Relation Algebras and CSPs

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- ... are a prominent way to formulate infinite-domain CSPs in the 90s.
- ... and fundamental problems in this context will be explained and discussed in the light of recent progress on infinite-domain CSP (in this talk).

Relation Algebras

Definition

A relation algebra $\mathcal{A}$ is an algebra $(A; \cup, \bar{}, 0, 1, 1', {}^{\smile}, \circ)$ of type $(2, 1, 0, 0, 0, 1, 2)$ with the following laws:

1. $(A; \cup, \bar{}, 0, 1)$ is a boolean algebra,
2. $(x \circ y) \circ z = x \circ (y \circ z)$,
3. $(x \cup y) \circ z = x \circ z \cup y \circ z$,
4. $x \circ 1' = x$,
5. $(x^{\smile})^{\smile} = x$,
6. $(x \cup y)^{\smile} = x^{\smile} \cup y^{\smile}$,
7. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$
8. $(x^{\smile} \circ \overline{(x \circ y)}) \cup \bar{y} = \bar{y}$.

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What does this mean?

Proper Relation Algebra

Definition

Let D be a set and $E \subset D^2$ an equivalence relation. Then $(\mathcal{P}(E); \cup, ^{-}, 0, 1, 1', ^{\smile}, \circ)$ is a relation algebra for the following interpretation of function symbols:

1. $A \cup B := A \cup B$,
2. $\bar{A} := E \setminus A$,
3. $0 := \varnothing$,
4. $1 := E$,
5. $1' := \{(x, x) \mid x \in D\}$,
6. $A^{\smile} := \{(x, y) \mid (y, x) \in A\}$,
7. $A \circ B := \{(x, z) \mid \exists y \in D : (x, y) \in A \text{ and } (y, z) \in B\}$.

A relation algebra is called proper if it is a subalgebra of $(\mathcal{P}(E); \cup, ^{-}, 0, 1, 1', ^{\smile}, \circ)$.

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For a proper relation algebra $\mathcal{R}$ on a set D exists the natural relational structure $\mathbb{R} = (D; \mathcal{R})$.

Examples

Point Algebra:

Consider the rational numbers $\mathbb{Q}$. The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the induced relation algebra operations is a proper relation algebra.

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Henson:

Let $\mathbb{H}$ be the universal, homogeneous, triangle-free graph. The set $\{=, E, N, E\cup =, E \cup N, N\cup =, \varnothing, H^2\}$ together with the induced relation algebra operations is a proper relation algebra.

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$\circ$	$=$	E	N
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E	E	$N\cup =$	$E \cup N$
N	N	$E \cup N$	H^2

Networks

Definition

Let $\mathcal{A}$ be a relation algebra. A relational structure $\mathbb{B}$ is called a representation of $\mathcal{A}$ if

- $\mathbb{B}$ is an A-structure,
- the induced proper relation algebra on a subset of $\mathcal{P}(B^2)$ is isomorphic to $\mathcal{A}$.

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Definitions

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.
Let $\mathbb{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathbb{B}$ if there exists an assignment $s\colon V \to B$ such that for all $x, y \in V$:

$$(s(x), s(y)) \in f(x, y)^{\mathbb{B}}$$

An $\mathcal{A}$-network $(V; f)$ is satisfiable if there exists some representation $\mathbb{C}$ of $\mathcal{A}$ such that $(V; f)$ is satisfiable in $\mathbb{C}$.

A Computational Problem

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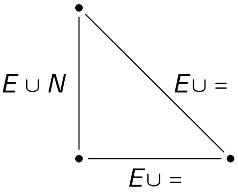
Example: Let $\mathcal{A}$ be the Henson relation algebra.



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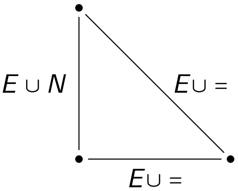

Satisfiable in the Henson graph.

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- Research Goal: Classifying those NSPs which are tractable.

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Relation algebras with symmetric relations and normal representations.

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Model theory: Classification of homogeneous multigraphs is open. (Cherlin)

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Definition

A relational structure $\mathbb{A}$ is called homogeneous if every isomorphism of finite substructures of $\mathbb{A}$ can be extended to an automorphism.

Normal Representations

Definition

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$$f(a, c) \leq f(a, b) \circ f(b, c)$$

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- square if $1^{\mathbb{B}} = B^2$
- normal if it is fully universal, square and homogeneous.

NSP as CSP

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Proposition

Let $\mathcal{A}$ be a finite relation algebra with normal representation $\mathbb{A}$. Then $\mathbb{A}$ is finitely bounded, homogeneous and the network satisfaction problem of $\mathcal{A}$ equals CSP($\mathbb{A}$) (up to some cosmetic differences in the formalisation) and is therefore in NP.

Proof sketch:

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The other direction: Use that $\mathbb{A}$ is square and "fill" the instance of the CSP to get a network.

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Let $\mathbb{A}$ be a normal representation of a finite relation algebra $\mathcal{A}$ with only symmetric relations.

? If Aut($\mathbb{A}$) is primitive and not NP-complete, then there exist canonical symmetric polymorphisms of all arities (and CSP($\mathbb{A}$) is in P).

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- ? If Aut($\mathbb{A}$) is primitive and not NP-complete, then there exist canonical symmetric polymorphisms of all arities (and CSP($\mathbb{A}$) is in P).
- ? Let E be a maximal definable nontrivial equivalence relation. Then $\mathcal{A} - E$ has a normal representation.

Thank you for your attention!