

Valued Constraint Satisfaction Problem and Resilience in Database Theory

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Resilience of queries

Database: a relational structure \mathfrak{A}

Conjunctive query: a primitive positive formula q , i.e.

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Fixed conjunctive query q .

Input: a finite database \mathcal{A}

Output: minimal number of tuples to be removed from relations of \mathcal{A} , so that $\mathcal{A} \not\models q$

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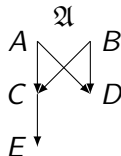
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Example: The resilience of

$$q = \exists x, y, z(\text{parent}(x, y) \wedge \text{parent}(y, z))$$

with respect to \mathfrak{A} is 1 – remove either (A, C) or (C, E) .

Goal: Classify complexity of resilience for all q .



Constraint satisfaction

Fixed τ -structure \mathfrak{A} (τ – finite relational signature)

Input: list of atomic τ -formulas (constraints)

Output:

- **CSP:** Decide whether there is a solution that satisfies **all** constraints.
- **MaxCSP:** Find the **maximal number** of constraints that can be satisfied at once.
- **VCSP:** Find the **minimal cost** with which the constraints can be satisfied (each constraint comes with a cost depending on the chosen values).

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Observation: VCSP **generalizes** CSP and MaxCSP.

Proof: Model the tuples in relations with cost 0 and outside with cost

- 1 and the same threshold (for MaxCSP);
- ∞ and threshold 0 (for CSP).

A valued structure Γ consists of:

- (countable) domain D
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k , a function $R^\Gamma: D^k \rightarrow \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \dots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ -formula

Question: Is

$$\inf_{\bar{a} \in D^n} \phi(\bar{a}) \leq u \text{ in } \Gamma?$$

Max-Cut as a VCSP

Example:

Input: $G = (V, E)$ – finite directed graph

Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.

Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ be a valued structure where:

- $D = \{0, 1\}$
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$$E(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 1 \\ 1 & \text{otherwise} \end{cases}$$

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Take vertices of G as variables. The **size of a maximal cut** of G is

$$\min_{\bar{x} \in D^n} \sum_{(x_i, x_j) \in E} E(x_i, x_j). \text{ The partition of } V \text{ is given by the values 0 and 1.}$$

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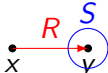
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Every instance of $\text{VCSP}(\Gamma)$ corresponds to a **digraph**, hence $\text{VCSP}(\Gamma)$ is the **Max-Cut** problem (NP-hard).

Homomorphism duality

Example (canonical structure): $\exists x, y(R(x, y) \wedge S(y)) \rightsquigarrow$ 

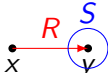
For a query q , take its canonical structure Ω .

Search for a structure \mathfrak{B}_q such that for every finite \mathfrak{A} :

$$\mathfrak{A} \not\models q \Leftrightarrow \Omega \not\rightarrow \mathfrak{A} \Leftrightarrow \mathfrak{A} \rightarrow \mathfrak{B}_q$$

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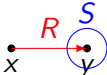
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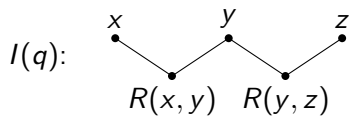
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\rightsquigarrow existence of \mathfrak{B}_q enables studying resilience of q using the results about (valued) constraint satisfaction problems

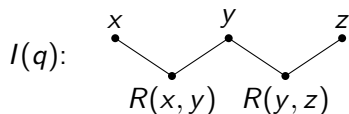
Existence of dual structures

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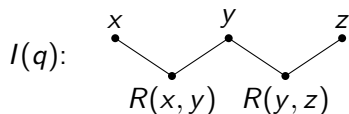


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Theorem (Cherlin, Shelah, Shi ('99))

If $I(q)$ is *connected*, then q has a countable dual \mathfrak{B}_q , which can be chosen so that $\text{Aut}(\mathfrak{B}_q)$ is *oligomorphic*.

oligomorphic – countable domain B_q and the action of $\text{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

Connection of resilience and VCSPs

query q with $I(q)$ connected (WLOG) \rightsquigarrow obtain the dual structure $\mathfrak{B}_q \rightsquigarrow$
turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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Example: Input $R(x, y) + R(x, y)$ for VCSP(Γ) corresponds to a database with multiplicity 2 for $R(x, y)$.

