Constraint Satisfaction over the Random Tournament

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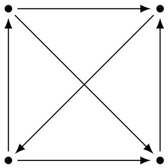
Tournament Graphs

Definition

A tournament graph is a directed graph, such that for each two distinct vertices there is exactly one edge and there are no loops.



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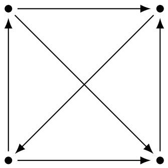

We write $T(x, y)$ for "there is a directed edge from x to y".

Tournament-SAT(Φ)

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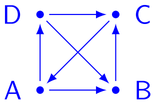

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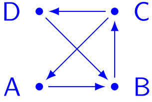

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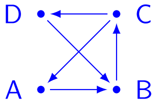

What is the computational complexity to solve Tournament-SAT(Φ) for a fixed Φ?

Result I

Theorem

Let $\Phi = \{\mathrm{T}, \varphi_1, \ldots, \varphi_l\}$ be a set of quantifier-free, first-order T-formulas. Then Tournament-SAT(Φ) is NP-complete or polynomial-time solvable.

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Main idea: Translation of Tournament-SAT to CSPs.

The Random Tournament

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- $\mathbb{T}$ is called homogeneous if every isomorphism of finite substructures extends to an automorphisms.

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- Fraisse's Theorem

Structures and Problems

Definitions:

A structure $\mathbb{A} = (V; R_1, \ldots, R_n)$ is called a first-order reduct of $\mathbb{T}$ if every relation R_i has a first-order definition in $\mathbb{T}$.

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Definition:

The CSP of a τ-structure $\mathbb{B}$ is the following computational problem:
INPUT: A finite τ-structure $\mathbb{A}$.
QUESTION: Does there exist a homomorphism $\mathbb{A} \to \mathbb{B}$?

Result II

Proposition

For every Φ there exists an expansion $\mathbb{A}$ of $\mathbb{T}$ by injective relations such that Tournament-SAT(Φ) and CSP($\mathbb{A}$) are polynomial-time equivalent.

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Complexity dichotomy for expansions of $\mathbb{T}$ by injective relations:

Theorem

Let Γ be an expansion of the random tournament $\mathbb{T}$ by injective relations that are first-order definable in $\mathbb{T}$. Then at least one of the following holds:

1. A relation H is primitive positive definable in Γ.
   In this case CSP(Γ) is NP-complete
2. Pol(Γ) contains a ternary function of type minority.
   In this case CSP(Γ) is in P.
3. Pol(Γ) contains a ternary function of type majority.
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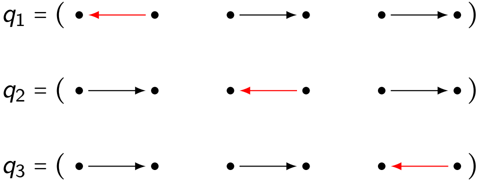
Proof Sketch: NP-completeness

Let H be the 6-ary relation on V that consists of elements of the following three types:



All missing edges are defined from left to right.

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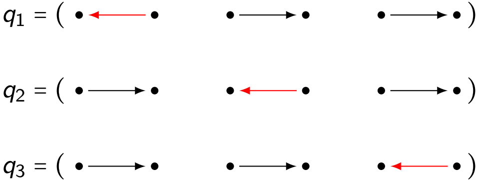

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Proposition

Let $\Gamma = (V; \mathrm{T}, H)$. Then there exists a pp-interpretation of $(\{0,1\}; 1\mathrm{IN}3)$ in Γ.

Proof Sketch: Tractability

Lemma (analogous to Bodirsky and Pinsker '10)

Let f be an operation on $\mathbb{T}$ that preserves T and violates H. Suppose moreover that all binary functions generated by f are of type projection. Then f generates a ternary function of type majority or minority.

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Theorem (Bodirsky and Mottet '18)

Let Γ be a first-order reduct of a finitely bounded homogeneous structure $\mathbb{B}$, and suppose that Γ has a weak near-unanimity polymorphism f modulo operations from $\overline{\mathrm{Aut}(\mathbb{B})}$ such that f is canonical with respect to $\mathrm{Aut}(\mathbb{B})$.
Then CSP(Γ) is in P.

Thank you for your attention!