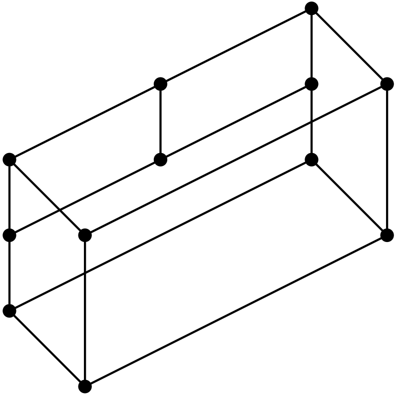
Proposition (, 2020)

The rank of $\mathfrak{u} \in \mathsf{Tri}(n)$ *in* $\mathbf{CLO}\big(\mathbf{Hoch}(n)\big)$ *equals* $\mathsf{in}(\mathfrak{u})$.

Outline

Recovering the Freehedron

Tamari



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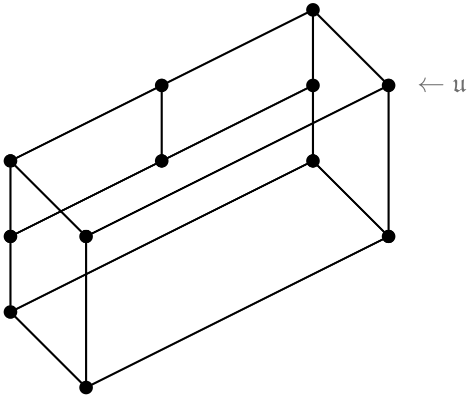

Recovering the Freehedron

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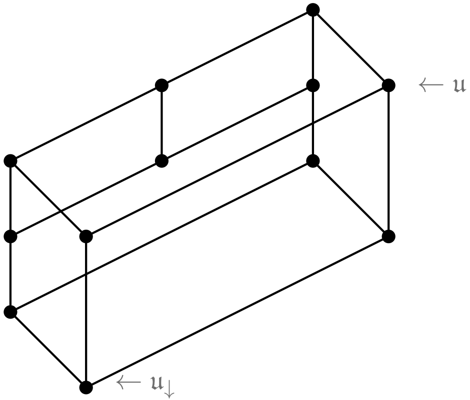
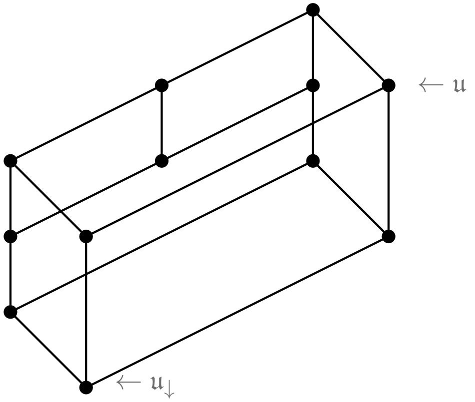


Recovering the Freehedron

Tamari

- $\mathsf{Pre}(\mathfrak{u}) = \{\mathfrak{u}' \in \mathsf{Tri}(n) \mid (\mathfrak{u}', \mathfrak{u}) \in \mathcal{E}(\mathbf{Hoch}(n))\}$



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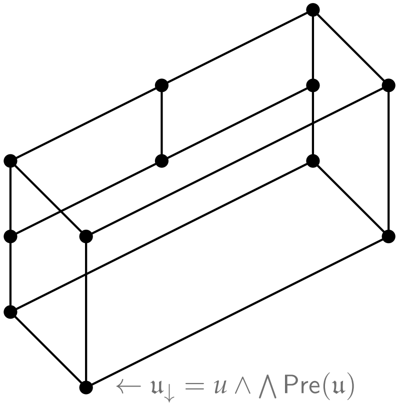

$\leftarrow \mathfrak{u}$

$\leftarrow \mathfrak{u}_{\downarrow} = u \wedge \bigwedge \mathsf{Pre}(\mathfrak{u})$

Hochschild, Shuffle, FHM

Henri Mühle

Recovering the Freehedron

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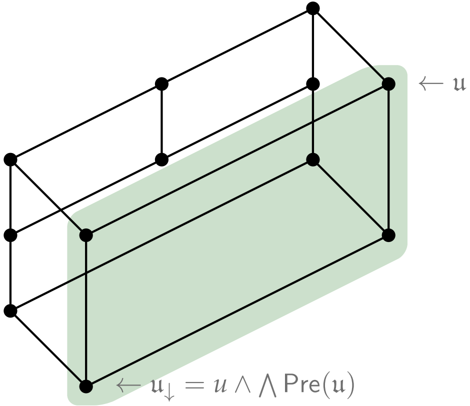

Recovering the Freehedron

Tamari

- $\mathsf{Pre}(\mathfrak{u}) = \{\mathfrak{u}' \in \mathsf{Tri}(n) \mid (\mathfrak{u}', \mathfrak{u}) \in \mathcal{E}(\mathbf{Hoch}(n))\}$
- $P \subseteq \mathsf{Pre}(\mathfrak{u})$: $\mathfrak{u}_{\downarrow P} \stackrel{\text{def}}{=} \mathfrak{u} \wedge \bigwedge\{\mathfrak{u}' \mid \mathfrak{u}' \in P\}$
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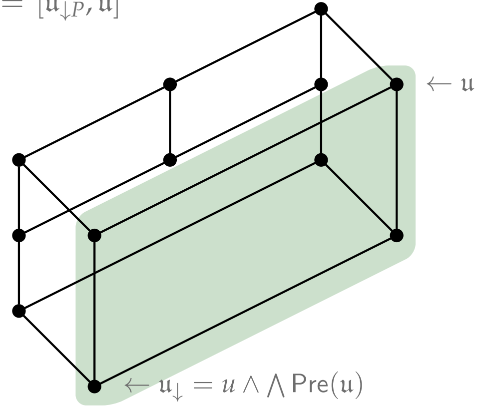

