Relation Algebras and CSPs

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Joint work with Manuel Bodirsky

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QuantLA Workshop 2019, Stolpen
What you can expect

- Relation Algebras

Intensively studied algebraic object (Tarski, Hodkinson, Maddux, ...).

Tool to model temporal and spatial reasoning problems in AI.

The Really Big Complexity Problem (RBCP)

Classification problem for relation algebras.

Introduced by Robin Hirsch in 1996.

Result: Partial Solution of RBCP.

A model theory perspective on relation algebras.

Labeled homogeneous graphs (Cherlin).

Translation of RBCP into a classification question about CSPs.

Use of structural Ramsey Theory (Hubička, Nešetřil).

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Definition

A relation algebra $\mathcal{A}$ is an algebra $(\mathcal{A}; \cup, \neg, 0, 1, 1', \sim, \circ)$ of type $(2, 1, 0, 0, 0, 1, 2)$ satisfying the following laws:

1. $(\mathcal{A}; \cup, \neg, 0, 1)$ is a boolean algebra,
2. $(x \circ y) \circ z = x \circ (y \circ z)$,
3. $(x \cup y) \circ z = x \circ z \cup y \circ z$,
4. $x \circ 1' = x$,
5. $(x\neg)\neg = x$,
6. $(x \cup y)\neg = x\neg \cup y\neg$,
7. $(x \circ y)\neg = y\neg \circ x\neg$,
8. $(x\neg \circ (x \circ y)) \cup \bar{y} = \bar{y}$.

What does this mean?
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3. \((x \cup y) \circ z = x \circ z \cup y \circ z\),
4. \(x \circ 1' = x\),
5. \((x')' = x\),
6. \((x \cup y)' = x' \cup y'\),
7. \((x \circ y)' = y' \circ x'\)
8. \((x' \circ (x \circ y)) \cup \bar{y} = \bar{y}\).

What does this mean?
Proper Relation Algebra

Definition

Let $D$ be a set and $E \subseteq D^2$ an equivalence relation. Then $(\mathcal{P}(E); \cup, \sim, 0, 1, 1', \circ)$ is a relation algebra for the following interpretation of function symbols:

1. $A \cup B := A \cup B$,
2. $\bar{A} := E \setminus A$,
3. $0 := \emptyset$,
4. $1 := E$,
5. $1' := \{(x, x) \mid x \in D\}$,
6. $A\sim := \{(x, y) \mid (y, x) \in A\}$,
7. $A \circ B := \{(x, z) \mid \exists y \in D: (x, y) \in A \text{ and } (y, z) \in B\}$.

A subalgebra of $(\mathcal{P}(E); \cup, \sim, 0, 1, 1', \circ)$ is called proper relation algebra.
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For model theorists:
For a proper relation algebra $\mathcal{R}$ we view $\mathcal{R} = (D; \mathcal{R})$ as a relational structure.
Point Algebra:
The set \( \{=, <, >, \leq, \geq, \emptyset, \neq, Q^2 \} \) together with the “natural” relation algebra operations and the table.

\[
| \circ | = | < | > \\
<table>
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Examples I

Point Algebra:
The set \{=, <, >, \leq, \geq, \emptyset, \neq, \mathbb{Q}^2\} together with the “natural” relation algebra operations and the table.

Forbidden Triangle:
Examples I

Point Algebra:
The set \{=, <, >, \leq, \geq, \emptyset, \neq, Q^2\} together with the “natural” relation algebra operations and the table.

Henson Algebra:
The set \{=, E, N, E\cup =, E\cup N, N\cup =, \emptyset, V^2\} together with the “natural” relation algebra operations and the table.

Forbidden Triangle:

\[
\begin{array}{c}
\circ \\
= \\
E \\
N \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\circ & = & E \\
= & = & N \\
E & E & N \cup = \\
N & N & E \cup N \\
& & V^2 \\
\end{array}
\]
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Forbidden Triangle:

Definition

The minimal non-trivial relations with respect to inclusion are called atoms.
Examples II

Metric spaces:
Let \{\neq, 1, 2, 3, 4\} be binary predicates associated with integer distances.
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Let \{=, 1, 2, 3, 4\} be binary predicates associated with integer distances. Consider the set of forbidden triangle inequalities.
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Define a relation algebra on \( \mathcal{P}(\{=, 1, 2, 3, 4\}) \) with the following multiplication table.
Examples II

