Promise constraint satisfaction

Jakub Opršal (TU Dresden)

Joint work with Jakub Bulín, Andrei Krokhin, and others

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Constraint Satisfaction Problem

CSP over a domain D

Given a conjunction of constraints over some variable set V of the form

$$(v_1, \dots, v_k) \in R$$

where $R \subseteq D^k$, decide whether there is an assignment $s \colon V \to D$ such that all constraints are satisfied (i.e., $(s(v_1), \dots, s(v_n)) \in R$).

CSP with fixed template $\mathbf{D}$

Fix a relational structure $\mathbf{D}$. $\mathrm{CSP}(\mathbf{D})$ is the problem to decide whether a given a structure $\mathbf{I}$ in the same language maps homomorphically to $\mathbf{D}$, or not.

Examples of CSPs

SAT

Given a CNF formula, e.g.

$$(x \vee y) \wedge (\neg x \vee z \vee \neg w) \wedge (\neg y \vee z \vee w),$$

decide whether there is a satisfying assignment.

3-coloring

Given a graph $\mathbf{G}$, decide whether it is 3-colorable. This is $\mathrm{CSP}(\mathbf{K}_3)$.

SAT and 3-coloring are NP-complete [Karp, "72]

What makes a problem easy?

Answer. Symmetry!

[Barto]

- Aut(**D**) No! (Aut($\mathbf{K}_3$) $=$ Sym(K_3), but CSP($\mathbf{K}_3$) is NP-hard.)
- Set of polymorphisms of **D**. [Jeavong, Cohen, Gyssens, “97]
  (Polymorphism of **D** is a homomorphism from $\mathbf{D}^n$ to **D**.)
- The abstract clone of polymorphisms of **D**. [Bulatov, Jeavons, ‘01; Bulatov, Jeavons, Krokhin, ‘05]
- Height 1 identities satisfied by polymorphisms of **D**. [Barto, Pinksker, __, ‘16]
  Height 1 identity is an identity of the form

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

Approximate graph coloring

Question

How hard is to color a given k-colorable graph by c colors?
[Garey, Johnson, “76]

- …a 3-colorable graph with 3 colors is NP-hard. [Karp, “72]
- …a 3-colorable graph with 4 colors is NP-hard.
  [Guruswami, Khanna, ‘04]
- …a k-colorable graph with $2k-2$ colors is NP-hard.
  [Brakensiek, Guruswami, ‘16]
- …a K-colorable graph with $2^{\Omega(K^{1/3})}$ colors is NP-hard for big-enough K. [Huang, ‘13]

Promise constraint satisfaction

Fix two finite relational structures $\mathbf{A}, \mathbf{B}$ in the same finite language with a homomorphism $\mathbf{A} \to \mathbf{B}$.

$\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is the following problem:

Search

Given a finite structure $\mathbf{I}$ that maps homomorphically to $\mathbf{A}$, find a homomorphism $h \colon \mathbf{I} \to \mathbf{B}$.

Decide

Given $\mathbf{I}$ arbitrary structure with the same language,

- ACCEPT if $\mathbf{I} \to \mathbf{A}$,
- REJECT if $\mathbf{I} \not\to \mathbf{B}$.

Example: 3-uniform hypergraph coloring

A valid coloring of a hypergraph $\mathbf{H}$ is a coloring of vertices of H such that no edge is monochromatic.

Fix $c \geq k \geq 2$. The goal is to find c-colouring for a given k-colourable 3-uniform hypergraph.

This is a PCSP with template $(\mathbf{H}_K, \mathbf{H}_c)$ where

$$\mathbf{H}_n = (\{1, \ldots, n\}; \mathrm{NAE}_n),$$

and $\mathrm{NAE}_n = \{(a, b, c) \in \{1, \ldots, n\}^3 \mid a \neq b \vee a \neq c \vee b \neq c\}$.

This was proven to be NP-hard [Dinur, Regev, Smyth, '05].

Example: 1-in-3- vs. NAE-SAT

- 1-in-3-SAT is CSP with the template $\mathbf{T}_2 = (\{0, 1\}; T)$ where T is the ternary relation satisfying 'exactly one is 1', i.e. $T = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.
- NAE-SAT is CSP with the template $\mathbf{H}_2 = (\{0, 1\}; \mathrm{NAE}_2)$

Clearly, $T \subseteq \mathrm{NAE}_2$, and therefore $\mathbf{T}_2 \to \mathbf{H}_2$.

The goal here is, given a solvable instance $\mathbf{I}$ of 1-in-3-SAT, find a solution to $\mathbf{I}$ as a NAE-SAT instance.

Both 1-in-3-SAT and NAE-SAT are NP-complete, but $\mathrm{PCSP}(\mathbf{T}_2, \mathbf{H}_2)$ is in P [Brakensiek, Guruswami, '16].