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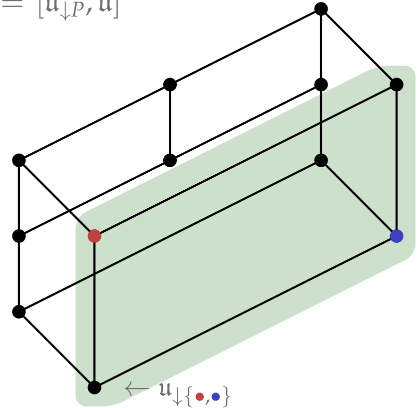

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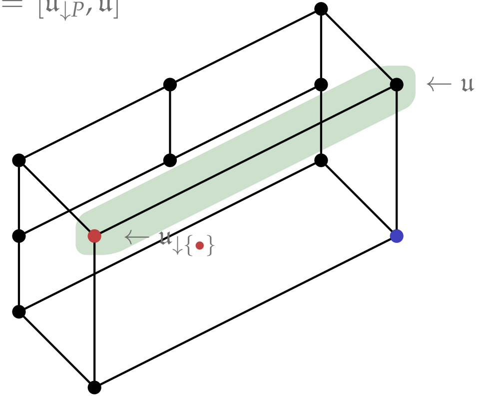

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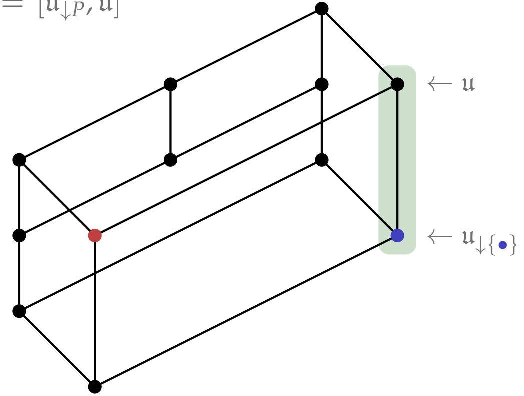

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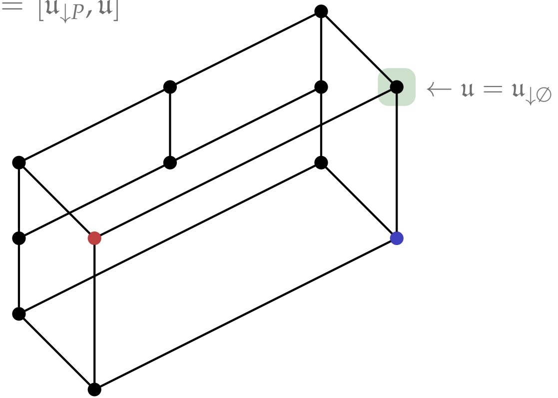

Recovering the Freehedron

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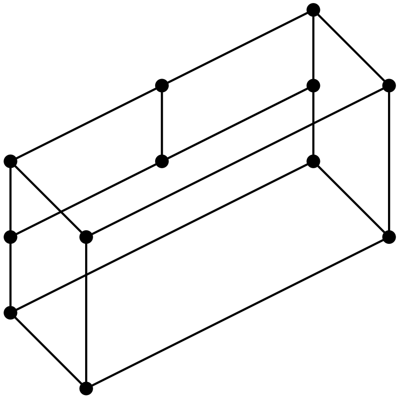
- $\mathsf{CP}\big(\mathbf{Hoch}(n)\big) \overset{\mathsf{def}}{=} \Big\{(\mathfrak{u}, P) \mid \mathfrak{u} \in \mathsf{Tri}(n), P \subseteq \mathsf{Pre}(\mathfrak{u})\Big\}$



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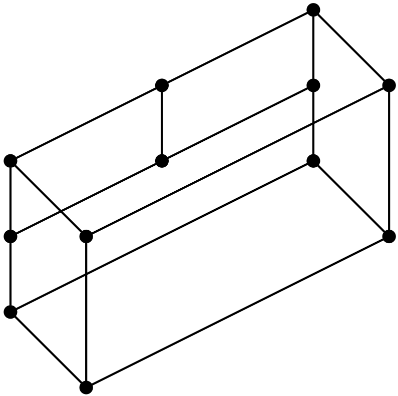

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Observation

The set $\mathsf{CP}\big(\mathbf{Hoch}(n)\big)$ *is in bijection with the faces of* $\mathsf{Free}(n)$.

Recovering the Freehedron

Tamari

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Recovering the Freehedron

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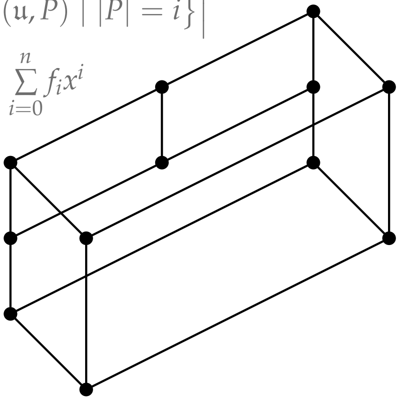

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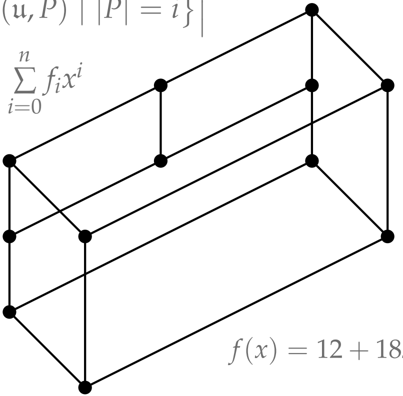

$f(x) = 12 + 18x + 8x^2 + x^3$

Recovering the Freehedron

Tamari

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Observation

$f(x)$ is the f-polynomial of the dual of $\mathsf{Free}(n)$.