Metric spaces:
Let \( \{=, 1, 2, 3, 4\} \) be binary predicates associated with integer distances. Consider the set of forbidden triangle inequalities.

\[
\begin{array}{ccc}
1 & 3 & 1 \\
1 & 1 & 3 \\
1 & 1 & 4 \\
1 & 4 & 2
\end{array}
\]

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<td>1 \cup 2 \cup 3</td>
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Simon Knäuer (TU Dresden)
Representations

Definition

A relational structure \( \mathbb{B} \) is called a representation of a relation algebra \( \mathcal{A} \) if

1. \( \mathbb{B} \) is an \( \mathcal{A} \)-structure,
2. the induced proper relation algebra on a subset of \( \mathcal{P}(B^2) \) is isomorphic to \( \mathcal{A} \).
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Examples

- $(\mathbb{Q}; =, <, >, \leq, \geq, \emptyset, \neq, \mathbb{Q}^2)$ is a representation of the Point Algebra.
- The countable, universal, homogeneous, triangle-free graph

$$\mathbb{H} = (V; =, E, N, E \cup =, E \cup N, N \cup =, \emptyset, V^2)$$

is a representation of the Henson Algebra.
Networks

Definitions

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes $V$ together with a function $f : V \times V \to A$.

Point Algebra Network:

\[
\begin{array}{c}
\wedge \\
\leq
\end{array}
\]

Henson Algebra Network:

\[
\begin{array}{c}
E \cup N \\
E \cup =
\end{array}
\]
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Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is a finite set of nodes $V$ together with a function $f: V \times V \to \mathcal{A}$.

Let $\mathcal{B}$ be a representation of $\mathcal{A}$. An $\mathcal{A}$-network $(V; f)$ is satisfiable in $\mathcal{B}$ if there exists an assignment $s: V \to \mathcal{B}$ such that for all $x, y \in V$:

$$(s(x), s(y)) \in f(x, y)^\mathcal{B}$$

Point Algebra Network:

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Not satisfiable in $\mathcal{Q}$!

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An $\mathcal{A}$-network $(V; f)$ is satisfiable if there exists some representation $\mathcal{C}$ of $\mathcal{A}$ such that $(V; f)$ is satisfiable in $\mathcal{C}$.

Point Algebra Network:

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Henson Algebra Network:

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The Really Big Complexity Problem

Definition

The Network Satisfaction Problem for a finite relation algebra $\mathcal{A}$ is the problem to decide whether a given $\mathcal{A}$-network is satisfiable. We denote this with $\text{NSP}(\mathcal{A})$. 
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Theorem (Partial RBCP)

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\text{NSP}(\mathcal{A})$ is in P or NP-complete. Moreover, it is decidable which of the two cases holds.
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Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\text{NSP}(\mathcal{A})$ is in P or NP-complete. Moreover, it is decidable which of the two cases holds.

Definition

Let $\mathcal{A}$ be a finite relation algebra. An atom $S \in A$ is flexible if for all $B, C \in A \setminus \{1\}$ it holds that $S \leq B \circ C$.

→ “All triangles that contain a $S$ are allowed.”
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Hirsch 1994

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Cherlin: Classification is open.
Normal Representations

Definition

Let $\mathcal{A}$ be a relation algebra. An $\mathcal{A}$-network $(V; f)$ is called atomic if the image of $f$ only contains atoms and if

$$f(a, c) \leq f(a, b) \circ f(b, c)$$
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- **square** if $\mathbf{1}_\mathcal{B} = \mathcal{B}^2$;
- **homogeneous** if every isomorphism of finite substructures of $\mathcal{B}$ can be extended to an automorphism;
- **normal** if it is fully universal, square and homogeneous.
NSP as CSP

Definition

Let $A$ be a $\tau$-structure. The Constraint Satisfaction Problem of $A$ is to decide for a given finite $\tau$-structure $C$ whether there exists a homomorphism from $C$ to $A$.
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**Proposition**

Let $\mathcal{A}$ be a finite relation algebra with normal representation $\mathcal{A}$. Then $\mathcal{A}$ is finitely bounded and $\text{NSP}(\mathcal{A})$ equals $\text{CSP}(\mathcal{A})$ (up to some cosmetic differences in the formalisation) and is therefore in $\text{NP}$.

Remark: There exists a finite relation algebra with undecidable NSP (Hirsch 1999)!
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Result restated

**Theorem**

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation $\Gamma$ and $\text{CSP}(\Gamma)$ is in $\Pi \text{ or } \text{NP-complete}$. Moreover, it is decidable which of the two cases holds.

