Relation Algebras and CSPs

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  - Translation of RBCP into a classification question about CSPs.
  - Use of structural Ramsey Theory (Hubička, Nešetřil).

Relation Algebras

Definition

A relation algebra $\mathcal{A}$ is an algebra $(A; \cup, \bar{\ }, 0, 1, 1', \breve{\ }, \circ)$ of type $(2, 1, 0, 0, 0, 1, 2)$ satisfying the following laws:

1. $(A; \cup, \bar{\ }, 0, 1)$ is a boolean algebra,
2. $(x \circ y) \circ z = x \circ (y \circ z)$,
3. $(x \cup y) \circ z = x \circ z \cup y \circ z$,
4. $x \circ 1' = x$,
5. $(x^{\smile})^{\smile} = x$,
6. $(x \cup y)^{\smile} = x^{\smile} \cup y^{\smile}$,
7. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$
8. $(x^{\smile} \circ \overline{(x \circ y)}) \cup \bar{y} = \bar{y}$.

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What does this mean?

Proper Relation Algebra

Definition

Let D be a set and $E \subseteq D^2$ an equivalence relation. Then $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is a relation algebra for the following interpretation of function symbols:

1. $A \cup B := A \cup B$,
2. $\bar{A} := E \setminus A$,
3. $0 := \varnothing$,
4. $1 := E$,
5. $1' := \{(x, x) \mid x \in D\}$,
6. $A^{\smile} := \{(x, y) \mid (y, x) \in A\}$,
7. $A \circ B := \{(x, z) \mid \exists y \in D : (x, y) \in A \text{ and } (y, z) \in B\}$.

A subalgebra of $(\mathcal{P}(E); \cup, \bar{}, 0, 1, 1', \breve{}, \circ)$ is called proper relation algebra.

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For model theorists:

For a proper relation algebra $\mathcal{R}$ we view $\mathbb{R} = (D; \mathcal{R})$ as a relational structure.

Examples I

Point Algebra:

The set $\{=, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2\}$ together with the "natural" relation algebra operations and the table.

$\circ$	$=$	$<$	$>$
$=$	$=$	$<$	$>$
$<$	$<$	$<$	$\mathbb{Q}^2$
$>$	$>$	$\mathbb{Q}^2$	$>$

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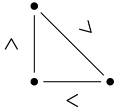
Forbidden Triangle:



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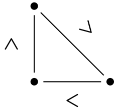

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The set $\{=, E, N, E \cup =, E \cup N, N \cup =, \varnothing, V^2\}$ together with the "natural" relation algebra operations and the table.

$\circ$	$=$	E	N
$=$	$=$	E	N
E	E	$N \cup =$	$E \cup N$
N	N	$E \cup N$	V^2

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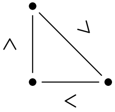

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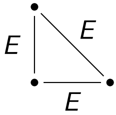

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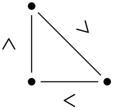

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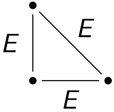

Definition

The minimal non-trivial relations with respect to inclusion are called atoms.

Examples II

Metric spaces:
Let $\{=, 1, 2, 3, 4\}$ be binary predicates associated with integer distances.

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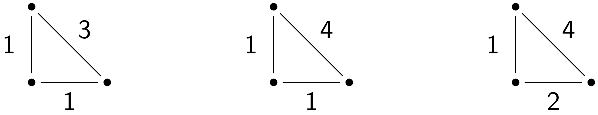

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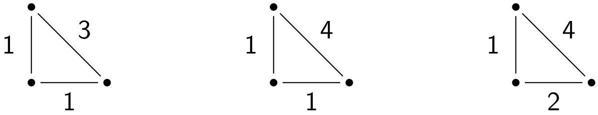

Define a relation algebra on $\mathcal{P}(\{=, 1, 2, 3, 4\})$ with the following multiplication table.

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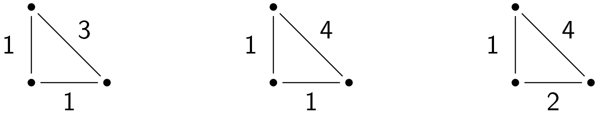

Define a relation algebra on $\mathcal{P}(\{=, 1, 2, 3, 4\})$ with the following multiplication table.

