On promise graph coloring and Mal'cev conditions

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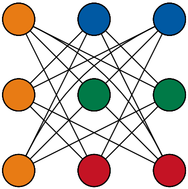
Darmstadt, June 1, 2018

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 681988)

From SAT to (3,4)-coloring

Given an instance ϕ of 3-SAT, we construct a graph $\mathbf{G}$.

1. For each variable we add to $\mathbf{G}$:



2. For each constraint, we add a copy of K_3^7 and glue ...

- If ϕ is solvable, then $\mathbf{G}$ is 3-colorable; and
- if $\mathbf{G}$ is 4-colorable, then ϕ is solvable.

Promise (d, k)-coloring

Theorem (Brakensiek, Guruswami, '16)

The following problem is NP-complete: Given a graph, output

- YES *if it is d-colorable,*
- NO *if it is not $(2d-2)$-colorable.*

Promise (d, k)-coloring is a promise problem which given a graph $\mathbf{G}$ decides between:

- YES $\mathbf{G}$ is d-colorable,
- NO $\mathbf{G}$ is not k-colorable.

Promise constraint satisfaction

Fix two finite relational structures $\mathbf{D}_1, \mathbf{D}_2$ in the same finite language with a homomorphism $\mathbf{D}_1 \to \mathbf{D}_2$. $\mathrm{PCSP}(\mathbf{D}_1, \mathbf{D}_2)$ is the following problem: Given a finite structure $\mathbf{I}$ in the same language, output

- YES if $\mathbf{I}$ maps homomorphically to $\mathbf{D}_1$,
- NO if $\mathbf{I}$ does not map homomorphically to $\mathbf{D}_2$.

Note. $\mathrm{CSP}(\mathbf{D}) \equiv \mathrm{PCSP}(\mathbf{D}, \mathbf{D})$.

An n-ary polymorphism of a template $(\mathbf{D}_1, \mathbf{D}_2)$ is a homomorphism from $\mathbf{D}_1^n \to \mathbf{D}_2$.

Siggers polymorphism

Siggers function is a 6-ary function s satisfying

$$s(x, y, z, x, y, z) \approx s(y, z, x, z, x, y).$$

Theorem (Siggers '10)

The following are equivalent for a finite structure $\mathbb{A}$.

1. $\mathbb{A}$ *has no Siggers polymorphism.*
2. *Polymorphisms of* $\mathbb{A}$ *do not satisfy any non-trivial minor Mal'cev condition.*

And if either of the cases happen, $\mathrm{CSP}(\mathbb{A})$ *is NP-hard.*

Siggers argument

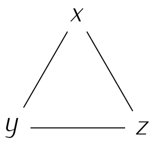
“The following rephrasing of the fact that CSP(H) is NP-complete for any graph H containing a triangle follows from Bulatov’s reproof of Hell and Nešetřil’s H-colouring Dichotomy.

“If some irreflexive symmetric binary relation (i.e. graph) containing a triangle is an invariant relation of a finite algebra A, then A omits Taylor operations.”

[Siggers, ‘10]

Proof.

Consider the free algebra generated by $\{x, y, z\}$ and the smallest compatible graph containing:



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Siggers argument for promise problems

We connect:

- Promise $(3, N)$-coloring, and
- Finite PCSPs without a Siggers polymorphism.

Suppose that there is no Siggers polymorphism from $\mathbf{A}$ to $\mathbf{B}$ (no 6-ary homomorphism from $\mathbf{A}^6$ to $\mathbf{B}$ satisfying Siggers identity).

1. Construct a graph F with vertex set $V_F = \mathrm{Pol}^{(3)}(\mathbf{A}, \mathbf{B})$, two vertices g and f are connected with an edge if there is a 6-ary polymoprhism s s.t.

$$s\begin{pmatrix} x, y, z, x, y, z \\ y, z, x, z, x, y \end{pmatrix} \approx \begin{pmatrix} g(x, y, z) \\ f(x, y, z) \end{pmatrix}.$$

Observation. Such F has no loop, and therefore is N-colorable for some N.

Siggers argument for promise problems (cont.)

2. Starting with a graph G, construct an instance $\mathbf{C}_G$ of $\mathrm{CSP}(\mathbf{A})$:
   - for each vertex v take a copy of $\mathbf{A}^3$ (expressing existence of ternary polymorphism g_v from $\mathbf{A}$),
   - for each edge (u, v) express that g_u and g_v are connected by a 6-ary polymoprhism as before.
3. If G is 3-colorable, then $\mathbf{C}_G$ maps to $\mathbf{A}$. And if $\mathbf{C}_G$ maps to $\mathbf{B}$, then G maps to F, and therefore is N-colorable.

Theorem

The following are equivalent.

1. *Promise $(3, k)$-coloring is NP-hard for each $k \geq 3$.*
2. *Every finite PCSP without a Siggers polymorphism is NP-hard.*
3. *Promise (d, k)-coloring is NP-hard for each $k \geq d \geq 3$.*

Hypergraph coloring

Theorem (Dinur, Regev, Safra, '05)

For each $K \geq 2$, it is NP-hard to distinguish between a 3-uniform hypergraph that is colorable by 2 colors, and one that is not colorable by K colors.

NAE_k is a relational structure with universe $[k]$ and a single ternary relation R_k saying 'the three entries are not all equal', i.e.,

$$R_k = \{(x, y, z) \in [k]^3 : x \neq y \text{ or } x \neq z\}.$$

Corollary

$\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_K)$ *is NP-hard for all K.*

Olšák function

Olšák function is a 6-ary function o satisfying

$$o(x,x,y,y,y,x) \approx$$
$$o(x,y,x,y,x,y) \approx$$
$$o(y,x,x,x,y,y)$$

Theorem (Olšák, '17)

Every (idempotent) clone that satisfies a non-trivial minor condition contains an Olšák function.

Theorem (Cor. to Dinur, Regev, Safra,'05)

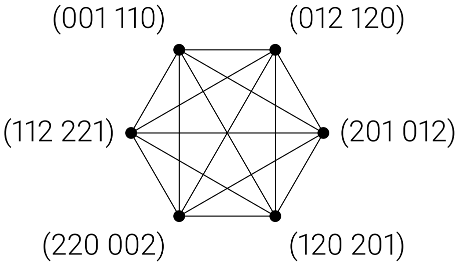
Every finite PCSP without an Olšák polymorphism is NP-hard.

Hardness of graph coloring

Lemma

$\mathrm{Pol}(K_d, K_{2d-1})$ *does not have an Olšák polymorphism.*

Olšák term is a coloring of $K_3^6/{\sim}$ where
$(x, y, y, y, x, x) \sim (y, x, y, x, y, x) \sim (y, y, x, x, x, y)$.



Theorem

For all $d > 2$, $\mathrm{PCSP}(K_d, K_{2d-1})$ *is NP-hard.*

Thank you for your attention!

Theorem

The following are equivalent.

1. *Promise* $(3, k)$*-coloring is NP-hard for each* $k \geq 3$*.*
2. *Every finite PCSP without a Siggers polymorphism is NP-hard.*

Theorem

Promise $(d, 2d-1)$*-coloring is NP-hard for each* $d \geq 3$*.* ∎