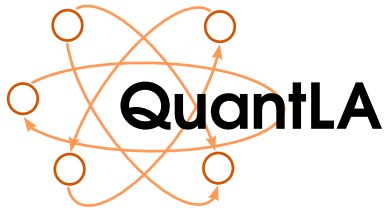
MSO ∩ Datalog

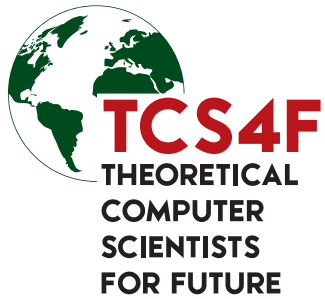
Simon Knäuer (TU Dresden)

Joint work with Manuel Bodirsky and Sebastian Rudolph

QuantLA Workshop 2020



DFG Research Training Group 1763



Introduction 1

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Can we describe such problems in a 'nice' way?

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- Necessary conditions for MSO definability in Datalog .
- Pebble game characterization of MSO ∩ Datalog. Not in this talk!

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Example

Monadic second-order $\{E\}$-sentence:

$$\forall R \exists x \in R \forall y \in R : (\exists z . z \in R) \Rightarrow \neg E(y, x)$$

MSO definable classes

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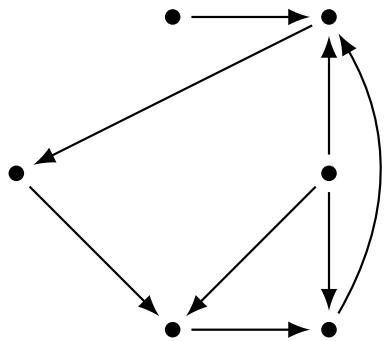
A structure that satisfies $\neg\Phi$:



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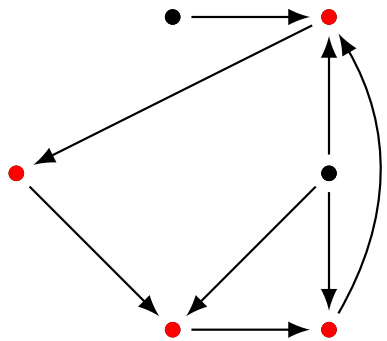

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The class $[\![\Phi]\!]$ consists of all directed graphs that do not contain a cycle.

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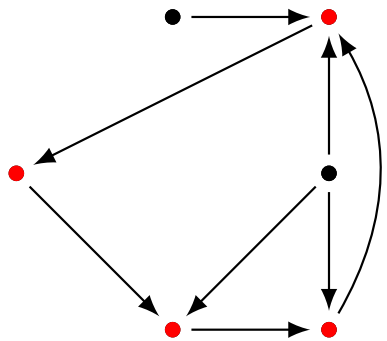

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We denote the class of accepted instances by $[\![\Pi]\!]$.

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Datalog program Π^{succ} with EDB Succ and IDBs L, S, T and **false**:

$$S(x,y) :- \mathrm{Succ}(x,y)$$
$$L(x,x) :- x$$
$$L(x',y') :- L(x,y), S(x,x'), S(y,y')$$
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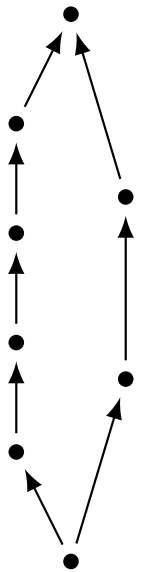

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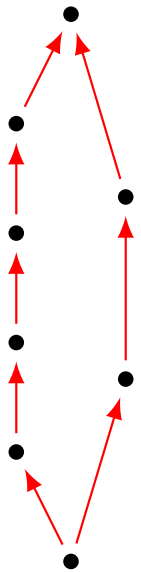

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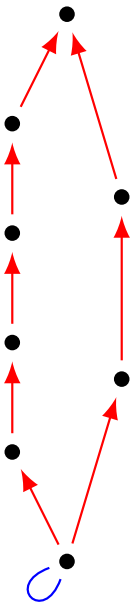

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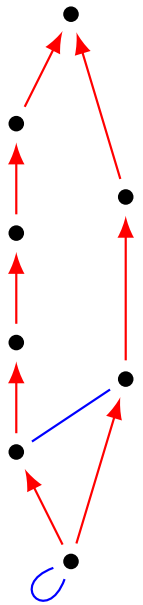

