Constraint Satisfaction Problems of First-Order Expansions of Algebraic Products

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Constraint Satisfaction Problems

(relational) structure $\mathfrak{A} = (A; R^{\mathfrak{A}} : R \in \tau)$; **finite** signature τ

Definition (CSP)

$\mathfrak{B}$ – τ-structure

Constraint Satisfaction Problem for $\mathfrak{B}$ (CSP($\mathfrak{B}$)):

Input: finite τ-structure $\mathfrak{A}$

Question: Is there a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$?

Example: complete graph on 3 vertices

$$K_3 = (\{0, 1, 2\}; \neq)$$

CSP(K_3) = 3-colorability problem for graphs

more generally: CSP(K_n) = n-colorability problem

Complexity dichotomy

Theorem (Bulatov (2017), Zhuk (2017))

For every finite structure $\mathfrak{B}$ with finite signature, $\mathrm{CSP}(\mathfrak{B})$ is in P or NP-complete.

Conjecture (Bodirsky, Pinsker (2011))

For a reduct $\mathfrak{B}$ of a finitely bounded homogeneous structure, $\mathrm{CSP}(\mathfrak{B})$ is in P or NP-complete.

Interesting infinite examples in the scope of the conjecture:
fo-expansions of (algebraic powers of) $(\mathbb{Q}; <)$

Primitive positive interpretations

primitive positive formula: $\exists y_1, \dots, y_l(\psi_1 \wedge \cdots \wedge \psi_m)$, ψ_i atomic formulas
Example: $\phi(x, y) = \exists z\, R(x, y, z) \wedge R(x, x, z)$

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Definition (pp-interpretation)

Primitive positive interpretation of $\mathfrak{C}$ in $\mathfrak{B}$:
a *partial surjection* I from B^d to C (for some d) such that for every k-ary *relation* R defined by an *atomic formula* in $\mathfrak{C}$, $I^{-1}(R)$ as a dk-ary *relation* over B is *pp-definable* in $\mathfrak{B}$

Example: closed intervals $[a, b]$ over $\mathbb{Q}$ are elements of $\mathbb{Q}^2$ such that $a < b$

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Proposition (folklore)

If $\mathfrak{C}$ has a pp-interpretation (in particular, pp-definition) in $\mathfrak{B}$, then there is a poly-time reduction from $\mathrm{CSP}(\mathfrak{C})$ *to* $\mathrm{CSP}(\mathfrak{B})$.

Cardinal Direction Calculus

- $\mathfrak{C} = (\mathbb{Q}^2; \mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}, \mathsf{NE}, \mathsf{SE}, \mathsf{SW}, \mathsf{NW})$ (North, East, etc.)

N	E	S	W	NE	SE	SW	NW
$(=,>)$	$(>,=)$	$(=,<)$	$(<,=)$	$(>,>)$	$(>,<)$	$(<,<)$	$(<,>)$

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- denote $(<,\top)$ by $<_1$ and similarly for $=_1, <_2, =_2$
- these relations are pp-definable in $\mathfrak{C}$
- view fo-expansions of $\mathfrak{C}$ as fo-expansions of $(\mathbb{Q}^2; <_1, =_1, <_2, =_2)$

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- CDC: relations are unions of the relations above – fo-expansions of $\mathfrak{C}$
- natural generalization: CDC_n with the domain $\mathbb{Q}^n$

Open problem (Balbiani, Condotta, 2002): complexity classification of the CSPs of reducts of CDC_n

$\longrightarrow$ we solve it by classifying fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

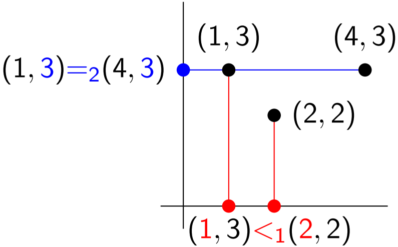
Algebraic products

Definition (algebraic product)

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures with signatures τ_1 and τ_2, respectively. The algebraic product $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is the structure with the domain $A_1 \times A_2$ which has the following relations:

- for every $R \in \tau_1 \cup \{=\}$, the relation $R_1 = (R, \top)$;
- for every $R \in \tau_2 \cup \{=\}$, the relation $R_2 = (\top, R)$.