Recovering the Freehedron

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Observation

$f(x)$ is the f-polynomial of the dual of $\mathsf{Free}(n)$.

$(\mathfrak{u}, P)$ with $|P| = i$ corresponds to an $(n-1-i)$-face of $\partial\mathsf{Free}(n)^{\mathsf{dual}}$.

Recovering the Freehedron

Tamari

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- $f_i \overset{\text{def}}{=} \Big|\big\{(\mathfrak{u},P) \mid |P| = i\big\}\Big|$

Proposition (🌬, 2020)

For $n > 0$ and $0 \leq i \leq n$, we have

$$f_i = \binom{n}{i} 2^{n-i-2}\frac{n(n+3) - i(i-1)}{n}.$$

Recovering the Freehedron

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- $f(x) \stackrel{\text{def}}{=} \sum_{i=0}^{n} f_i x^i$

Corollary (, 2020)

For $n > 0$, we have

$$f(x) = (x+2)^{n-2}\Big(x^2 + (n+3)x + n + 3\Big),$$

$$h(x) = (x+1)^{n-2}\Big(x^2 + (n+1)x + 1\Big).$$

Back to the Hochschild Lattice

- to prove the proposition, we observe:

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$$h(x) = f(x-1)$$

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Corollary (, 2020)

The number of $\mathfrak{u} \in \mathsf{Tri}(n)$ *with* $\mathsf{in}(\mathfrak{u}) = i$ *is*

$$\binom{n-1}{i} + \binom{n-1}{i-1} + (n-1)\binom{n-2}{i-1}.$$

Back to the Hochschild Lattice

- to prove the proposition, we observe:

$$\tilde{f}(x) = x^n f\left(\frac{1}{x}\right)$$

$$h(x) = \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{\mathsf{in}(\mathfrak{u})}$$

Back to the Hochschild Lattice

- to prove the proposition, we observe:

$$\tilde{f}(x) = \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{n-\mathsf{in}(\mathfrak{u})}(x+1)^{\mathsf{in}(\mathfrak{u})}$$

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Back to the Hochschild Lattice

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$$c(x) = \sum_{\mathfrak{u} \in \mathsf{Tri}(n)} x^{n - \mathsf{in}(\mathfrak{u})}(x+1)^{\mathsf{in}(\mathfrak{u})}$$

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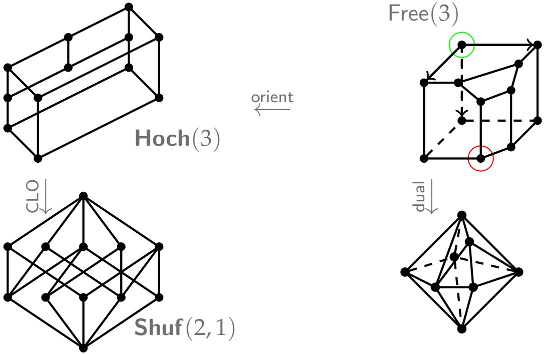
Corollary (🌬, 2020)

For $n > 0$, we have

$$c(x) = x^n(2x+1)^{n-2}\Big((n+3)(x^2+x)+1\Big),$$

$$r(x) = x^n(x+1)^{n-2}\Big(x^2+(n+1)x+1\Big).$$

The Freehedron (dimension $d = 3$)



face numbers:

$f_{-1} = 1$
$f_0 = 12$
$f_1 = 18$
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$f_{-1} = 1$
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$$c(x) = (2x+1)^{d-2}\big((d+3)(x^2+x)+1\big)$$

$$r(x) = (x+1)^{d-2}\big(x^2+(d+1)x+1\big)$$

$$c(x) = x^d r\left(\frac{x+1}{x}\right)$$

$$\tilde{f}(x) = (2x+1)^{d-2}\big((d+3)(x^2+x)+1\big)$$

$$h(x) = (x+1)^{d-2}\big(x^2+(d+1)x+1\big)$$

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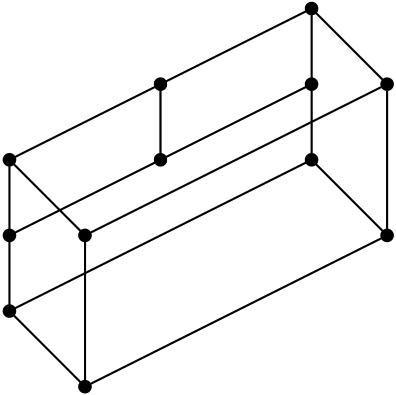
Outline

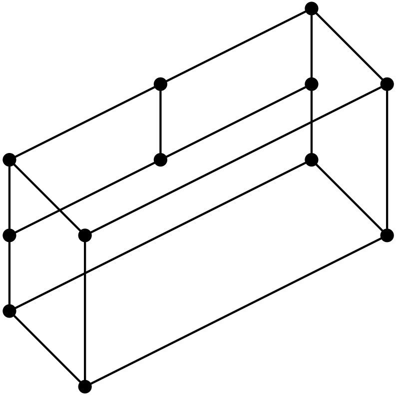
Refined Face Enumeration



Refined Face Enumeration

- $c(x) = \sum_{\mathfrak{u} \in \mathsf{Tri}(n)} x^{n-\mathsf{in}(\mathfrak{u})}(x+1)^{\mathsf{in}(\mathfrak{u})}$

