Adjunct to the polymorphism functor

Jakub Opršal

Technische Universität Dresden

Bratislava, Feb 10, 2018

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 681988)

Endopolymorphisms

Given a relational structure $\mathbf{A}$.

We say that $f\colon A^n \to A$ is a polymorphism of $\mathbf{A}$ if one of the following equivalent conditions is satisfied:

- f is a homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$,
- for each relation $R^{\mathbf{A}}$ and all tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R^{\mathbf{A}}$ we have

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in R^{\mathbf{A}},$$

- each relation $R^{\mathbf{A}}$ is a subuniverse of $(A; f)^k$ where k is the arity of R.

The set $\operatorname{Pol}(\mathbf{A})$ of all (endo)polymorphisms of $\mathbf{A}$ is a clone.

Polymorphisms

Given relational structures $\mathbf{A}$ and $\mathbf{B}$ that share a signature.

We say that $f: A^n \to B$ is a polymorphism from $\mathbf{A}$ to $\mathbf{B}$ if one of the following equivalent conditions is satisfied:

- f is a homomorphism from $\mathbf{A}^n$ to $\mathbf{B}$,
- for each relation $R^{\mathbf{A}}$ and all tuples $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R^{\mathbf{A}}$ we have

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R^{\mathbf{B}},$$

- each relational pair $(R^{\mathbf{A}}, R^{\mathbf{B}})$ is a subuniverse of $(A, B; f)^k$ where k is the arity of R.

The set $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ of all polymorphisms from $\mathbf{A}$ to $\mathbf{B}$ is not a clone, but it is closed under taking minors.

Minor closed sets

a.k.a. clonoids

Let $f\colon A^n \to B$ be a function. Any function g of the form

$$g(x_1, \dots, x_m) = f(x_{e(1)}, \dots, x_{e(n)}).$$

for some $e\colon [n] \to [m]$ is called a minor of f.

Theorem (Pippenger, 2002; Brakiensiek, Guruswami, 2016)

For all finite sets A, B and every minor closed set $\mathscr{A} \subseteq \mathscr{O}(A, B)$ there exist relational structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathrm{Pol}(\mathbf{A}, \mathbf{B}) = \mathscr{A}$.

$(\mathscr{O}(A, B) = \{f \mid f\colon A^n \to B, n \in \mathbb{N}\})$

Minor preserving maps

a.k.a. h1 homomorphisms, clonoid homomorphisms

Let $\mathscr{A}$ and $\mathscr{B}$ be minor closed. A map $\xi \colon \mathscr{A} \to \mathscr{B}$ that preserves arities is minor preserving if for each $f \in \mathscr{A}^{(n)}$ and each $e \colon [n] \to [m]$ we have

$$\xi(f(\pi^m_{e(1)}, \dots, \pi^m_{e(n)})) \approx \xi(f)(\pi^m_{e(1)}, \dots, \pi^m_{e(n)}).$$

Theorem (Barto, O, Pinsker, 2017)

If $\mathbf{A}$ *and* $\mathbf{B}$ *are finite (ω-categorical) structures such that there is a (uniformly continuous) minor preserving map* $\xi \colon \operatorname{Pol}(\mathbf{A}) \to \operatorname{Pol}(\mathbf{B})$, *then* $\operatorname{CSP}(\mathbf{A}) \geq_{\mathrm{L}} \operatorname{CSP}(\mathbf{B})$.

The polymorphism functor

Note. Even if $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent, there is no clone homomorphism from $\mathrm{Pol}(\mathbb{A})$ to $\mathrm{Pol}(\mathbb{B})$.

Fix a relational signature, and let $\mathbb{A}, \mathbb{B}, \mathbb{A}', \mathbb{B}'$ be structures in this signatures, and $a\colon \mathbb{A}' \to \mathbb{A}$, $b\colon \mathbb{B} \to \mathbb{B}'$ homomorphisms.

$$\begin{array}{ccc} \mathbb{A}^n & \xleftarrow{\;a\;} & (\mathbb{A}')^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \xi_{a,b}(f)} \\ \mathbb{B} & \xrightarrow[\;b\;]{} & \mathbb{B}' \end{array}$$

$$\xi_{a,b}(f)(x_1, \ldots, x_n) = bf(a(x_1), \ldots, a(x_n))$$

Note. $\mathrm{Pol}(\mathbb{A}', \mathbb{B}')$ contains a reflection of $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$.

The adjunct

For each structure $\mathbf{A}$, $\mathrm{Pol}(\mathbf{A}, -)$ has a left adjunct $\mathrm{Free}(-, \mathbf{A})$.

Given a minor closed set $\mathscr{C}$ and a relational structure $\mathbf{A}$. We define the free structure ('free action of $\mathscr{C}$ on $\mathbf{A}$') $\mathbf{F}$:

- Let $F = \{f(a_1, \dots, a_n) : n \in \mathbb{N}, f \in \mathscr{C}^{(n)}, a_1, \dots, a_n \in A\}/\approx$
- for a relation R, we define $R^{\mathbf{F}} = \{f(\mathbf{a}_1, \dots, \mathbf{a}_n) : n \in \mathbb{N}, f \in \mathscr{C}^{(n)}, \mathbf{a}_1, \dots, \mathbf{a}_n \in R^{\mathbf{A}}\}$.

