Relation Algebras and CSPs

Simon Knäuer

Joint work with Manuel Bodirsky

Institut für Algebra,
TU Dresden

DMV 2019, Karlsruhe

What you can expect

- Relation Algebras

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.
- The Really Big Complexity Problem (RBCP)

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, ...).
  - Tool to model temporal and spatial reasoning problems.

- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, ...).
  - Tool to model temporal and spatial reasoning problems.

- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, ...).
  - Tool to model temporal and spatial reasoning problems.

- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.

- Result: Partial Solution of RBCP.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.
- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.
- Result: Partial Solution of RBCP.
- A model theory perspective on relation algebras.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.
- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.
- Result: Partial Solution of RBCP.
- A model theory perspective on relation algebras.
  - Labeled homogeneous graphs (Cherlin).

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.
- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.
- Result: Partial Solution of RBCP.
- A model theory perspective on relation algebras.
  - Labeled homogeneous graphs (Cherlin).
  - Translation of RBCP into a classification question about CSPs.

What you can expect

- Relation Algebras
  - Intensively studied algebraic object (Tarski, Hodkinson, Maddux, . . .).
  - Tool to model temporal and spatial reasoning problems.
- The Really Big Complexity Problem (RBCP)
  - Classification problem for relation algebras.
  - Introduced by Robin Hirsch in 1996.
- Result: Partial Solution of RBCP.
- A model theory perspective on relation algebras.
  - Labeled homogeneous graphs (Cherlin).
  - Translation of RBCP into a classification question about CSPs.
  - Use of structural Ramsey Theory (Hubička, Nešetřil).

Relation Algebras

Definition

A relation algebra $\mathcal{A}$ is an algebra $(A; \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ of type $(2, 1, 0, 0, 0, 1, 2)$ satisfying the following laws:

1. $(A; \cup, \bar{}, 0, 1)$ is a boolean algebra,
2. $(x \circ y) \circ z = x \circ (y \circ z)$,
3. $(x \cup y) \circ z = x \circ z \cup y \circ z$,
4. $x \circ 1' = x$,
5. $(x^{\smile})^{\smile} = x$,
6. $(x \cup y)^{\smile} = x^{\smile} \cup y^{\smile}$,
7. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$
8. $(x^{\smile} \circ \overline{(x \circ y)}) \cup \bar{y} = \bar{y}$.

Relation Algebras

Definition

A relation algebra $\mathcal{A}$ is an algebra $(A; \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ of type $(2, 1, 0, 0, 0, 1, 2)$ satisfying the following laws:

1. $(A; \cup, \bar{}, 0, 1)$ is a boolean algebra,
2. $(x \circ y) \circ z = x \circ (y \circ z)$,
3. $(x \cup y) \circ z = x \circ z \cup y \circ z$,
4. $x \circ 1' = x$,
5. $(x^{\smile})^{\smile} = x$,
6. $(x \cup y)^{\smile} = x^{\smile} \cup y^{\smile}$,
7. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$
8. $(x^{\smile} \circ \overline{(x \circ y)}) \cup \bar{y} = \bar{y}$.

What does this mean?

Proper Relation Algebra

Definition

Let D be a set and $E \subseteq D^2$ an equivalence relation. Then $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is a relation algebra for the following interpretation of function symbols:

1. $A \cup B := A \cup B$,
2. $\bar{A} := E \setminus A$,
3. $0 := \emptyset$,
4. $1 := E$,
5. $1' := \{(x, x) \mid x \in D\}$,
6. $A^{\smile} := \{(x, y) \mid (y, x) \in A\}$,
7. $A \circ B := \{(x, z) \mid \exists y \in D : (x, y) \in A \text{ and } (y, z) \in B\}$.

A subalgebra of $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is called proper relation algebra.

Proper Relation Algebra

Definition

Let D be a set and $E \subseteq D^2$ an equivalence relation. Then $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is a relation algebra for the following interpretation of function symbols:

1. $A \cup B := A \cup B$,
2. $\bar{A} := E \setminus A$,
3. $0 := \varnothing$,
4. $1 := E$,
5. $1' := \{(x, x) \mid x \in D\}$,
6. $A^{\breve{}} := \{(x, y) \mid (y, x) \in A\}$,
7. $A \circ B := \{(x, z) \mid \exists y \in D : (x, y) \in A \text{ and } (y, z) \in B\}$.