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Example: Input $R(x, y) + R(x, y)$ for $\text{VCSP}(\Gamma)$ corresponds to a database with multiplicity 2 for $R(x, y)$.

Corollary: $I(q)$ is a tree $\Rightarrow \Gamma_q$ can be taken finite

\rightsquigarrow complexity dichotomy for finite-domain VCSPs applies

\Rightarrow the resilience problem for q is in P or NP-complete (holds even when $I(q)$ is acyclic)

Hard resilience problems

pp-construction – a notion of ‘expressing’ one valued structure in another
(generalizes pp-constructions for classical structures)

Fact: If $\text{Aut}(\Gamma)$ is **oligomorphic** and Γ **pp-constructs** Δ , then $\text{VCSP}(\Delta)$ **reduces** to $\text{VCSP}(\Gamma)$ in **poly-time**.

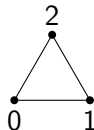
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K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{if } x = y \end{cases}$$



Observation: $\text{VCSP}(K_3)$ is the 3-colorability problem and hence NP-hard.

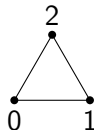
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Fractional polymorphisms

polymorphism f of \mathfrak{B} – an operation $f : B^n \rightarrow B$ that preserves all relations of \mathfrak{B}

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Definition (fractional polymorphism)

Γ – valued τ -structure with domain D

A **fractional polymorphism** of Γ of arity n is a **probability distribution** ω on operations $D^n \rightarrow D$ such that for every k -ary $R \in \tau$ and $a^1, \dots, a^n \in D^k$

$$\underbrace{E_{\omega}[f \mapsto R(f(a^1, \dots, a^n))]}_{\text{expected value}} \leq \underbrace{\frac{1}{n} \sum_{j=1}^n R(a^j)}_{\text{arithmetic mean}} .$$

Tractability conjecture

Known for finite-domain VCSPs:

Theorem

Γ – a *finite-domain* valued structure

- If Γ does not *pp-construct* K_3 , then Γ has *cyclic fractional polymorphism* of arity ≥ 2 (essentially Kozik, Ochremiak ('15)).
- If Γ has a *cyclic fractional polymorphism* of arity ≥ 2 , then $\text{VCSP}(\Gamma)$ is in P (Kolmogorov, Krokhin, Rolínek ('15)).

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If Γ_q has a *fractional polymorphism* of arity ≥ 2 which is *canonical* and *pseudo-cyclic* with respect to $\text{Aut}(\Gamma_q)$, then $\text{VCSP}(\Gamma_q)$ is in P .

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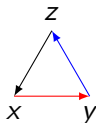
Conjecture: If Γ_q does not pp-construct K_3 , then the tractability theorem applies and $\text{VCSP}(\Gamma_q)$ and hence resilience of q is in P .

Examples

Example (hardness):

$$q := \exists x, y, z (R(x, y) \wedge S(y, z) \wedge T(z, x))$$

- resilience of q is known to be NP-hard (Freire, Gatterbauer, Immerman, Meliou ('15))
- Γ_q pp-constructs K_3

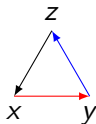


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Example (tractability):

$$q := \exists x, y (R(x, y) \wedge R(y, y) \wedge R(y, x) \wedge S(x))$$

- complexity left open in Freire, Gatterbauer, Immerman, Meliou ('20)
- Γ_q has a **canonical** and **pseudo-cyclic fractional polymorphism**



Thank you for your attention

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