Adjunct to the polymorphism functor

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Endopolymorphisms

Given a relational structure $A$.

We say that $f : A^n \rightarrow A$ is a polymorphism of $A$ if one of the following equivalent conditions is satisfied:

- $f$ is a homomorphism from $A^n$ to $A$,
- for each relation $R^A$ and all tuples $a_1, \ldots, a_n \in R^A$ we have
  $$f(a_1, \ldots, a_n) \in R^A,$$
- each relation $R^A$ is a subuniverse of $(A; f)^k$ where $k$ is the arity of $R$.

The set $\text{Pol}(A)$ of all (endo)polymorphisms of $A$ is a clone.
Polymorphisms

Given relational structures $A$ and $B$ that share a signature. We say that $f : A^n \rightarrow B$ is a polymorphism from $A$ to $B$ if one of the following equivalent conditions is satisfied:

- $f$ is a homomorphism from $A^n$ to $B$,
- for each relation $R^A$ and all tuples $a_1, \ldots, a_n \in R^A$ we have $f(a_1, \ldots, a_n) \in R^B$,
- each relational pair $(R^A, R^B)$ is a subuniverse of $(A, B; f)^k$ where $k$ is the arity of $R$.

The set $\text{Pol}(A, B)$ of all polymorphisms from $A$ to $B$ is not a clone, but it is closed under taking minors.
Minor closed sets
a.k.a. clonoids

Let \( f : A^n \to B \) be a function. Any function \( g \) of the form

\[
g(x_1, \ldots, x_m) = f(x_{e(1)}, \ldots, x_{e(n)}).
\]

for some \( e : [n] \to [m] \) is called a minor of \( f \).

Theorem (Pippenger, 2002; Brakiensiek, Guruswami, 2016)

For all finite sets \( A, B \) and every minor closed set \( \mathcal{A} \subseteq \mathcal{O}(A, B) \) there exist relational structures \( A \) and \( B \) such that \( \text{Pol}(A, B) = \mathcal{A} \).

\((\mathcal{O}(A, B) = \{ f \mid f : A^n \to B, n \in \mathbb{N} \})\)
Minor preserving maps
a.k.a. h1 homomorphisms, clonoid homomorphisms

Let \( \mathcal{A} \) and \( \mathcal{B} \) be minor closed. A map \( \xi : \mathcal{A} \to \mathcal{B} \) that preserves arities is **minor preserving** if for each \( f \in \mathcal{A}^{(n)} \) and each \( e : [n] \to [m] \) we have

\[
\xi(f(\pi^m_{e(1)}, \ldots, \pi^m_{e(n)})) \approx \xi(f(\pi^m_{e(1)}, \ldots, \pi^m_{e(n)})).
\]

**Theorem (Barto, O, Pinsker, 2017)**

*If \( \mathcal{A} \) and \( \mathcal{B} \) are finite (\( \omega \)-categorical) structures such that there is a (uniformly continuous) minor preserving map \( \xi : \text{Pol}(\mathcal{A}) \to \text{Pol}(\mathcal{B}) \), then \( \text{CSP}(\mathcal{A}) \geq_L \text{CSP}(\mathcal{B}) \).*
The polymorphism functor

Note. Even if $A$ and $B$ are homomorphically equivalent, there is no clone homomorphism from $\text{Pol}(A)$ to $\text{Pol}(B)$.

Fix a relational signature, and let $A, B, A', B'$ be structures in this signatures, and $a : A' \rightarrow A$, $b : B \rightarrow B'$ homomorphisms.

\[
\begin{align*}
A^n & \xleftarrow{a} (A')^n \\
B & \xrightarrow{b} B'
\end{align*}
\]

\[\xi_{a,b}(f)(x_1, \ldots, x_n) = b f (a(x_1), \ldots, a(x_n))\]

Note. $\text{Pol}(A', B')$ contains a reflection of $\text{Pol}(A, B)$. 
The adjunct

For each structure \( A, \text{Pol}(A, -) \) has a left adjunct \( \text{Free}(-, A) \).