Symmetries of PCSP: Polymorphisms

Given relational structures $\mathbf{A}$ and $\mathbf{B}$ that share a signature.

We say that $f\colon A^n \to B$ is a polymorphism from $\mathbf{A}$ to $\mathbf{B}$ if one of the following equivalent conditions is satisfied:

- f is a homomorphism from $\mathbf{A}^n$ to $\mathbf{B}$,
- for each relation $R^{\mathbf{A}}$ and all tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R^{\mathbf{A}}$ we have

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in R^{\mathbf{B}}.$$

The set of all polymorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$.

$\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ is **not** closed under composition!

Minors and minions

Let $f\colon A^n \to B$ be a function. Any function g of the form

$$g(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

for some $\pi\colon [n] \to [m]$ is called a minor of f.

We call a set of functions from A to B, that is closed under taking minors, a minion.

Theorem [Pippenger, '02; Brakiensiek, Guruswami, '16]

For all finite sets A, B and minion $\mathscr{A}$ on A and B there exist relational structures $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\operatorname{Pol}(\mathbf{A}, \mathbf{B}) = \mathscr{A}$.

PCSP and Minions

The complexity of $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is determined (up to poly-time reductions) by:

- Set of polymorphisms from $\mathbf{A}$ to $\mathbf{B}$. [Brakensiek, Guruswami, ‘16–‘18]
- The abstract minion of polymorphisms from $\mathbf{A}$ to $\mathbf{B}$. [Bulín, Krokhin, __, ‘18]

Height 1 identities are natural for minions!

The main result

Given minions $\mathscr{M}$ and $\mathscr{N}$, a minor homomorphism is a map $\xi\colon \mathscr{M} \to \mathscr{N}$ that preserves arities, and preserves minors, i.e.,

$$\xi(f)(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \xi(f(x_{\pi(1)}, \ldots, x_{\pi(n)}))$$

for all $f \in \mathscr{M}^{(n)}$ and $\pi\colon [n] \to [m]$.

Minor homomorphisms preserve height 1 identities.

Theorem [Bulín, Krokhin, __, '18]

If there is a minor homomorphism $\xi\colon \operatorname{Pol}(\mathbf{A}_1, \mathbf{B}_1) \to \operatorname{Pol}(\mathbf{A}_2, \mathbf{B}_2)$, then $\mathrm{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ is log-space reducible to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$.

Example: Graph coloring from hypergraph coloring

Claim. It is NP-hard to distinguish between a graph that is 3-colorable and one that is not 5-colorable. Equivalently, $\mathrm{PCSP}(\mathbf{K}_3, \mathbf{K}_5)$ is NP-hard.

Theorem [Dinur, Regev, Smyth, '05]

$\mathrm{PCSP}(\mathbf{H}_2, \mathbf{H}_K)$ *is NP-hard for all* $K \geq 2$.

Key point. There is a minor homomorphism from $\mathrm{Pol}(\mathbf{K}_3, \mathbf{K}_5)$ to $\mathrm{Pol}(\mathbf{H}_2, \mathbf{H}_K)$.

Intermediate problem: Deciding identities

A minor (Maltsev) condition is a finite set of identities (functional equations) of the form

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \approx g(x_1, \ldots, x_m)$$

for some $\pi\colon [n] \to [m]$.

Function symbols are variables! I.e., we usually ask for functions that satisfy the identities.

$\mathrm{MC}(N)$:

Given is a minor condition Σ that involves at most N-ary function symbols, decide whether the condition is satisfied by projections.

Example: From $\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_K)$ to $\mathrm{MC}(6)$

- For each vertex v introduce a binary symbol t_v into $\mathcal{V}$.
- For each edge $e = (v_1, v_2, v_3)$, introduce a 6-ary f_e into U, and add constraints:

$$f_e(x, x, y, y, y, x) \approx t_{v_1}(x, y)$$
$$f_e(x, y, x, y, x, y) \approx t_{v_2}(x, y)$$
$$f_e(y, x, x, x, y, y) \approx t_{v_3}(x, y)$$

Few observations.

- A solution to the MC instance gives a solution to $\mathrm{CSP}(\mathrm{NAE}_2)$.
- It is enough to have a solution in $\mathrm{Pol}(\mathrm{NAE}_2, \mathrm{NAE}_K)$: The assignment $v \mapsto t_v(0, 1)$ is a solution.

From minor conditions to PCSP

Hint

We can ask Is this minor condition satisfied by polymorphisms from $\mathbf{A}$ to $\mathbf{B}$? as an instance of $\mathrm{CSP}(\mathbf{B})$.

- We use just $\mathbf{A}$ to construct the instance!
- Warning! The structure is of exponential size in N.