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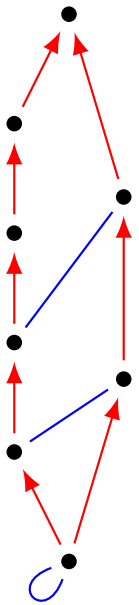

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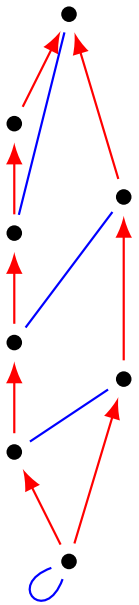

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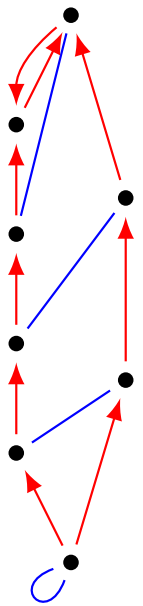

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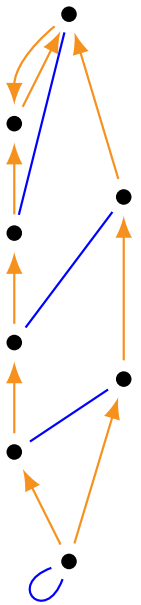

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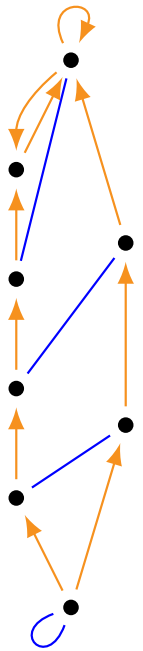

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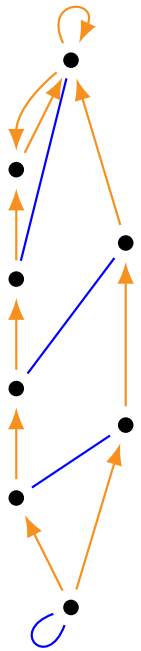

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Π^{succ} accepts exactly the structures where for each two nodes all connecting paths have the same length.

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Datalog program Π with EDB E and IDB R:

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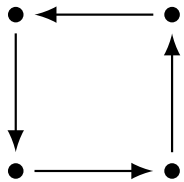

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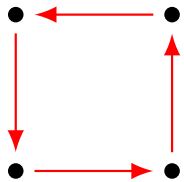

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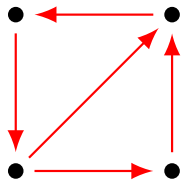

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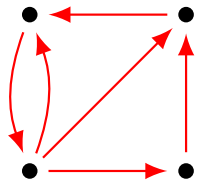

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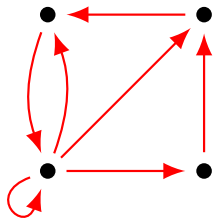

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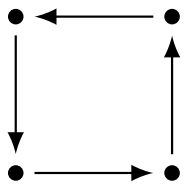
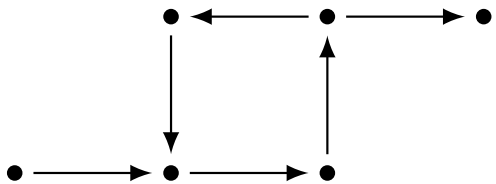


homomorphism

Instance **J**:



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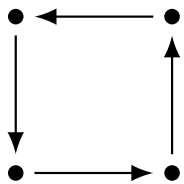

homomorphism

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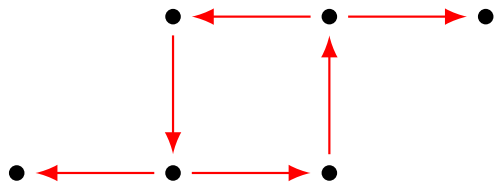

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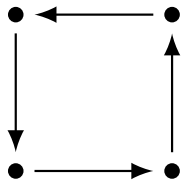

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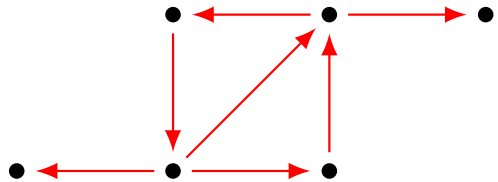

