Promise constraint satisfaction

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Constraint Satisfaction Problem

CSP over a domain $D$
Given a conjunction of constraints over some variable set $V$ of the form

$$(v_1, \ldots, v_k) \in R$$

where $R \subseteq D^k$, decide whether there is an assignment $s : V \rightarrow D$ such that all constraints are satisfied (i.e., $(s(v_1), \ldots, s(v_n)) \in R$).

CSP with fixed template $D$
Fix a relational structure $D$. CSP($D$) is the problem to decide whether a given a structure $I$ in the same language maps homomorphically to $D$, or not.
Examples of CSPs

**SAT**
Given a CNF formula, e.g.

\[(x \lor y) \land (\lnot x \lor z \lor \lnot w) \land (\lnot y \lor z \lor w),\]

decide whether there is a satisfying assignment.

**3-coloring**
Given a graph \(G\), decide whether it is 3-colorable. This is \(\text{CSP}(K_3)\).

\(\text{SAT}\) and \(3\)-coloring are NP-complete [Karp, “72]
What makes a problem easy?

Answer. Symmetry!

- $\text{Aut}(D)$ No! ($\text{Aut}(K_3) = \text{Sym}(K_3)$, but $\text{CSP}(K_3)$ is NP-hard.)
- Set of polymorphisms of $D$. [Jeavons, Cohen, Gyssens, “97] (Polymorphism of $D$ is a homomorphism from $D^n$ to $D$.)
- The abstract clone of polymorphisms of $D$. [Bulatov, Jeavons, ‘01; Bulatov, Jeavons, Krokhin, ‘05]
- Height 1 identities satisfied by polymorphisms of $D$. [Barto, Pinksker, __, ‘16]

Height 1 identity is an identity of the form

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \approx g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}).$$
Approximate graph coloring

Question
How hard is to color a given \( k \)-colorable graph by \( c \) colors?

[Garey, Johnson, “76]

- … a 3-colorable graph with 3 colors is NP-hard. [Karp, “72]
- … a 3-colorable graph with 4 colors is NP-hard. [Guruswami, Khanna, ‘04]
- … a \( k \)-colorable graph with \( 2k - 2 \) colors is NP-hard. [Brakensiek, Guruswami, ‘16]
- … a \( K \)-colorable graph with \( 2^{\Omega(K^{1/3})} \) colors is NP-hard for big-enough \( K \). [Huang, ‘13]
Promise constraint satisfaction

Fix two finite relational structures $A, B$ in the same finite language with a homomorphism $A \rightarrow B$.

$PCSP(A, B)$ is the following problem:

**Search**
Given a finite structure $I$ that maps homomorphically to $A$, find a homomorphism $h : I \rightarrow B$.

**Decide**
Given $I$ arbitrary structure with the same language,

- ACCEPT if $I \rightarrow A$,
- REJECT if $I \not\rightarrow B$. 

Example: 3-uniform hypergraph coloring

A valid coloring of a hypergraph $H$ is a coloring of vertices of $H$ such that no edge is monochromatic.

Fix $c \geq k \geq 2$. The goal is to find $c$-colouring for a given $k$-colourable 3-uniform hypergraph.

This is a PCSP with template $(H_K, H_c)$ where

$$H_n = (\{1, \ldots, n\}; \text{NAE}_n),$$

and $\text{NAE}_n = \{(a, b, c) \in \{1, \ldots, n\}^3 \mid a \neq b \lor a \neq c \lor b \neq c\}$.

This was proven to be NP-hard [Dinur, Regev, Smyth, ‘05].
Example: 1-in-3- vs. NAE-SAT

▶ 1-in-3-SAT is CSP with the template $T_2 = (\{0, 1\}; T)$ where $T$ is the ternary relation satisfying ‘exactly one is 1’, i.e. $T = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

▶ NAE-SAT is CSP with the template $H_2 = (\{0, 1\}; \text{NAE}_2)$

Clearly, $T \subseteq \text{NAE}_2$, and therefore $T_2 \rightarrow H_2$.

The goal here is, given a solvable instance $I$ of 1-in-3-SAT, find a solution to $I$ as a NAE-SAT instance.