Example: The reduction (Step 1)

1. Construct a graph F with vertex set $V_F = \mathrm{Pol}^{(2)}(\mathsf{K}_3, \mathsf{K}_5)$, three vertices f, g, and h are connected with an edge if there is a 6-ary polymorphism o s.t.

$$o(x,x,y,y,y,x) \approx f(x,y)$$
$$o(x,y,x,y,x,y) \approx g(x,y)$$
$$o(y,x,x,x,y,y) \approx h(x,y)$$

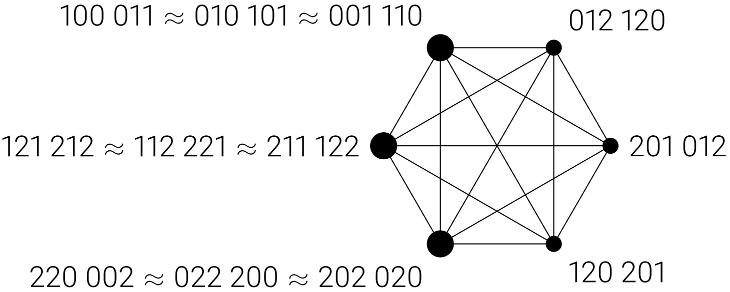
Observation. As long as such F has no loop (does not contain edge (a,a,a)), it is K-colorable for some K.

Example: A graph that is not 5-colorable

Claim. $\mathrm{Pol}(\mathbf{K}_3, \mathbf{K}_5)$ does not have a polymorphism o satisfying (Olšák polymorphism)

$$o(x,x,y,y,y,x) \approx o(x,y,x,y,x,y) \approx o(y,x,x,x,y,y).$$

Such polymorphism would give a 5-coloring of:



Free structure

Given a minion $\mathscr{M}$ and a PCSP template $(\mathbf{A}, \mathbf{B})$. Assume $A = [n]$. We define the free structure of $\mathscr{M}$ generated by $\mathbf{A}$ to be a structure $\mathbf{F}$ similar to $\mathbf{A}$:

- $F = \mathscr{M}^n$.
- $R^{\mathbf{F}}$ consists of those k-tuples of functions $(f_1, \dots, f_k)$ for which there exists $g \in \mathscr{M}$ and $\mathbf{r}_1, \dots, \mathbf{r}_m \in R^{\mathbf{A}}$ s.t.

  $$g(x_{\mathbf{r}_1(i)}, \dots, x_{\mathbf{r}_m(i)}) \approx f_i(x_1, \dots, x_n)$$

  for each $i = 1, \dots, k$.

The graph before was a free hypergraph of $\mathrm{Pol}(\mathbf{K}_3, \mathbf{K}_5)$ generated by $\mathbf{H}_2$.

Free structure (cont.)

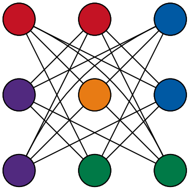
Theorem [Bulín, Krokhin, __, '18]

There is a 1-to-1 correspondence between homomorphisms form the free structure of $\mathbf{M}$ generated by $\mathbf{A}$ to $\mathbf{B}$ and minor homomorphisms from $\mathbf{M}$ to $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$.

In particular, this shows that there is a minor homomorphism from $\mathrm{Pol}(\mathbf{K}_3, \mathbf{K}_5)$ to $\mathrm{Pol}(\mathbf{H}_2, \mathbf{H}_{458})$.

Example: The reduction (Step 2)

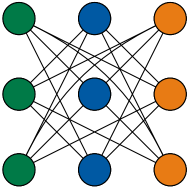
2. Starting with a hypergraph $\mathbf{G}$, construct a graph $\mathbf{C}_G$:
   - for each vertex v take a copy of K_3^2 (expressing existence of binary polymorphism g_v from K_3),



   - for each edge (u, v, w) express that g_u, g_v, and g_w are connected by a 6-ary Olšák-like polymorphism.

Example: The reduction (Step 3)

3. If $\mathbf{G}$ is 2-colorable hypergraph, then $\mathbf{C}_G$ is a 3-colorable graph.



And if $\mathbf{C}_G$ maps to $\mathbf{B}$, then $\mathbf{G}$ maps to $\mathbf{F}$, and therefore it is K-colorable.

Theorem [Bulín, Krokhin, __, '18]

It is NP-hard to color a k-colorable graph with $2k - 1$ colors.

Conclusions

Theorem [Bulín, Krokhin, __, '18]

If there is a minor homomorphism $\xi\colon \operatorname{Pol}(\mathbf{A}_1, \mathbf{B}_1) \to \operatorname{Pol}(\mathbf{A}_2, \mathbf{B}_2)$, *then* $\operatorname{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ *is log-space reducible to* $\operatorname{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$.

Theorem [Bulín, Krokhin, __, '18]

For all $k \geq 3$*, it is NP-hard to color a* k*-colorable graph with* $2k - 1$ *colors.*

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