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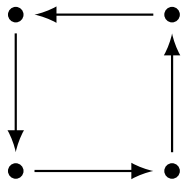

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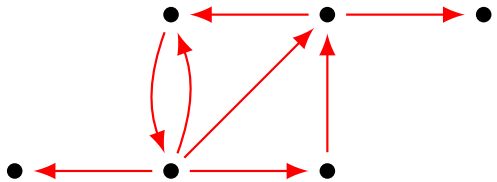

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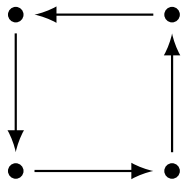

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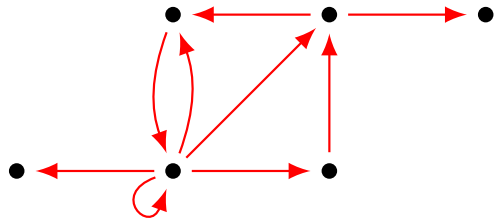

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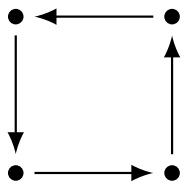

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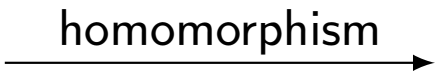

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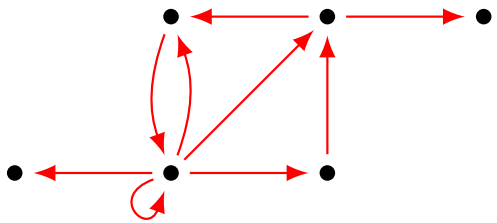

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Observation

Let $\mathcal{C}$ be defined by a Datalog program. Let **A** and **B** be structures where **B** is in $\mathcal{C}$ and **A** has a homomorphism to **B**. Then **A** is in $\mathcal{C}$.
We say that $\mathcal{C}$ is closed under inverse homomorphisms.

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- $\mathrm{CSP}(\mathbb{Z}; \mathrm{Succ} = \{(x, y) \mid x + 1 = y\}) = [\![\Pi^{succ}]\!]$ is the class of all graphs where for each two nodes all connecting paths have the same length.

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Proposition

A class of τ-structures $\mathcal{C}$ is a CSP for some structure iff it is closed under inverse homomorphisms and closed under disjoint unions., i.e. if $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, then $\mathbf{A} \uplus \mathbf{B} \in \mathcal{C}$.

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Let B and R be unary and let E be a binary relation symbol.

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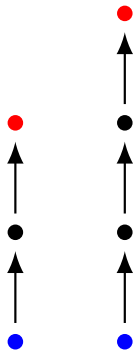

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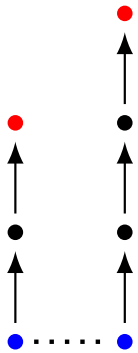

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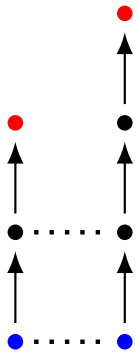

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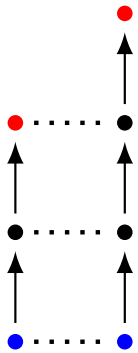

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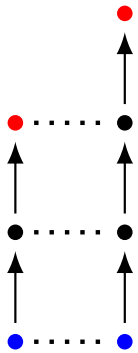

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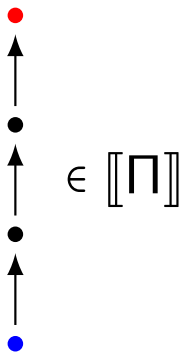

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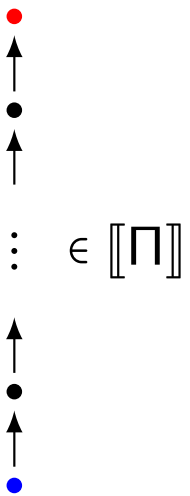

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$\longrightarrow$ By Result 1 and the observation about Datalog, the class $[\![\Pi]\!]$ is not MSO definable.

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Remark

Result 2 can be used to achieve a characterization of MSO $\cap$ Datalog in terms of *existential pebble games*.