Given a minor closed set \( C \) and a relational structure \( A \). We define the free structure (‘free action of \( C \) on \( A \)’) \( F \):

- Let \( F = \{f(a_1, \ldots, a_n) : n \in \mathbb{N}, f \in C^{(n)}, a_1, \ldots, a_n \in A\} / \approx \)
- for a relation \( R \), we define
  \[ R^F = \{f(a_1, \ldots, a_n) : n \in \mathbb{N}, f \in C^{(n)}, a_1, \ldots, a_n \in RA\} \]

We use the symbol \( \text{Free}(C, A) \) for \( F \).

Observation

For each \( A, B \), relational structures and \( C \), minor closed, there is a natural isomorphism \( \eta_{C,A,B} \):

\[ \{h : \text{Free}(C, A) \to B\} \cong \{\xi : C \to \text{Pol}(A, B)\} \]
Linear Mal’cev conditions

A coloring of a minor closed set $\mathcal{C}$ by $A$ is a homomorphism $c: \text{Free}(\mathcal{C}, A) \rightarrow A$. A coloring of a clone is strong if $c(a) = a$ for all $a \in A$.

Note. $\mathcal{C}$ is strongly colorable by $A$ iff there is a minor preserving map from $\mathcal{C}$ to $\text{Pol}(A)$ that maps the identity map to itself.

Theorem (Sequeira, Greenwell & Lovász, …)

Let $\mathcal{C}$ be a clone.

- $\mathcal{C}$ has a Mal’cev term iff it is not strongly colorable by $L = (\{0, 1, 2\}, 01|2, 0|12)$.
- $\mathcal{C}$ satisfies a non-trivial Mal’cev condition iff it is not colorable by $K_3$. 
Deciding triviality of Mal’cev conditions

Label Cover

Mal’cev condition is **linear** if it contains only identities of the form

\[ f(x_{\pi(1)}, \ldots, x_{\pi(m)}) \approx g(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \text{ or } f(x_{\pi(1)}, \ldots, x_{\pi(m)}) \approx x_1. \]

**Corollary**

*Given a linear Mal’cev condition \( \Sigma \) of arity at most \( N \).*

- For each \( N \), deciding whether \( \Sigma \) implies the Mal’cev term is solvable in \( \text{Ptime} \).
- For each \( N \geq 6 \), deciding whether \( \Sigma \) is trivial is \( \text{NP-complete} \).

**Proof.** Construct \( \text{Free}(\Sigma, A) \) in \( \text{Ptime} \), then decide existence of a homomorphism \( \text{Free}(\Sigma, A) \rightarrow A \) by \( \text{CSP}(A) \).
Promise constraint satisfaction

Fix two finite relational structures $A$ and $B$ with the same finite signature. $\text{PCSP}(A, B)$ is the following problem: Given a structure $Q$ in the common language, output

- **YES** if $Q$ maps homomorphically into $A$,
- **NO** if $Q$ does not map homomorphically into $B$.

Note. $\text{CSP}(A) \equiv \text{PCSP}(A, A)$ and $\text{PCSP}(A, B) \leq \text{CSP}(A), \text{CSP}(B)$. 
Gap Label Cover

Fix $A$, $B$. Given a minor closed set $C$, we know

- $C \to \text{Pol}(A, A)$ iff $\text{Free}(C, A) \to A$, and
- $C \not\to \text{Pol}(A, B)$ iff $\text{Free}(C, A) \not\to B$.

**Gap Mal’cev Sat.** Fix two minor closed sets $A$ and $B$. Given $\Sigma$ of maximal arity $N$, output

- **YES** if $\Sigma$ is satisfied in $A$,
- **NO** if $\Sigma$ is not satisfied in $B$.

Denote this problem $\text{GMS}_A,B(N)$. 
Reduction between LC and PCSP

Theorem (O, Bulin, 2017*)

Let $A$ and $B$ be relational structures, $A = Pol(A, A)$, and $Pol(A, B)$. Then

- For all $N > 0$, $GMS_{A,B}(N) \leq_{L} PCSP(A, B)$.
- There exists $N > 0$ s.t. $PCSP(A, B) \leq_{L} GMS_{A,B}(N)$.

Corollary

The complexity of $PCSP(A, B)$ depends only on minor Mal’cev conditions satisfied by $Pol(A, B)$.

Thanks! ■