$\circ$	$=$	1	2	3	4
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1	1	$1 \cup 2 \cup =$	$1 \cup 2 \cup 3$	$3 \cup 4$	$3 \cup 4$
2	2	$1 \cup 2 \cup 3$	$1 \cup 2 \cup 3 \cup 4 \cup =$	$1 \cup 2 \cup 3 \cup 4$	$2 \cup 3 \cup 4$
3	3	$2 \cup 3 \cup 4$	$1 \cup 2 \cup 3 \cup 4$	$1 \cup 2 \cup 3 \cup 4 \cup =$	$1 \cup 2 \cup 3 \cup 4$
4	4	$3 \cup 4$	$2 \cup 3 \cup 4$	$1 \cup 2 \cup 3 \cup 4$	$1 \cup 2 \cup 3 \cup 4 \cup =$

Representations

Definition

A relational structure $\mathbb{B}$ is called a representation of a relation algebra $\mathcal{A}$ if

- $\mathbb{B}$ is an A-structure,
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Examples

- $(\mathbb{Q}; =, <, >, \leq, \geq, \varnothing, \neq, \mathbb{Q}^2)$ is a representation of the Point Algebra.
- The countable, universal, homogeneous, triangle-free graph

$$\mathbb{H} = (V; =, E, N, E\cup =, E \cup N, N\cup =, \varnothing, V^2)$$

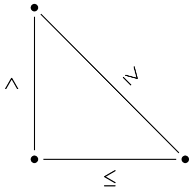
is a representation of the Henson Algebra.

Networks

Definitions

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes V together with a function $f\colon V \times V \to A$.

Point Algebra Network:



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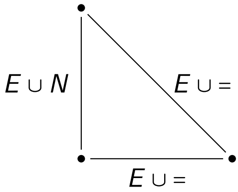

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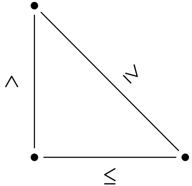

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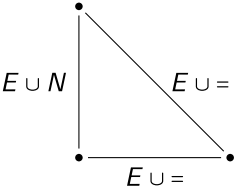

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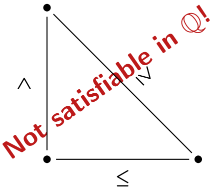

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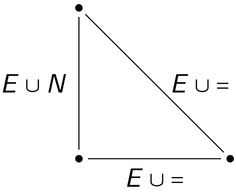

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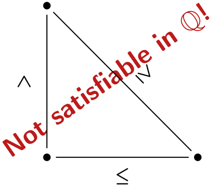

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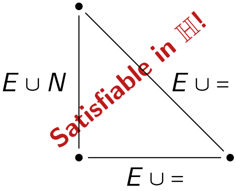

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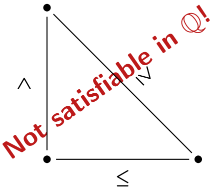

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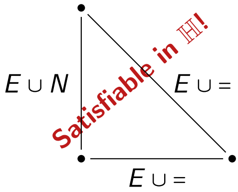

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- Cristani, Hirsch 2004: 18 small relation algebras.

Result: A Complexity Classification

Theorem (Partial RBCP)

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom.
Then NSP($\mathcal{A}$) is in P or NP-complete.
Moreover, it is decidable which of the two cases holds.

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Definition

Let $\mathcal{A}$ be a finite relation algebra. An atom $S \in A$ is flexible if for all $B, C \in A \setminus \{1'\}$ it holds that $S \leq B \circ C$.
→ "All triangles that contain a S are allowed."

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Cherlin: Classification is open.

Normal Representations

Definition

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- homogeneous if every isomorphism of finite substructures of $\mathbb{B}$ can be extended to an automorphism;
- normal if it is fully universal, square and homogeneous.