Both 1-in-3-SAT and NAE-SAT are NP-complete, but $\text{PCSP}(T_2, H_2)$ is in P [Brakensiek, Guruswami, ‘16].
Symmetries of PCSP: Polymorphisms

Given relational structures $A$ and $B$ that share a signature.

We say that $f : A^n \rightarrow B$ is a polymorphism from $A$ to $B$ if one of the following equivalent conditions is satisfied:

- $f$ is a homomorphism from $A^n$ to $B$,
- for each relation $R^A$ and all tuples $a_1, \ldots, a_n \in R^A$ we have

$$f(a_1, \ldots, a_n) \in R^B.$$ 

The set of all polymorphisms from $A$ to $B$ is denoted by $\operatorname{Pol}(A, B)$.

$\operatorname{Pol}(A, B)$ is not closed under composition!
Minors and minions

Let \( f : A^n \to B \) be a function. Any function \( g \) of the form

\[
g(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}).
\]

for some \( \pi : [n] \to [m] \) is called a minor of \( f \).

We call a set of functions from \( A \) to \( B \), that is closed under taking minors, a minion.

**Theorem [Pippenger, ‘02; Brakensiek, Guruswami, ‘16]**

For all finite sets \( A, B \) and minion \( \mathcal{A} \) on \( A \) and \( B \) there exist relational structures \( A \) and \( B \) such that \( \text{Pol}(A, B) = \mathcal{A} \).
PCSP and Minions

The complexity of PCSP(A, B) is determined (up to poly-time reductions) by:

- Set of polymorphisms from A to B. [Brakensiek, Guruswami, ‘16–‘18]
- The abstract minion of polymorphisms from A to B. [Bulín, Krokhin, __, ‘18]

Height 1 identities are natural for minions!
The main result

Given minions $\mathcal{M}$ and $\mathcal{N}$, a minor homomorphism is a map $\xi: \mathcal{M} \to \mathcal{N}$ that preserves arities, and preserves minors, i.e.,

$$\xi(f)(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \xi(f(x_{\pi(1)}, \ldots, x_{\pi(n)}))$$

for all $f \in \mathcal{M}^{(n)}$ and $\pi: [n] \to [m]$.

Minor homomorphisms preserve height 1 identities.

**Theorem [Bulín, Krokhin, __, ‘18]**

If there is a minor homomorphism $\xi: \text{Pol}(A_1, B_1) \to \text{Pol}(A_2, B_2)$, then $\text{PCSP}(A_2, B_2)$ is log-space reducible to $\text{PCSP}(A_1, B_1)$. 
Claim. It is NP-hard to distinguish between a graph that is 3-colorable and one that is not 5-colorable. Equivalently, PCSP($K_3, K_5$) is NP-hard.

Theorem [Dinur, Regev, Smyth, ’05] 
PCSP($H_2, H_K$) is NP-hard for all $K \geq 2$.

Key point. There is a minor homomorphism from $Pol(K_3, K_5)$ to $Pol(H_2, H_K)$. 
Intermediate problem: Deciding identities

A minor (Maltsev) condition is a finite set of identities (functional equations) of the form

\[ f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \approx g(x_1, \ldots, x_m) \]

for some \( \pi: [n] \to [m] \).

**Function symbols are variables!** I.e., we usually ask for functions that satisfy the identities.

**MC(\( N \)):**
Given is a minor condition \( \Sigma \) that involves at most \( N \)-ary function symbols, decide whether the condition is satisfied by projections.
Example: From PCSP(NAE$_2$, NAE$_K$) to MC(6)

- For each vertex $v$ introduce a binary symbol $t_v$ into $V$.
- For each edge $e = (v_1, v_2, v_3)$, introduce a 6-ary $f_e$ into $U$, and add constraints:

\[
\begin{align*}
  f_e(x, x, y, y, y, x) &\approx t_{v_1}(x, y) \\
  f_e(x, y, x, y, x, y) &\approx t_{v_2}(x, y) \\
  f_e(y, x, x, x, y, y) &\approx t_{v_3}(x, y)
\end{align*}
\]

Few observations.