$n = 3$



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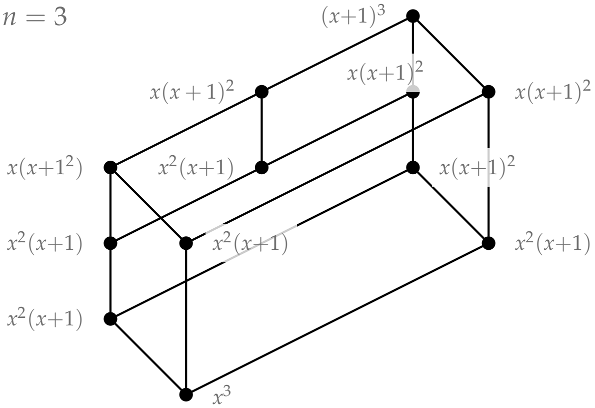

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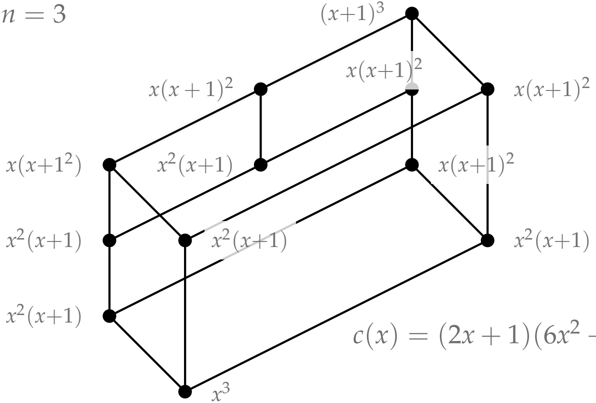

$c(x) = (2x+1)(6x^2+6x+1)$

Refined Face Enumeration

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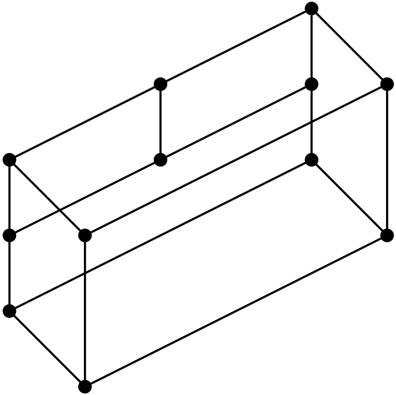

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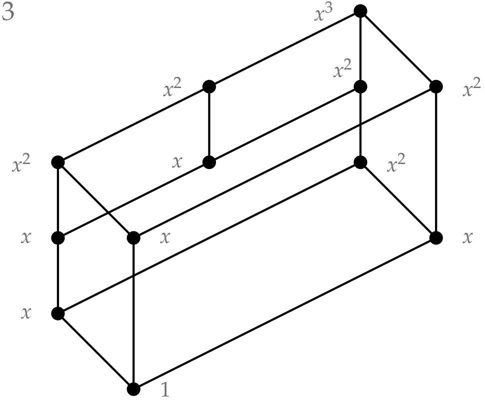

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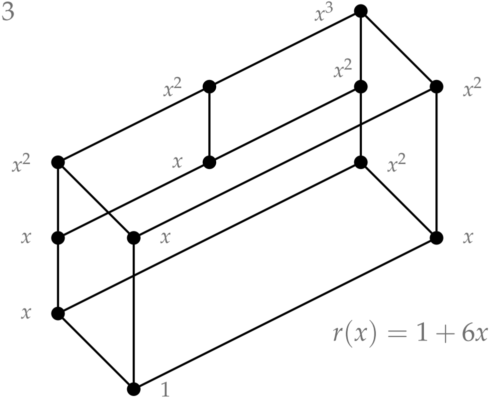
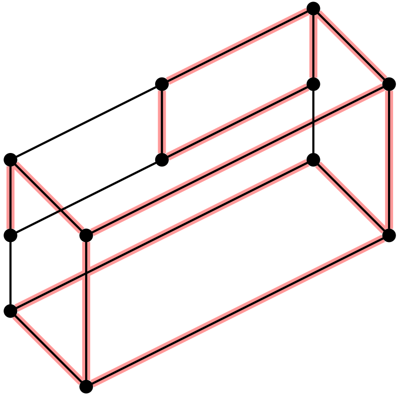


$r(x) = 1 + 6x + 6x^2 + x^3$

Refined Face Enumeration

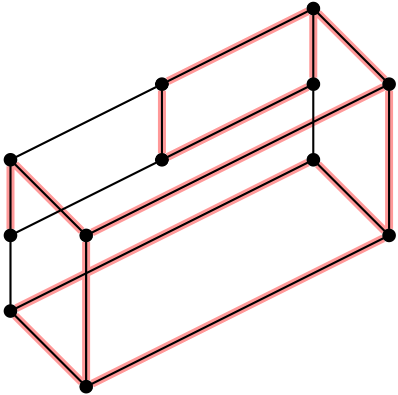
$n = 3$



Refined Face Enumeration

- $F_{\mathsf{Hoch}(n)}(x,y) \overset{\text{def}}{=} \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{n-\mathsf{in}(\mathfrak{u})}(x+1)^{\mathsf{in}(\mathfrak{u})-\mathsf{in}(\mathfrak{u})}(y+1)^{\mathsf{in}(\mathfrak{u})}$