NSP as CSP

Definition

Let $\mathbb{A}$ be a τ-structure. The Constraint Satisfaction Problem of $\mathbb{A}$ is to decide for a given finite τ-structure $\mathbb{C}$ whether there exists a homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

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Let $\mathbb{A}$ be a τ-structure. The Constraint Satisfaction Problem of $\mathbb{A}$ is to decide for a given finite τ-structure $\mathbb{C}$ whether there exists a homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

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Let $\mathcal{A}$ be a finite relation algebra with normal representation $\mathbb{A}$.
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Remark: There exists a finite relation algebra with undecidable NSP (Hirsch 1999)!

Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom.
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Let $\mathcal{A}$ be a finite relation algebra. The class of atomic $\mathcal{A}$-networks (considered as structures) is finitely bounded by all forbidden (with respect to $\circ$) triangles.

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(AP) **Amalgamation property:** For $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{C}$ and embeddings $e: A \to B$ and $f: A \to C$ there exists a structure $\mathbb{D} \in \mathcal{C}$ and embeddings $g: B \to D$ and $h: C \to D$ such that $g \circ e = h \circ f$.

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All finite linear orders, all finite undirected graphs,...

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- This structure is fully-universal since the age of Γ contains all atomic $\mathcal{A}$-networks.

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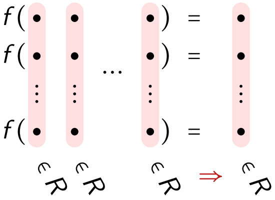

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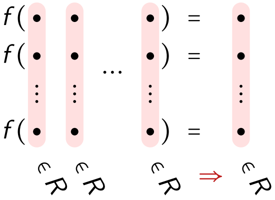

Proposition

Let $\mathbb{A}$ and $\mathbb{B}$ be ω-categorical structures. If $\mathrm{Pol}(\mathbb{A}) \subset \mathrm{Pol}(\mathbb{B})$ holds, then there exists a polynomial-time reduction from $\mathrm{CSP}(\mathbb{B})$ to $\mathrm{CSP}(\mathbb{A})$.

Canonical Polymorphisms

Motivation: Reduction of infinite-domain CSPs to finite-domain CSPs.

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Let X be a subset of the set of atomic relations Y of Γ. Then a polymorphism $\Gamma^n \to \Gamma$ is called X-canonical if it induces a function $X^n \to Y$.
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A polymorphism of a normal representation is edge conservative:
Let $a_1, \ldots, a_n, b_1, \ldots b_n \in V$ with $X_i(a_i, b_i)$ for atomic relations X_i, then

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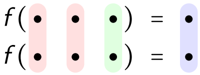

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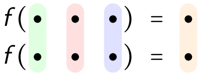

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Analysis of Atomic Relations

Theorem

Let Γ be a normal representation of a finite relation algebra with a flexible atom. If CSP(Γ) is not NP-complete then for every two atomic relations A and B there exists an $\{A, B\}$-canonical polymorphism $f_{A,B}$ of Γ such that the induced function on $\{A, B\}$ is one of the following:

- A binary symmetric function;
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We call these functions of Schaefer-type.

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These functions give tractability results for finite-domain CSPs!

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Theorem (Bodirsky and Mottet '18 + Finite-domain Dichotomy '17)

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Definition

A function f is Siggers modulo operations from $\mathrm{End}(\Gamma)$ if there exist $e_1, e_2 \in \mathrm{End}(\Gamma)$ such that the following holds:

$$\forall x, y, z : e_1(f(x, y, x, z, y, z)) = e_2(f(z, z, y, y, x, x))$$

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Proposition

Let Γ be a normal representation of a finite relation algebra with a flexible atom. Assume that Γ has for every two atomic relations A and B a polymorphism that is canonical and of Schaefer-type on $\{A, B\}$.
Then Γ has a polymorphism that is Siggers modulo operations from $\mathrm{End}(\Gamma)$.

Thank you for your attention!

Result

Theorem

Let Γ be a normal representation of a finite relation algebra with a flexible atom. One of the following holds:

1. There exists for every two atoms A and B of the algebra a polymorphism $f_{A,B}$ of Γ that is canonical and the induced function on $\{A, B\}$ is of Schaefer-type, then Γ has a canonical pseudo-Siggers polymorphism. Then CSP(Γ) is in P.
2. CSP(Γ) is NP-complete.