Valued Constraint Satisfaction Problem and Resilience in Database Theory

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The Constraint Satisfaction Problem: Complexity and Approximability
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Outline

1. Connection between resilience and valued CSPs
2. Complexity results for resilience (new results for digraphs!)
3. Valued cores for resilience

Resilience of queries

database – a relational structure $\mathfrak{A}$

conjunctive query – a formula q of the form $\exists y_1, \ldots, y_l \, (\psi_1 \wedge \cdots \wedge \psi_m)$, where ψ_i are atomic

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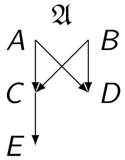
Example: The resilience of

$$q = \exists x, y, z (R(x, y) \wedge R(y, z))$$

with respect to $\mathfrak{A}$ is 1 – remove (C, E).



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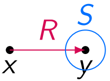
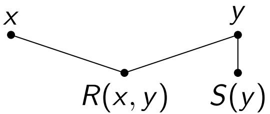
Goal: Classify complexity of resilience for all q.

Translation to a dual problem

Example:
$q := \exists x, y(R(x,y) \wedge S(y))$



canonical structure



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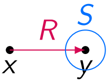

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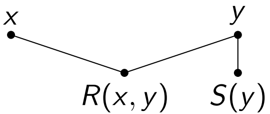

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Theorem (Cherlin, Shelah, Shi '99)

Let q be a query and $\mathfrak{Q}$ its canonical structure. If $I(q)$ is connected, then there exists a dual structure $\mathfrak{B}_q$, such that for every finite $\mathfrak{A}$:

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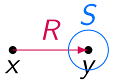

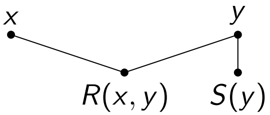

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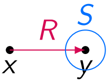

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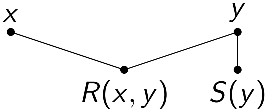

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- *$\mathfrak{B}_q$ can be chosen finite iff q is homomorphically equivalent to q' such that $I(q')$ is a tree. (Nešetřil, Tardiff '00; Larose, Loten, Tardiff '07)*

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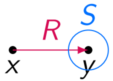

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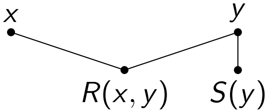

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oligomorphic – countable domain B_q and the action of $\mathrm{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

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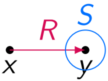

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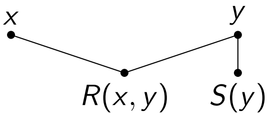

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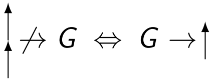
Example: For every finite directed graph G we have:



Connection to min-CSPs

Resilience of q:
Input: a finite database $\mathfrak{A}$, $u \in \mathbb{N}$
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More flexible approach: valued CSPs

Valued CSPs

A valued structure Γ consists of:

- (countable) domain C
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^{\Gamma}\colon C^k \to \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Question: Is

$$\inf_{t \in C^n} \phi(t) \leq u \text{ in } \Gamma?$$

Example: Max-Cut as a VCSP

Input: $G = (V, E)$ – finite directed (multi)graph
Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.
Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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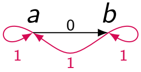
Let Γ_{MC} be a valued structure where:

- $C = \{a, b\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = a \text{ and } y = b \\ 1 \text{ otherwise} \end{cases}$$



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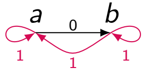

Take vertices of G as variables. The size of a maximal cut of G is

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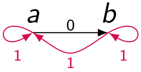

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every instance of VCSP(Γ_{MC}) corresponds to a directed multigraph
$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Connection of resilience and VCSPs

query q (WLOG $I(q)$ connected) $\rightsquigarrow$ dual $\mathfrak{B}_q$ $\rightsquigarrow$ 0-1-valued structure Γ_q

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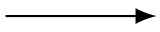

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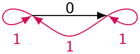

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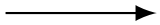
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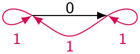

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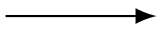
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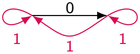

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Remark: We have to consider bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1 (differs from the original setting).

Example: Input $R(x,y) + R(x,y)$ for VCSP(Γ) corresponds to a database with multiplicity 2 for $R(x,y)$.

Complexity dichotomy for resilience of acyclic queries

Recall: $I(q)$ is a tree $\Rightarrow$ Γ_q can be taken finite

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Question: For general queries, choose $\mathfrak{B}_q$ with oligomorphic Aut($\mathfrak{B}_q$).
Can we use adapt some results for finite domains?

Fractional homomorphisms

Definition

Let Γ and Δ be valued τ-structures with domains C and D, respectively. A fractional homomorphism from Δ to Γ is a probability distribution ω on the maps $f : D \to C$ such that for every k-ary $R \in \tau$ and tuple $t \in D^k$

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- Dirac fractional homomorphism – if $\omega(f) = 1$ for some $f : D \to C$

Fractional polymorphisms

$\Gamma^\ell = (C^\ell; (R^{\Gamma^\ell})_{R \in \tau})$ where

$$R^{\Gamma^\ell}((t_1^1, \dots, t_\ell^1), \dots, (t_1^k, \dots, t_\ell^k)) := \frac{1}{\ell} \sum_{i=1}^{\ell} R^\Gamma(t_i^1, \dots, t_i^k).$$

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Theorem (Kolmogorov, Krokhin, Rolínek '15)

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Theorem (Bodirsky, S., Lutz '24)

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K_3 – complete graph on 3 vertices, viewed as a 0-∞ valued structure

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Conjecture: If Γ_q does not pp-construct K_3, then the tractability theorem applies and $\text{VCSP}(\Gamma_q)$ and hence resilience of q is in P.

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Let q be a minimal connected conjunctive query over the signature $\{R\}$. Then either q is equal to q_ℓ, q_1 or q_c and the resilience of q is in P, or the resilience problem of q is NP-complete.

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Assume oligomorphic automorphism groups.

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Theorem (Bodirsky, Jahel, S. '25)

Let Γ be a valued structure such that all of its valued relations attain values from $\{0, 1, \infty\}$. Then there exists a valued core Δ with $\mathrm{Aut}(\Delta)$ oligomorphic such that:

- *Δ is a substructure of Γ;*
- *Γ and Δ are fractionally homomorphically equivalent and the fractional homomorphisms can be chosen to be Dirac;*
- *every valued core which is frac. hom. equiv. to Γ is isomorphic to Δ.*

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Question: Can the results on valued cores be generalized to all valued structures with an oligomorphic automorphism group?

Thank you for your attention

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