*Proof:*

- **Finitely bounded structures:** Nice description for $\Gamma$.
- **Fraisse’s Theorem:** $\Gamma$ exists, because of free amalgamation.
- **Universal algebra:** Study homomorphisms $\Gamma_n \to \Gamma$ (Polymorphisms).
- **Ramsey theory:** $\Gamma$ with a generic order is a Ramsey structure by a result of Hubiška and Nešetřil (2016).
- **Finite-domain CSP:** Use the Bulatov-Zhuk Dichotomy Theorem (2017) for tractability results.
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Finitely Bounded Structures

**Definition**

Let $F$ be a finite set of finite $\tau$-structures. $\text{Forb}(F)$ is the class of all finite $\tau$-structures that embed no $B \in F$.

A class $C$ of finite $\tau$-structures is called **finitely bounded** if $C = \text{Forb}(F)$ for a finite set $F$. 

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Observation
Let $A$ be a finite relation algebra. The class of atomic $A$-networks (considered as structures) is finitely bounded by all forbidden (with respect to $\circ$) triangles.
Theorem

Let $A$ be a finite relation algebra with a flexible atom. Then $A$ has a normal representation $\Gamma$ and $\text{CSP}(\Gamma)$ is in P or NP-complete. Moreover, it is decidable which of the two cases holds.

Proof:

- Finitely bounded structures: Nice $\Gamma$.
- Fraisse’s Theorem: $\Gamma$ exists, because of free amalgamation.
- Universal algebra: Study homomorphisms $\Gamma^n \rightarrow \Gamma$ (Polymorphisms).
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Fraisse’s Theorem

Theorem

Let $\tau$ be a finite signature and let $C$ be an amalgamation class. Then there exists a unique countable $\tau$-structure $F$ which is homogeneous and the age of $F$ is exactly $C$. 

(I) Isomorphism-closed: For every $A \in C$ every isomorphic copy $A$ is also in $C$.

(HP) Hereditary property: For $A \in C$ and an arbitrary substructure $B$ of $A$ the structure $B$ is in $C$.

(AP) Amalgamation property: For $A, B, C \in C$ and embeddings $e : A \to B$ and $f : A \to C$ there exists a structure $D \in C$ and embeddings $g : B \to D$ and $h : C \to D$ such that $g \circ e = h \circ f$. 

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Amalgamation

Examples
All finite linear orders, all finite undirected graphs,...
Amalgamation

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Proposition
Let \( \mathcal{A} \) be a finite relation algebra with a flexible atom. Then \( \mathcal{A} \) has a normal representation \( \Gamma \) (and CSP(\( \Gamma \)) is in NP).

Proof:
The class of atomic \( \mathcal{A} \)-networks is an amalgamation class by free amalgamation with the flexible atom. Fraisse’s Theorem states that a countable, homogeneous limit structure \( \Gamma \) exists and is unique up to isomorphism. This structure is fully-universal since the age of \( \Gamma \) contains all atomic \( \mathcal{A} \)-networks.
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Universal Algebra

Definition

A function $f : A^k \to A$ preserves a relation $R \subseteq A^s$ if for every $r_1, \ldots, r_k \in R$ the tuple $(f(r_1^1, \ldots, r_k^1), \ldots, f(r_1^s, \ldots, r_k^s))$ is in the relation $R$.

A function $f : A^k \to A$ is called a polymorphism of a $\tau$-structure $\mathbb{A}$ if $f$ preserves every relation $R_i^A$ in $\mathbb{A}$. 

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$$
\begin{align*}
  f( \bullet & \bullet) = \bullet \\
  f( \bullet & \bullet \ldots \bullet) = \bullet \\
  \vdots & \vdots \\
  f( \bullet & \bullet) = \bullet \\
  \in R & \in R \quad \in R \quad \Rightarrow \quad \in R
\end{align*}
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Proposition

Let \( A \) and \( B \) be \( \omega \)-categorical structures. If \( \text{Pol}(A) \subseteq \text{Pol}(B) \) holds, then there exists a polynomial-time reduction from \( \text{CSP}(B) \) to \( \text{CSP}(A) \).
Canonical Polymorphisms

Motivation: Reduction of infinite-domain CSPs to finite-domain CSPs.