Example: $(\mathbb{Q}; <) \boxtimes (\mathbb{Q}; <) = (\mathbb{Q}^2; <_1, =_1, <_2, =_2)$



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$\longrightarrow$ natural generalization to n-fold algebraic products

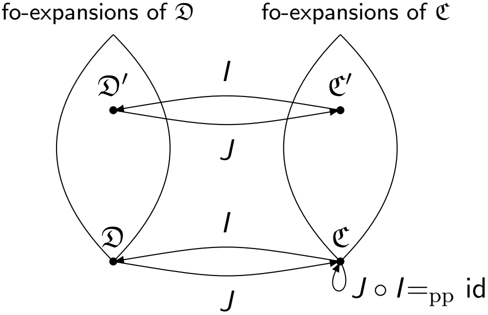
Observation: Complexity classification of CSPs of fo-expansions of

$$\underbrace{(\mathbb{Q}; <) \boxtimes \cdots \boxtimes (\mathbb{Q}; <)}_{n} = (\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$$

leads to classification for reducts of CDC_n!

Complexity classification transfer

- I – pp-interpretation of $\mathfrak{D}$ in $\mathfrak{C}$
- J – pp-interpretation of $\mathfrak{C}$ in $\mathfrak{D}$
- $J \circ I$ is pp-homotopic to the identity interpretation of $\mathfrak{C}$ (i.e., $\{(\bar{x}, \bar{y}) \mid J \circ I(\bar{x}) = \bar{y}\}$ is pp-definable in $\mathfrak{C}$)



$\Rightarrow$ for every fo-expansion $\mathfrak{C}'$ of $\mathfrak{C}$ there is an fo-expansion $\mathfrak{D}'$ of $\mathfrak{D}$ such that CSP($\mathfrak{C}'$) and CSP($\mathfrak{D}'$) are poly-time equivalent

Allen's Interval Algebra and Block Algebra

Allen's Interval Algebra:

- $\mathbb{I} = \{(a, b) \in \mathbb{Q}^2 \mid a < b\}$ – closed intervals
- 13 basic relations correspond to relative positions of intervals, e.g.:

```
s(X,Y):   XXX          f(X,Y):      XXX      m(X,Y):   XXXX
starts    YYYYYY       finishes  YYYYYY      meets         YYYY
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Block Algebra:

- domain: $\mathbb{I}^n$
- basic relations: n-tuples of Allen's basic relations
- all relations: unions of basic relations

Complexity classification transfer for Block Algebras

Open problem (Balbiani, Condotta, del Cerro, 1999 ($n = 2$) and 2002 ($n \geq 2$)): complexity classification of the CSPs of reducts of the n-dim. Block Algebra

Solution:

- Block Algebra with the basic relations is pp-interpretable in $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$ and vice versa
- all relations are fo-definable in basic relations
- we solve the problem by transfering the complexity classification for fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

Polymorphisms

Definition (polymorphism)

An operation $f : A^k \to A$ is a polymorphism of (or preserves) a structure $\mathfrak{A}$ if for every relation R of $\mathfrak{A}$ and for all tuples $\bar{r}_1, \dots, \bar{r}_k \in R$ also $f(\bar{r}_1, \dots, \bar{r}_k) \in R$ (computed row-wise).
Pol($\mathfrak{A}$) – the set of all polymorphisms of $\mathfrak{A}$

Example: $+$ is a polymorphism of $(\mathbb{Q}; <)$

$$\begin{pmatrix} 1 \\ \wedge \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ \wedge \\ 3 \end{pmatrix} \begin{matrix} \to \\ \\ \to \end{matrix} \begin{pmatrix} 3 \\ \wedge \\ 8 \end{pmatrix}$$

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Theorem (Bodirsky, Nešetřil (2006))

A relation $R \subseteq A^l$ is preserved by all polymorphisms of an ω-categorical structure $\mathfrak{A}$ iff R is has a pp-definition in $\mathfrak{A}$.

Complexity of CSPs of (fo-expansions) of alg. products

$\mathfrak{A}_1, \mathfrak{A}_2$ – countable ω-categorical structures
$\mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) = \mathrm{Pol}(\mathfrak{A}_1) \times \mathrm{Pol}(\mathfrak{A}_2) \Rightarrow$ the complexity of the CSP (of an fo-expansion) of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is related to "the complexity in each dimension"

Proposition

If $\mathrm{CSP}(\mathfrak{A}_1)$ *is in P and* $\mathrm{CSP}(\mathfrak{A}_2)$ *is in P, then* $\mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$ *is in P.*

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$\theta_i : \mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) \to \mathrm{Pol}(\mathfrak{A}_i)$ (projects on the i-th coordinate)

Follows from the results by Barto, Opršal, Pinsker (2018):

Proposition

Let $\mathfrak{D}$ *be an fo-expansion of* $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$*. Let* i *be such that* $\theta_i(\mathrm{Pol}(\mathfrak{D}))$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$*. Then* $\mathrm{Pol}(\mathfrak{D})$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$ *as well and* $\mathrm{CSP}(\mathfrak{D})$ *is NP-complete.*

CSPs of fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

pwnu polymorphism = pseudo weak near unaminity polymorphism

Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ be an fo-expansion of $(\mathbb{Q}; <)$. If $\mathfrak{B}$ contains a pwnu polymorphism, then $\mathrm{CSP}(\mathfrak{B})$ is in P. Otherwise, $\mathrm{Pol}(\mathfrak{B})$ has a uniformly continuous minor-preserving map to $\mathrm{Pol}(K_3)$ and $\mathrm{CSP}(\mathfrak{B})$ is NP-complete.

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Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ be an fo-expansion of $(\mathbb{Q}; <)$. If $\mathfrak{B}$ contains a pwnu polymorphism, then CSP($\mathfrak{B}$) *is in P. Otherwise,* Pol($\mathfrak{B}$) *has a uniformly continuous minor-preserving map to* Pol(K_3) *and* CSP($\mathfrak{B}$) *is NP-complete.*

Theorem (Bodirsky, Jonsson, Martin, Mottet, S. (2022))

Let $\mathfrak{D}$ be an fo-expansion of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$. Exactly one of the following two cases applies.

- θ_i(Pol($\mathfrak{D}$)) *contains a pwnu polymorphism for each i. In this case $\mathfrak{D}$ has a pwnu polymorphism and* CSP($\mathfrak{D}$) *is in P.*
- *There is i such that* θ_i(Pol($\mathfrak{D}$)) *has a uniformly continuous minor-preserving map to* Pol(K_3) *and* CSP($\mathfrak{D}$) *is NP-complete.*

Proof idea for $n = 2$

NP-complete:

- follows directly from the previous proposition

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P:

- relations of $\mathfrak{D}$ are defined by fo-formulas in $<_i$ and $=_i$
- we may assume quantifier-free definitions in conjunctive normal form
- key: have conjunctions of clauses which are (almost) i-determined (contains literals only with index i)
- aim is to run the poly-time algorithm to decide satisfiability of:
  1. the 1-determined constraints
  2. the (possibly modified) 2-determined constraints
- existence of such poly-time algorithms follows from the theorem for $(\mathbb{Q}; <)$

What is next

Classify the complexity of:

- CSPs of (reducts) of fo-expansions of

$$\underbrace{(\{0,1\};\{0\},\{1\}) \boxtimes \cdots \boxtimes (\{0,1\};\{0\},\{1\})}_{n} \boxtimes (\mathbb{Q};<)$$

  for $n = 1$ and general n
- more generally: CSPs of fo-expansions of $\mathfrak{B} \boxtimes (\mathbb{Q};<)$, where $\mathfrak{B}$ is a finite structure
- challenge: CSPs of structures fo-interpretable over $(\mathbb{Q};<)$

All of the above is in the scope of the infinite-domain dichotomy conjecture.

Thank you for your attention