$n=3$



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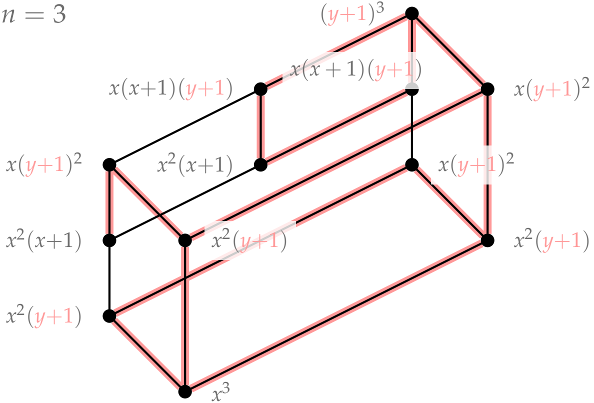

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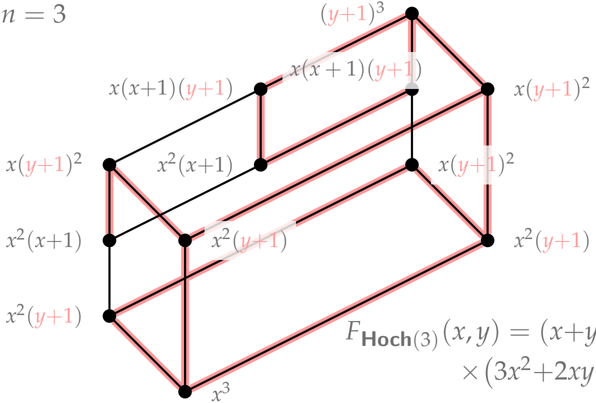
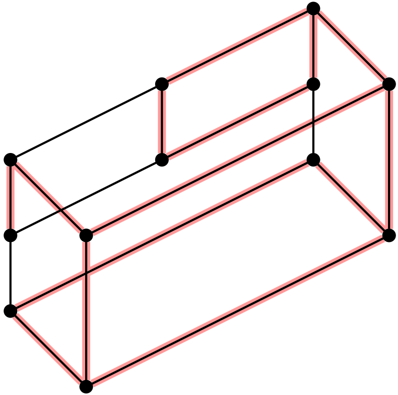


$$F_{\mathsf{Hoch}(3)}(x,y) = (x+y+1) \times \big(3x^2+2xy+4x+(y+1)^2\big)$$

Refined Face Enumeration

- $F_{\mathsf{Hoch}(n)}(x,x) = c(x)$

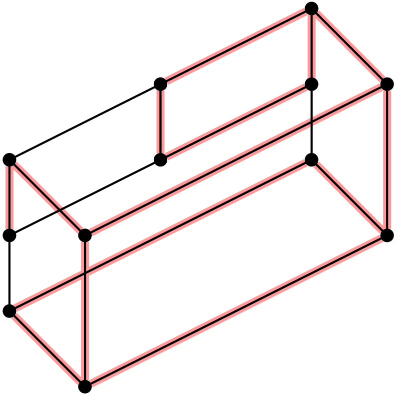
$n = 3$



Refined Face Enumeration

- $H_{\mathsf{Hoch}(n)}(x,y) \overset{\mathsf{def}}{=} \sum_{\mathfrak{u}\in\mathsf{Tri}(n)} x^{\mathsf{in}(\mathfrak{u})} y^{\mathsf{in}(\mathfrak{u})}$

$n = 3$



Refined Face Enumeration

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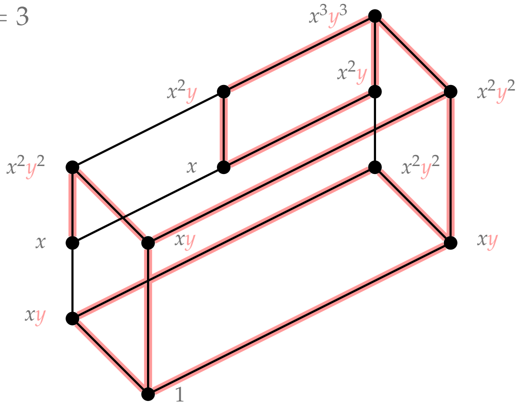

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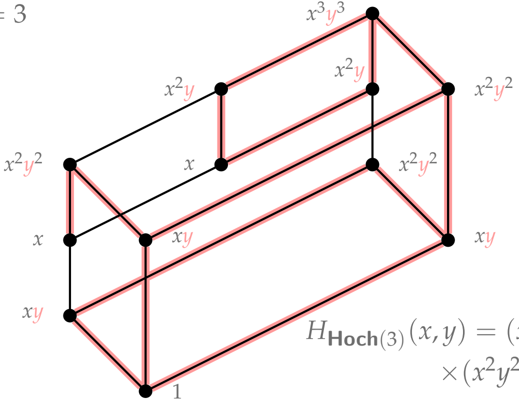
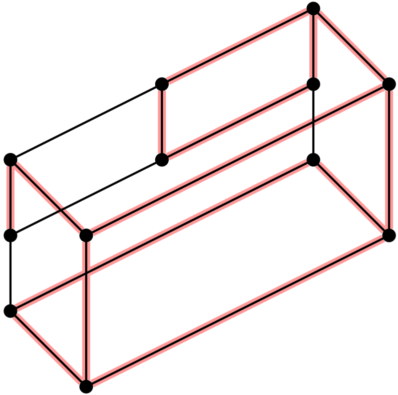


$$H_{\mathsf{Hoch}(3)}(x,y) = (xy+1) \times (x^2y^2+2xy+2x+1)$$

Refined Face Enumeration

- $H_{\mathsf{Hoch}(n)}(x,1) = r(x)$

$n = 3$



Refined Face Enumeration

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  - $\mathsf{in}(\mathfrak{u}) = \left|\mathsf{Can}(\mathfrak{u})\right|$
  - $\mathsf{Can}(\mathfrak{u})$ consists of join-irreducible triwords
- we want a distinguished subset of join-irreducible triwords to realize $\mathsf{in}(\mathfrak{u})$ canonically