- A solution to the MC instance gives a solution to CSP(NAE$_2$).
- It is enough to have a solution in Pol(NAE$_2$, NAE$_K$): The assignment $v \mapsto t_v(0, 1)$ is a solution.
Hint
We can ask Is this minor condition satisfied by polymorphisms from $A$ to $B$? as an instance of $\text{CSP}(B)$.

- We use just $A$ to construct the instance!
- **Warning!** The structure is of exponential size in $N$. 
Example: The reduction (Step 1)

1. Construct a graph $F$ with vertex set $V_F = \text{Pol}^{(2)}(K_3, K_5)$, three vertices $f$, $g$, and $h$ are connected with an edge if there is a 6-ary polymorphism $o$ s.t.

$$o(x, x, y, y, y, x) \approx f(x, y)$$
$$o(x, y, x, y, x, y) \approx g(x, y)$$
$$o(y, x, x, x, y, y) \approx h(x, y)$$

**Observation.** As long as such $F$ has no loop (does not contain edge $(a, a, a)$), it is $K$-colorable for some $K$. 
Example: A graph that is not 5-colorable

Claim. \( Pol(K_3, K_5) \) does not have a polymorphism \( o \) satisfying (Olšák polymorphism)

\[
o(x, x, y, y, y, x) \approx o(x, y, x, y, x, y) \approx o(y, x, x, x, y, y).
\]

Such polymorphism would give a 5-coloring of:

\[
\begin{align*}
100 & \; 011 \; \approx \; 010 & \; 101 \; \approx \; 001 & \; 110 \\
121 & \; 212 \; \approx \; 112 & \; 221 \; \approx \; 211 & \; 122 \\
220 & \; 002 \; \approx \; 022 & \; 200 \; \approx \; 202 & \; 020 \\
120 & \; 201
\end{align*}
\]
Free structure

Given a minion $\mathcal{M}$ and a PCSP template $(A, B)$. Assume $A = [n]$. We define the free structure of $\mathcal{M}$ generated by $A$ to be a structure $F$ similar to $A$:

- $F = \mathcal{M}^n$.
- $R^F$ consists of those $k$-tuples of functions $(f_1, \ldots, f_k)$ for which there exists $g \in \mathcal{M}$ and $r_1, \ldots, r_m \in R^A$ s.t.

$$g(x_{r_1(i)}, \ldots, x_{r_m(i)}) \approx f_i(x_1, \ldots, x_n)$$

for each $i = 1, \ldots, k$.

The graph before was a free hypergraph of $\text{Pol}(K_3, K_5)$ generated by $H_2$. 
Theorem [Bulín, Krokhin, __, ‘18]
There is a 1-to-1 correspondence between homomorphisms form the free structure of $M$ generated by $A$ to $B$ and minor homomorphisms from $M$ to $Pol(A, B)$.

In particular, this shows that there is a minor homomorphism from $Pol(K_3, K_5)$ to $Pol(H_2, H_{458})$. 
Example: The reduction (Step 2)

2. Starting with a hypergraph $G$, construct a graph $C_G$:
   - for each vertex $v$ take a copy of $K_3^2$ (expressing existence of binary polymorphism $g_v$ from $K_3$),
   - for each edge $(u, v, w)$ express that $g_u$, $g_v$, and $g_w$ are connected by a 6-ary Olšák-like polymorphism.
Example: The reduction (Step 3)

3. If \( G \) is 2-colorable hypergraph, then \( C_G \) is a 3-colorable graph.

And if \( C_G \) maps to \( B \), then \( G \) maps to \( F \), and therefore it is \( K \)-colorable.

Theorem [Bulín, Krokhin, __, ‘18]
It is NP-hard to color a \( k \)-colorable graph with \( 2k - 1 \) colors.
Conclusions

Theorem [Bulín, Krokhin, __, ‘18]
If there is a minor homomorphism $\xi : \text{Pol}(A_1, B_1) \to \text{Pol}(A_2, B_2)$, then PCSP($A_2, B_2$) is log-space reducible to PCSP($A_1, B_1$).

Theorem [Bulín, Krokhin, __, ‘18]
For all $k \geq 3$, it is NP-hard to color a $k$-colorable graph with $2k - 1$ colors.