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Result 3

Every problem in MSO ∩ Datalog is the finite union of ω-categorical CSPs.

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- Is the intersection of GSO and Datalog describable by some logic?
- Is there an example of a CSP for a reduct of a finitely bounded homogeneous structure that is not in GSO?

Thank you for your attention!

Publications

- *Hardness of Network Satisfaction for Relation Algebras with Normal Representations*, Manuel Bodirsky and Simon Knäuer.
  In Proceedings of RAMICS 2020.
- *ASNP: A Tame Fragment of Existential Second-Order Logic*
  Manuel Bodirsky, Simon Knäuer and Florian Starke
  In Proceedings of CiE 2020.
- *Network satisfaction for symmetric relation algebras with a flexible atom*,
  Manuel Bodirsky and Simon Knäuer.
  Submitted.
- *Datalog-Expressibility for Monadic and Guarded Second-Order Logic*
  Manuel Bodirsky, Simon Knäuer and Sebastian Rudolph
  Submitted.

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- Induction argument over the number of classes of $\sim$.

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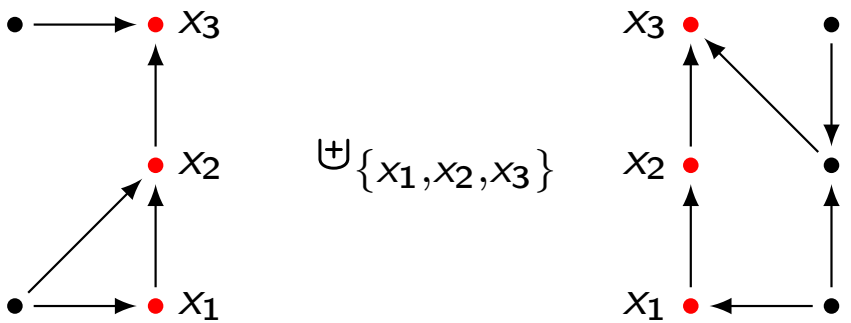

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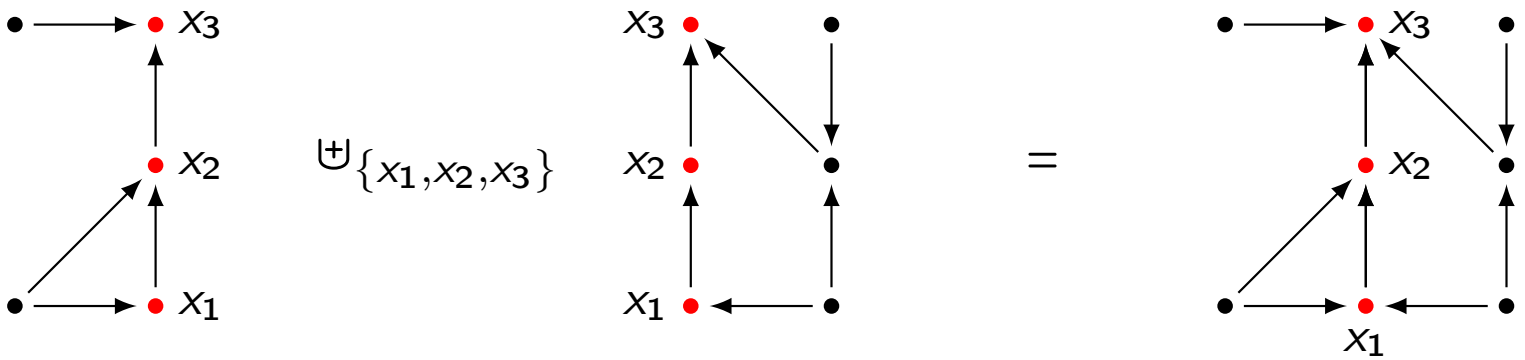

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Use of the following theorem:

Theorem (Bodirsky, Hils, Martin)

Let $\mathcal{C}$ be closed under inverse homomorphisms and disjoint unions. Then there exists an ω-categorical structure $\mathbf{B}$ such that $\mathrm{CSP}(\mathbf{B}) = \mathcal{C}$ if and only if $\sim_n^{\mathcal{C}}$ has finitely many equivalence classes for each $n \in \mathbb{N}$.