Refined Face Enumeration

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice; $\hat{0}$ .. least element
- **atom**: $a \in L$ such that $(\hat{0}, a) \in \mathcal{E}(\mathbf{L})$ $\rightsquigarrow \mathcal{A}(\mathbf{L})$

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Proposition (, 2020)

For $n > 0$, we have $\mathcal{A}\big(\mathsf{Hoch}(n)\big) = \big\{\mathfrak{a}^{(1)}, \mathfrak{b}^{(2)}, \ldots, \mathfrak{b}^{(n)}\big\}$.

Refined Face Enumeration

- $\mathsf{pos}(\mathfrak{u}) \stackrel{\mathsf{def}}{=} \bigl|\mathsf{Can}(\mathfrak{u}) \setminus \mathcal{A}\bigl(\mathbf{Hoch}(n)\bigr)\bigr|$
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$F = H$

Corollary

For $n > 0$, we have

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Joint with C. Ceballos in the context of ν-Tamari lattices.

$F = H$

Corollary

For $n > 0$, we have

$$F_{\mathsf{Hoch}(n)}(x,y) = x^n H_{\mathsf{Hoch}(n)}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right).$$

Remark

F. Chapoton has conjectured the analogous relation for $\mathsf{Tam}(n)$ in the early 2000s. This was proven by M. Thiel in 2014 using generating functions.

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Chapoton also introduced a polynomial $M(x,y)$, defined on $\mathsf{Nonc}(n)$, which could be obtained by variable substitutions from F or H.

Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. (finite) poset

Möbius Polynomials

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- **Möbius function**:

$$\mu_{\mathbf{P}}(a,b) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } a = b \\ -\sum_{a \leq c < b} \mu_{\mathbf{P}}(a,c), & \text{if } a < b \\ 0, & \text{otherwise} \end{cases}$$

Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. graded (finite) poset with bounds $\hat{0}$ and $\hat{1}$
- (reverse) **characteristic polynomial**:

$$\chi_{\mathbf{P}}(x) \stackrel{\mathsf{def}}{=} \sum_{a \in P} \mu_{\mathbf{P}}(\hat{0}, a) x^{\mathsf{rk}(a)}$$

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Lemma

- $M_{\mathbf{P}}(x, y) = \sum_{a \in P} (xy)^{\mathsf{rk}(a)} \chi_{[a,\hat{1}]}(y)$.
- $\chi_{\mathbf{P}}(x) = M_{\mathbf{P}}(0, x)$.

The FHM-Correspondence

- formerly conjectured by F. Chapoton (2004)

Theorem (C. Athanasiadis, 2007)

For $n > 0$, we have

$$M_{\mathsf{Nonc}(n)}(x,y) = (xy-1)^n F_{\mathsf{Tam}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right).$$

The FHM-Correspondence

- formerly conjectured by F. Chapoton (2004)

Theorem (C. Athanasiadis, 2007)

For $n > 0$, we have

$$M_{\mathsf{CLO}(\mathsf{Tam}(n))}(x,y) = (xy-1)^n F_{\mathsf{Tam}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right).$$

The FHM-Correspondence

- $\tilde{M}(x,y) \overset{\mathsf{def}}{=} (xy-1)^n F_{\mathsf{Hoch}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$

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- $\tilde{M}(x,y) \stackrel{\mathsf{def}}{=} (xy-1)^n F_{\mathsf{Hoch}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$

Corollary (, 2020)

For $n > 0$, we have

$$\tilde{M}(x,y) = (xy-y+1)^{n-2} \times \Big((n+1)\big((x-1)y-xy^2\big)+(n+x^2)y^2+1\Big).$$

The FHM-Correspondence

- $\mathfrak{t} \overset{\text{def}}{=} (1, 2, 2, \ldots, 2)$ .. top element of $\mathsf{CLO}\big(\mathsf{Hoch}(n)\big)$
- if $\mathsf{in}(\mathfrak{u}) = i$, then

$$[\mathfrak{u}, \mathfrak{t}]_{\mathsf{CLO}\big(\mathsf{Hoch}(n)\big)} \cong \begin{cases} \mathsf{CLO}\big(\mathsf{Hoch}(n-i)\big), & \text{if } l_1(\mathfrak{u}) = 0\\ \mathsf{Bool}(n-i), & \text{otherwise}\end{cases}$$

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Proposition (C. Greene, 1988)

For $n > 0$, we have

$$\chi_{\mathsf{Bool}(n)}(x) = (1-x)^n,$$

$$\chi_{\mathsf{Shuf}(n-1,1)}(x) = (1-x)^{n-1}(1-nx).$$

The FHM-Correspondence

Proposition (, 2020)

For $n > 0$, we have

$$M_{\mathsf{CLO}\left(\mathsf{Hoch}(n)\right)}(x,y) = \tilde{M}(x,y).$$

The FHM-Correspondence

Theorem (🏠, 2020)