A subalgebra of $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is called proper relation algebra.

For model theorists:

For a proper relation algebra $\mathcal{R}$ we view $\mathbb{R} = (D; \mathcal{R})$ as a relational structure.

Examples

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the "natural" relation algebra operations and the table.

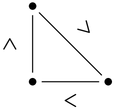
$\circ$	$=$	$<$	$>$
$=$	$=$	$<$	$>$
$<$	$<$	$<$	$\mathbb{Q}^2$
$>$	$>$	$\mathbb{Q}^2$	$>$

Examples

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the "natural" relation algebra operations and the table.

Forbidden Triangle:

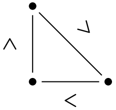


Examples

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the "natural" relation algebra operations and the table.

Forbidden Triangle:



Henson Algebra:

The set $\{=, E, N, E \cup =, E \cup N, N \cup =, \varnothing, V^2\}$ together with the "natural" relation algebra operations and the table.

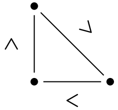
$\circ$	$=$	E	N
$=$	$=$	E	N
E	E	$N \cup =$	$E \cup N$
N	N	$E \cup N$	V^2

Examples

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the "natural" relation algebra operations and the table.

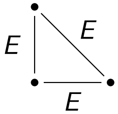
Forbidden Triangle:



Henson Algebra:

The set $\{=, E, N, E \cup =, E \cup N, N \cup =, \varnothing, V^2\}$ together with the "natural" relation algebra operations and the table.

Forbidden Triangle:

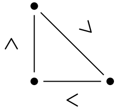


Examples

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the “natural” relation algebra operations and the table.

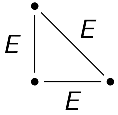
Forbidden Triangle:



Henson Algebra:

The set $\{=, E, N, E \cup =, E \cup N, N \cup =, \varnothing, V^2\}$ together with the “natural” relation algebra operations and the table.

Forbidden Triangle:



Definition

The minimal relations with respect to inclusion are called atoms.

Representations

Definition

Let $\mathcal{A}$ be a relation algebra. A relational structure $\mathbb{B}$ is called a representation of $\mathcal{A}$ if

- $\mathbb{B}$ is an A-structure,
- the induced proper relation algebra on a subset of $\mathcal{P}(B^2)$ is isomorphic to $\mathcal{A}$.

Representations

Definition

Let $\mathcal{A}$ be a relation algebra. A relational structure $\mathbb{B}$ is called a representation of $\mathcal{A}$ if

- $\mathbb{B}$ is an A-structure,
- the induced proper relation algebra on a subset of $\mathcal{P}(B^2)$ is isomorphic to $\mathcal{A}$.

Examples

- $(\mathbb{Q}; =, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2)$ is a representation of the Point Algebra.
- The countable, universal, homogeneous, triangle-free graph

$$\mathbb{H} = (V; =, E, N, E\cup =, E \cup N, N\cup =, \varnothing, V^2)$$

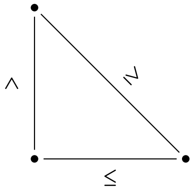
is a representation of the Henson Algebra.

Networks

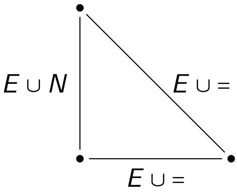
Definitions

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.

Point Algebra Network:



Henson Algebra Network:



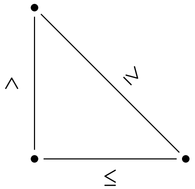
Networks

Definitions

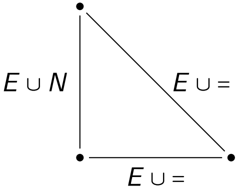
Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f \colon V \times V \to A$.
Let $\mathbb{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathbb{B}$ if there exists an assignment $s \colon V \to B$ such that for all $x, y \in V$:

$$(s(x), s(y)) \in f(x, y)^{\mathbb{B}}$$

Point Algebra Network:



Henson Algebra Network:



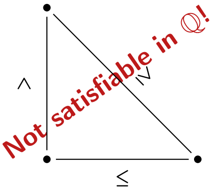
Networks

Definitions

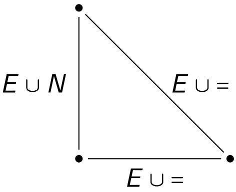
Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.
Let $\mathbb{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathbb{B}$ if there exists an assignment $s\colon V \to B$ such that for all $x, y \in V$:

$$(s(x), s(y)) \in f(x, y)^{\mathbb{B}}$$

Point Algebra Network:



Henson Algebra Network:



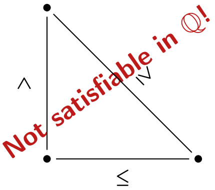
Networks

Definitions

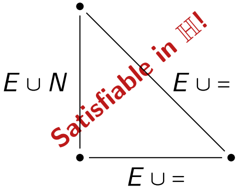
Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.
Let $\mathbb{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathbb{B}$ if there exists an assignment $s\colon V \to B$ such that for all $x, y \in V$:

$$(s(x), s(y)) \in f(x, y)^{\mathbb{B}}$$

Point Algebra Network:



Henson Algebra Network:



Networks

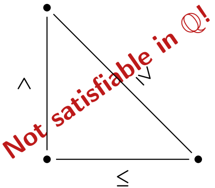
Definitions

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.
Let $\mathbb{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathbb{B}$ if there exists an assignment $s\colon V \to B$ such that for all $x, y \in V$:

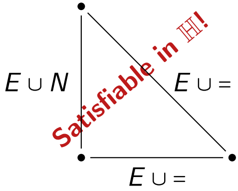
$$(s(x), s(y)) \in f(x, y)^{\mathbb{B}}$$

An $\mathcal{A}$-network $(V; f)$ is satisfiable if there exists some representation $\mathbb{C}$ of $\mathcal{A}$ such that $(V; f)$ is satisfiable in $\mathbb{C}$.

Point Algebra Network:



Henson Algebra Network:



Result: A Complexity Classification

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with $\mathrm{NSP}(\mathcal{A})$.

Result: A Complexity Classification

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with $\mathrm{NSP}(\mathcal{A})$.

- Research Goal: Classifying those NSPs which are polynomial-time tractable.

Result: A Complexity Classification

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with $\text{NSP}(\mathcal{A})$.

- Research Goal: Classifying those NSPs which are polynomial-time tractable.
- Robin Hirsch 1996: Really Big Complexity Problem (RBCP).

Result: A Complexity Classification

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with $\mathrm{NSP}(\mathcal{A})$.

- Research Goal: Classifying those NSPs which are polynomial-time tractable.
- Robin Hirsch 1996: Really Big Complexity Problem (RBCP).

Theorem (Partial RBCP)

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom.
Then $\mathrm{NSP}(\mathcal{A})$ is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Result: A Complexity Classification

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with NSP($\mathcal{A}$).

- Research Goal: Classifying those NSPs which are polynomial-time tractable.
- Robin Hirsch 1996: Really Big Complexity Problem (RBCP).

Theorem (Partial RBCP)

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom.
Then NSP($\mathcal{A}$) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Definition

Let $\mathcal{A}$ be a finite relation algebra. An atom $S \in A$ is flexible if for all $B, C \in A \setminus \{1'\}$ it holds that $S \leq B \circ C$.
→ "All triangles that contain a S are allowed."

Approach

- To solve RBCP in full generality seems very difficult.

Approach

- To solve RBCP in full generality seems very difficult.
- Hirsch introduced a subclass of finite relation algebras with “nice” representations:

Approach

- To solve RBCP in full generality seems very difficult.
- Hirsch introduced a subclass of finite relation algebras with “nice” representations:

Finite relation algebras with normal representations.

Approach

- To solve RBCP in full generality seems very difficult.
- Hirsch introduced a subclass of finite relation algebras with “nice” representations:

Finite relation algebras with normal representations.

↓ Hirsch 1994

Homogeneous edge-labeled graphs defined by forbidden triangles.

Approach

- To solve RBCP in full generality seems very difficult.
- Hirsch introduced a subclass of finite relation algebras with “nice” representations:

Finite relation algebras with normal representations.

↓ Hirsch 1994

Homogeneous edge-labeled graphs defined by forbidden triangles.

↗

Cherlin: Classification is open.

Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of f only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$

Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of f only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$

Definitions

A representation $\mathbb{B}$ of a relation algebra $\mathcal{A}$ is called

- fully universal if every atomic $\mathcal{A}$-network is satisfiable in $\mathbb{B}$;

Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of f only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$

Definitions

A representation $\mathbb{B}$ of a relation algebra $\mathcal{A}$ is called

- fully universal if every atomic $\mathcal{A}$-network is satisfiable in $\mathbb{B}$;
- square if $1^{\mathbb{B}} = B^2$;

Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of f only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$

Definitions

A representation $\mathbb{B}$ of a relation algebra $\mathcal{A}$ is called

- fully universal if every atomic $\mathcal{A}$-network is satisfiable in $\mathbb{B}$;
- square if $1^{\mathbb{B}} = B^2$;
- homogeneous if every isomorphism of finite substructures of $\mathbb{B}$ can be extended to an automorphism;

Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of f only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$

Definitions

A representation $\mathbb{B}$ of a relation algebra $\mathcal{A}$ is called

- fully universal if every atomic $\mathcal{A}$-network is satisfiable in $\mathbb{B}$;
- square if $1^{\mathbb{B}} = B^2$;
- homogeneous if every isomorphism of finite substructures of $\mathbb{B}$ can be extended to an automorphism;
- normal if it is fully universal, square and homogeneous.

NSP as CSP

Definition

Let $\mathbb{A}$ be a τ-structure. The Constraint Satisfaction Problem of $\mathbb{A}$ is to decide for a given finite τ-structure $\mathbb{C}$ whether there exists a homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

NSP as CSP

Definition

Let $\mathbb{A}$ be a τ-structure. The Constraint Satisfaction Problem of $\mathbb{A}$ is to decide for a given finite τ-structure $\mathbb{C}$ whether there exists a homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

Proposition

Let $\mathcal{A}$ be a finite relation algebra with normal representation $\mathbb{A}$.
Then $\mathbb{A}$ is finitely bounded and NSP($\mathcal{A}$) equals CSP($\mathbb{A}$) (up to some cosmetic differences in the formalisation) and is therefore in NP.

NSP as CSP

Definition

Let $\mathbb{A}$ be a τ-structure. The Constraint Satisfaction Problem of $\mathbb{A}$ is to decide for a given finite τ-structure $\mathbb{C}$ whether there exists a homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

Proposition

Let $\mathcal{A}$ be a finite relation algebra with normal representation $\mathbb{A}$.
Then $\mathbb{A}$ is finitely bounded and NSP($\mathcal{A}$) equals CSP($\mathbb{A}$) (up to some cosmetic differences in the formalisation) and is therefore in NP.

Remark: There exists a finite relation algebra with undecidable NSP (Hirsch 1999)!

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation Γ and $\mathrm{CSP}(\Gamma)$ is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation Γ and CSP(Γ) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Comments on the Proof:

- Fraisse's Theorem: Γ exists, because of free amalgamation.

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation Γ and CSP(Γ) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Comments on the Proof:

- Fraisse's Theorem: Γ exists, because of free amalgamation.
- Universal algebra: Study homomorphisms $\Gamma^n \to \Gamma$.

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation Γ and CSP(Γ) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

<u>Comments on the Proof:</u>

- Fraisse's Theorem: Γ exists, because of free amalgamation.
- Universal algebra: Study homomorphisms $\Gamma^n \to \Gamma$.
- Important result by Hubička and Nešetřil: Γ with the generic order is a Ramsey structure.

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation Γ and CSP(Γ) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

Comments on the Proof:

- Fraisse's Theorem: Γ exists, because of free amalgamation.
- Universal algebra: Study homomorphisms $\Gamma^n \to \Gamma$.
- Important result by Hubička and Nešetřil: Γ with the generic order is a Ramsey structure.
- Use of the Bulatov-Zhuk Dichotomy Theorem for finite-domain CSPs.

Thank you for your attention!

Result

Theorem

Let Γ be a normal representation of a finite integral relation algebra with a flexible atom. One of the following holds:

1. There exists for every two atoms A and B of the algebra a polymorphism $f_{A,B}$ of Γ that is canonical and the induced function on $\{A, B\}$ is of Schaefer-type, then Γ has a canonical pseudo-Siggers polymorphism. Then CSP(Γ) is in P.
2. CSP(Γ) is NP-complete.