Definition

Let $X$ be a subset of the set of atomic relations $Y$ of $\Gamma$. Then a polymorphism $\Gamma_n \rightarrow \Gamma$ is called $X$-canonical if it induces a function $X_n \rightarrow Y$.

If $X$ is the set of all atomic relations, $f$ is called canonical.

Observation

A polymorphism of a normal representation is edge conservative: Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in V$ with $X_i(a_i, b_i)$ for atomic relations $X_i$, then $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in X_1 \cup \ldots \cup X_n$. 
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Theorem

Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. If $\text{CSP}(\Gamma)$ is not NP-complete then for every two atomic relations $A$ and $B$ there exists an $\{A, B\}$-canonical polymorphism $f_{A,B}$ of $\Gamma$ such that the induced function on $\{A, B\}$ is one of the following:

- A binary symmetric function;
- The Boolean majority function;
- The Boolean minority function.

We call these functions of Schaefer-type.
Analysis of Atomic Relations

**Theorem**

Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. If $\text{CSP}(\Gamma)$ is not NP-complete then for every two atomic relations $A$ and $B$ there exists an $\{A, B\}$-canonical polymorphism $f_{A,B}$ of $\Gamma$ such that the induced function on $\{A, B\}$ is one of the following:

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We call these functions of Schaefer-type.

These functions give tractability results for finite-domain CSPs!
Result restated

Theorem

Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Then $\mathcal{A}$ has a normal representation $\Gamma$ and $\text{CSP}(\Gamma)$ is in P or NP-complete. Moreover, it is decidable which of the two cases holds.

Proof:

- Finitely bounded structures: Nice description for $\Gamma$.
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Ramsey Theory

The essence of this part is the following deletion:

Theorem
Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. If $CSP(\Gamma)$ is not NP-complete then for every two atomic relations $A$ and $B$ there exists a $\{A, B\}$-canonical polymorphism $f_{A,B}$ of $\Gamma$ such that the induced function on $\{A, B\}$ is one of the following:

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**Theorem**

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Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. If $\text{CSP}(\Gamma)$ is not NP-complete then for every two atomic relations $A$ and $B$ there exists a canonical polymorphism $f_{A,B}$ of $\Gamma$ such that the induced function on $\{A, B\}$ is one of the following:

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Reduction to Finite-Domain CSP

Theorem (Bodirsky and Mottet ’18 + Finite-domain Dichotomy ’17)

Let \( \Gamma \) be a finitely bounded homogeneous structure and suppose that \( \Gamma \) has a Siggers polymorphism \( f \) modulo operations from \( \text{End}(\Gamma) \) such that \( f \) is canonical. Then \( \text{CSP}(\Gamma) \) is in \( \text{P} \).
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Let $\Gamma$ be a finitely bounded homogeneous structure and suppose that $\Gamma$ has a Siggers polymorphism $f$ modulo operations from $\text{End}(\Gamma)$ such that $f$ is canonical. Then CSP($\Gamma$) is in P.

Definition

A function $f$ is Siggers modulo operations from $\text{End}(\Gamma)$ if there exist $e_1, e_2 \in \text{End}(\Gamma)$ such that the following holds:

$$\forall x, y, z : e_1(f(x, y, x, z, y, z)) = e_2(f(z, z, y, y, x, x))$$
**Reduction to Finite-Domain CSP**

**Theorem (Bodirsky and Mottet ’18 + Finite-domain Dichotomy ’17)**

Let $\Gamma$ be a finitely bounded homogeneous structure and suppose that $\Gamma$ has a Siggers polymorphism $f$ modulo operations from $\text{End}(\Gamma)$ such that $f$ is canonical. Then $\text{CSP}(\Gamma)$ is in P.

**Proposition**

Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. Assume that $\Gamma$ has for every two atomic relations $A$ and $B$ a polymorphism that is canonical and of Schaefer-type on $\{A, B\}$. Then $\Gamma$ has a polymorphism that is Siggers modulo operations from $\text{End}(\Gamma)$. 
Thank you for your attention!
Result

Theorem

Let $\Gamma$ be a normal representation of a finite relation algebra with a flexible atom. One of the following holds:

1. There exists for every two atoms $A$ and $B$ of the algebra a polymorphism $f_{A,B}$ of $\Gamma$ that is canonical and the induced function on $\{A, B\}$ is of Schaefer-type, then $\Gamma$ has a canonical pseudo-Siggers polymorphism. Then CSP($\Gamma$) is in P.

2. CSP($\Gamma$) is NP-complete.