For $n > 0$, we have

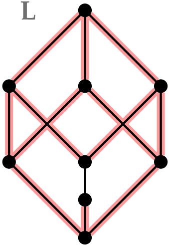
$$M_{\mathsf{CLO}(\mathsf{Hoch}(n))}(x,y) = (xy-1)^n F_{\mathsf{Hoch}(n)}\left(\frac{1-y}{xy-1}, \frac{1}{xy-1}\right)$$
$$= (1-y)^n H_{\mathsf{Hoch}(n)}\left(\frac{y(x-1)}{1-y}, \frac{x}{x-1}\right).$$

Open Questions

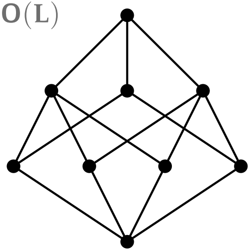
- what is the relation between $\chi_{\mathsf{CLO}\left(\mathsf{Hoch}(n)\right)}(x)$, $c(x)$ and $r(x)$?
- what is the geometric nature of $M_{\mathsf{CLO}\left(\mathsf{Hoch}(n)\right)}(x,y)$?
- can we characterize lattices satisfying the FHM-correspondence?

Thank You.

L



$\mathbf{CLO}(\mathbf{L})$



$$F(x,y) = (x+y+1)^3 + x^2(x+1)$$

$$H(x,y) = (xy+1)^3 + x$$

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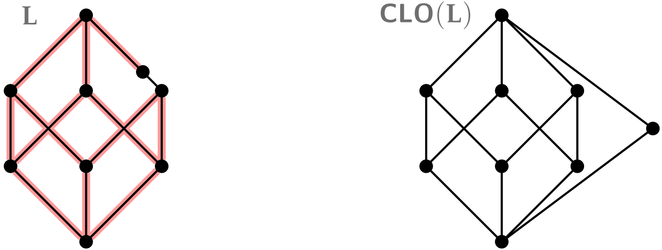
Abstract Examples



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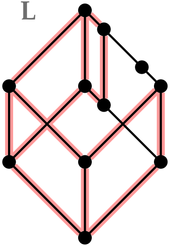
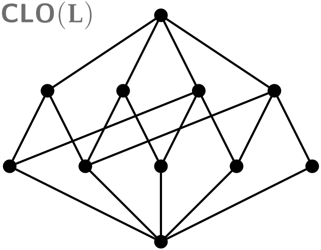




$$F(x,y) = (x+y+1)^3 + x(x+1)(2x+y+1)$$

$$H(x,y) = (xy+1)^3 + x^2y + 2x$$

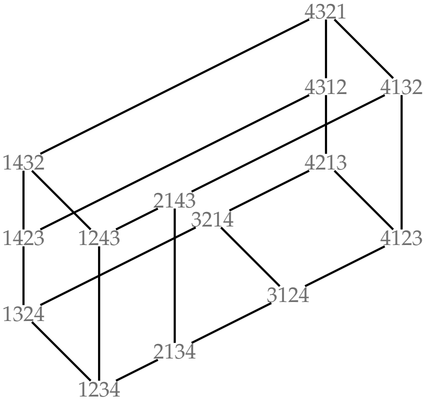
$$M(x,y) = (xy-y+1)^3 - (x-1)y(y-1)(xy-y+2)$$

$$\tilde{M}(x,y) = (xy-y+1)^3 - (x-1)y(y-1)(xy-2y+2)$$

The Tamari Lattice

Hochschild

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$



The Tamari Lattice

Hochschild

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Theorem (A. Björner & M. Wachs, 1997)

For $n > 0$, the weak order on $\mathfrak{S}_n(231)$ realizes the Tamari lattice of order $n-1$.

The Tamari Lattice

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Lemma (D. Knuth, 1968)

For $n > 0$, the cardinality of $\mathfrak{S}_n(231)$ is $\frac{1}{n+1}\binom{2n}{n}$.

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For $n > 0$, the cardinality of $\mathfrak{S}_n(231)$ is $\frac{1}{n+1}\binom{2n}{n}$.

$1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$ (A000108 in OEIS)

The Tamari Lattice

- **231-avoiding permutation**: a permutation without subwords standardizing to 231 $\rightsquigarrow \mathfrak{S}_n(231)$

Theorem (A. Urquhart, 1978)

For $n > 0$, the Tamari lattice $\mathsf{Tam}(n)$ is semidistributive.

A Bijection

- $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n(231)$
- $\mathsf{nc}(w)$ is the noncrossing partition whose bumps are the descents of w

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Proposition (P. Biane, 1997)

For $n > 0$, the map $\mathsf{nc}\colon \mathfrak{S}_n(231) \to \mathsf{Nonc}(n)$ is a bijection.

A Bijection

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Theorem (N. Reading, 2011)

For $n > 0$, the map nc *extends to an isomorphism from* $\mathbf{CLO}\big(\mathbf{Tam}(n)\big)$ *to* $\mathbf{Nonc}(n)$.

A Bijection

Hochschild

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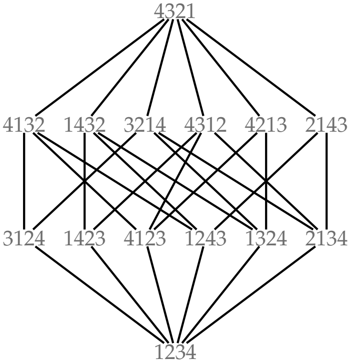
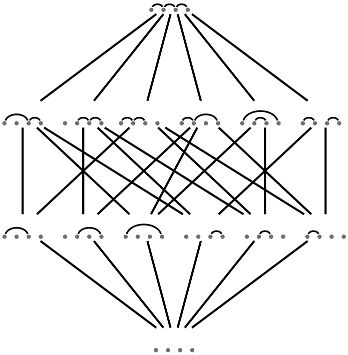


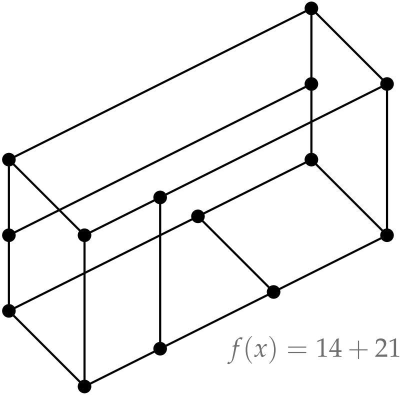


Recovering the Associahedron

Observation

The set $\mathsf{CP}\big(\mathbf{Tam}(n)\big)$ *is in bijection with the faces of* $\mathsf{Asso}(n)$.