We use the symbol $\mathrm{Free}(\mathscr{C}, \mathbf{A})$ for $\mathbf{F}$.

Observation

For each $\mathbf{A}$, $\mathbf{B}$, *relational structures and* $\mathscr{C}$, *minor closed, there is a natural isomorphism* $\eta_{\mathscr{C},\mathbf{A},\mathbf{B}}$:

$$\{h \colon \mathrm{Free}(\mathscr{C}, \mathbf{A}) \to \mathbf{B}\} \overset{\eta_{\mathscr{C},\mathbf{A},\mathbf{B}}}{\simeq} \{\xi \colon \mathscr{C} \to \mathrm{Pol}(\mathbf{A}, \mathbf{B})\}.$$

Linear Mal'cev conditions

A coloring of a minor closed set $\mathscr{C}$ by $\mathbf{A}$ is a homomorphism $c\colon \operatorname{Free}(\mathscr{C}, \mathbf{A}) \to \mathbf{A}$.
A coloring of a clone is strong if $c(a) = a$ for all $a \in A$.

Note. $\mathscr{C}$ is strongly colorable by $\mathbf{A}$ iff there is a minor preserving map from $\mathscr{C}$ to $\operatorname{Pol}(\mathbf{A})$ that maps the identity map to itself.

Theorem (Sequeira, Greenwell & Lovász, ...)

Let $\mathscr{C}$ be a clone.

- *$\mathscr{C}$ has a Mal'cev term iff it is not strongly colorable by* $\mathbf{L} = (\{0, 1, 2\}, 01|2, 0|12)$.
- *$\mathscr{C}$ satisfies a non-trivial Mal'cev condition iff it is not colorable by* $\mathbf{K}_3$.

Deciding triviality of Mal'cev conditions

Label Cover

Mal'cev condition is linear if it contains only identities of the form

$$f(x_{\pi(1)}, \dots, x_{\pi(m)}) \approx g(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \text{or} \quad f(x_{\pi(1)}, \dots, x_{\pi(m)}) \approx x_1.$$

Corollary

Given a linear Mal'cev condition Σ of arity at most N.

- *For each N, deciding whether Σ implies the Mal'cev term is solvable in* Ptime.
- *For each $N \geq 6$, deciding whether Σ is trivial is* NP*-complete.*

Proof. Construct $\mathrm{Free}(\Sigma, \mathbf{A})$ in Ptime, then decide existence of a homomorphism $\mathrm{Free}(\Sigma, \mathbf{A}) \to \mathbf{A}$ by $\mathrm{CSP}(\mathbf{A})$. ∎

Promise constraint satisfaction

Fix two finite relational structures $\mathbb{A}$ and $\mathbb{B}$ with the same finite signature. $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is the following problem: Given a structure $\mathbb{Q}$ in the common language, output

- YES if $\mathbb{Q}$ maps homomorphically into $\mathbb{A}$,
- NO if $\mathbb{Q}$ does not map homomorphically into $\mathbb{B}$.

Note. $\mathrm{CSP}(\mathbb{A}) \equiv \mathrm{PCSP}(\mathbb{A}, \mathbb{A})$ and $\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \leq \mathrm{CSP}(\mathbb{A}), \mathrm{CSP}(\mathbb{B})$.

Gap Label Cover

Fix $\mathbf{A}$, $\mathbf{B}$. Given a minor closed set $\mathscr{C}$, we know

- $\mathscr{C} \to \mathrm{Pol}(\mathbf{A}, \mathbf{A})$ iff $\mathrm{Free}(\mathscr{C}, \mathbf{A}) \to \mathbf{A}$, and
- $\mathscr{C} \not\to \mathrm{Pol}(\mathbf{A}, \mathbf{B})$ iff $\mathrm{Free}(\mathscr{C}, \mathbf{A}) \not\to \mathbf{B}$.

Gap Mal'cev Sat. Fix two minor closed sets $\mathscr{A}$ and $\mathscr{B}$. Given Σ of maximal arity N, output

- YES if Σ is satisfied in $\mathscr{A}$,
- NO if Σ is not satisfied in $\mathscr{B}$.

Denote this problem $\mathrm{GMS}_{\mathscr{A},\mathscr{B}}(N)$.

Reduction between LC and PCSP

Theorem (O, Bulin, 2017*)

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be relational structures,* $\mathscr{A} = \mathrm{Pol}(\mathbf{A}, \mathbf{A})$*, and* $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$*. Then*

- *For all* $N > 0$*,* $\mathrm{GMS}_{\mathscr{A},\mathscr{B}}(N) \leq_L \mathrm{PCSP}(\mathbf{A}, \mathbf{B})$*.*
- *There exists* $N > 0$ *s.t.* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B}) \leq_L \mathrm{GMS}_{\mathscr{A},\mathscr{B}}(N)$*.*

Corollary

The complexity of $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ *depends only on minor Mal'cev conditions satisfied by* $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$*.*

Thanks! ■