Recovering the Associahedron



$$f(x) = 14 + 21x + 9x^2 + x^3$$

Recovering the Associahedron

Proposition (C. Lee, 1989)

For $n > 0$ and $0 \leq i \leq n$, we have

$$f_i = \frac{1}{n+1-i}\binom{n}{i}\binom{2n+2-i}{n-i}.$$

Recovering the Associahedron

Corollary

For $n > 0$, we have

$$f(x) = \sum_{i=0}^{n} \frac{1}{n+1-i} \binom{n}{i} \binom{2n+2-i}{n-i} x^i,$$

$$h(x) = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} \binom{n+1}{i} x^i.$$

Perspectivity

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Perspectivity

Hochschild

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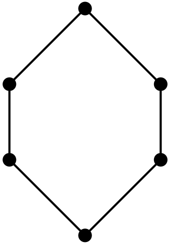

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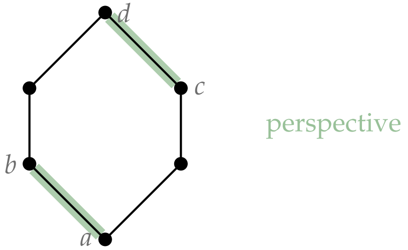

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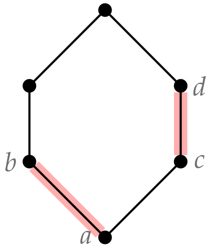

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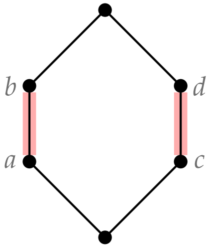

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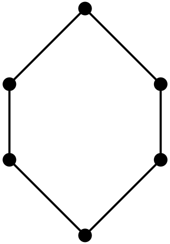

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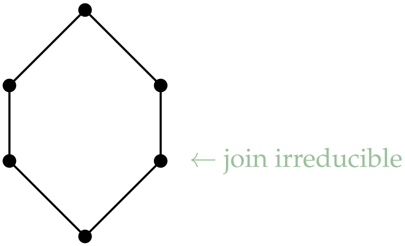

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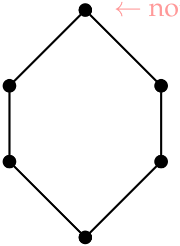

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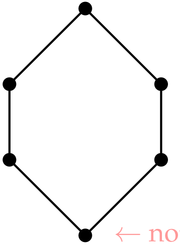

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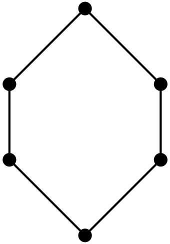

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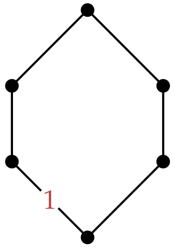

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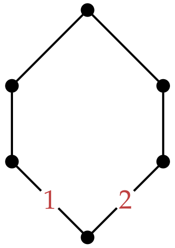

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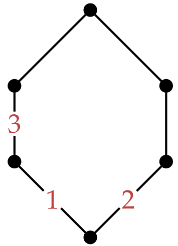

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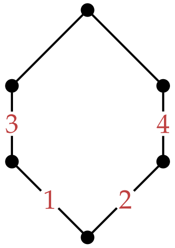

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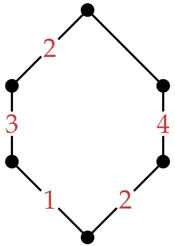

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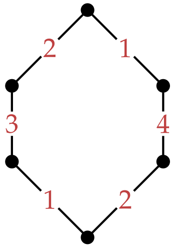

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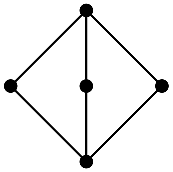

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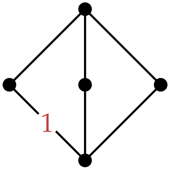

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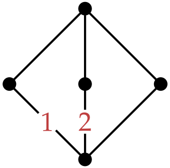

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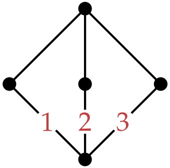

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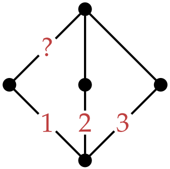

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Proposition

Every semidistributive lattice is edge determined.

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- **perspectivity labeling**: $\lambda\colon \mathcal{E}(\mathbf{L}) \to \mathcal{J}(\mathbf{L}), (a, b) \mapsto j$ such that $(a, b) \overline{\overline{\wedge}} (j_*, j)$

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- if **L** is semidistributive, then

$$\lambda(a, b) = \min\{c \mid a \vee c = b\}$$

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If **L** *is semidistributive, then*

$$\mathsf{Can}(a) = \Big\{\lambda(a',a) \mid (a',a) \in \mathcal{E}(\mathbf{L})\Big\}.$$

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