Valued Constraints over Infinite Domains

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- **least correlation clustering**

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P = class of efficiently solvable problems
NP = class of problems with efficiently verifiable solution
NP-complete problems = hardest problems in NP

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Observation: VCSP generalizes CSP and MinCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MinCSP) or ∞ (for CSP).

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \dots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Output: Is

$$\inf_{t \in D^n} \phi(t) \leq u \text{ in } \Gamma?$$

Example: Max-Cut as a VCSP

Input: $G = (V, E)$ – finite directed (multi)graph
Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.
Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $D = \{0, 1\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

$\min_{x \in D^n} \sum_{(x_i, x_j) \in E} R(x_i, x_j)$. The partition of V is given by the values 0 and 1.

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every instance of VCSP(Γ_{MC}) corresponds to a directed multigraph
$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Revisiting problems from the start

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  $\hookrightarrow$ not obvious how to model as a VCSP

Complexity of VCSPs

Theorem (Kozik, Ochremiak '15; Kolmogorov, Rolínek, Krokhin '15; Bulatov '17; Zhuk '17)

Let Γ be a valued structure with a finite domain. Then $\mathrm{VCSP}(\Gamma)$ *is in P or NP-complete.*

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Definition

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- automorphism of Γ – permutation α of C such that for $R \in \tau$ of arity k and every $t \in C^k$, $R(\alpha(t)) = R(t)$

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Example: $\mathrm{Aut}(\mathbb{Q}; (<)_0^1) = \mathrm{Aut}(\mathbb{Q}; <)$ is oligomorphic.

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Fact (Bodirsky, S., Lutz '24): If $\text{Aut}(\Gamma)$ is oligomorphic and $R \in \langle \Gamma \rangle$, $\text{VCSP}(\Gamma; R)$ reduces to $\text{VCSP}(\Gamma)$ in poly-time.

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pp-construction – a notion of 'translating' relations of one valued structure into relations of another (generalizes expressibility to different domains)

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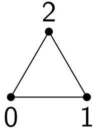
Fact: If Aut(Γ) is oligomorphic and Γ pp-constructs Δ, then VCSP(Δ) reduces to VCSP(Γ) in poly-time.

K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ \infty \text{ if } x = y \end{cases}$$



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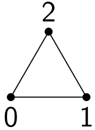

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Corollary (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\Gamma)$ *is oligomorphic and* Γ *pp-constructs* K_3*, then* $\mathrm{VCSP}(\Gamma)$ *is NP-hard.*

Fractional polymorphisms

polymorphism of a relational structure $\mathfrak{A}$ – $f : A^n \to A$ such that for all relations R of $\mathfrak{A}$ and $t^1, \dots, t^n \in R$, $f(t^1, \dots, t^n) \in R$ (applied row-wise)

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Example: The operation min is a polymorphism of $(\mathbb{Q}; <)$.

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Definition (fractional polymorphism)

A fractional polymorphism of Γ of arity n is a probability distribution ω on the maps $f\colon C^n \to C$ such that for every k-ary $R \in \tau$ and $t^1, \ldots, t^n \in C^k$

$$E_\omega[f \mapsto R(f(t^1, \ldots, t^n))] \leq \frac{1}{n} \sum_{j=1}^{n} R(t^j) \quad (\omega \text{ improves } R).$$

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Proposition (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\Gamma)$ *is oligomorphic and* $R \in \langle \Gamma \rangle$, *then* $\mathrm{fPol}(\Gamma)$ *improves* R.

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Equality and temporal valued structures

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A relational structure $\mathfrak{A}$ is

- an equality structure if $\mathfrak{A}$ is fo-definable in $(\mathbb{Q};=)$ $\Leftrightarrow$ $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q};=) = \mathrm{Sym}(\mathbb{Q})$;

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- temporal: $(\mathbb{Q};(<)^1_0)$ (models minimum feedback arc set problem)

Classification of equality VCSPs

Known for CSPs:

Theorem (Bodirsky, Kára '08)

If $\mathfrak{A}$ is an equality relational structure, then exactly one of the following:

- $\mathrm{Pol}(\mathfrak{A})$ *contains a unary constant operation or a binary injection and* $\mathrm{CSP}(\mathfrak{A})$ *is in P.*
- $\mathfrak{A}$ *pp-constructs* K_3 *and* $\mathrm{CSP}(\mathfrak{A})$ *is NP-complete.*

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$\hookrightarrow$ the considered probability distributions put all weight on one operation

Classification of temporal CSPs

Theorem (Bodirsky, Kára '10)

Let $\mathfrak{A}$ be a temporal relational structure. Then exactly one of the following holds:

- *At least one of the operations* const, min, mx, mi, ll, *or one of their duals lies in* $\mathrm{Pol}(\mathfrak{A})$ *and* $\mathrm{CSP}(\mathfrak{A})$ *is P.*
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Classification of temporal CSPs

Theorem (Bodirsky, Kára '10)

Let $\mathfrak{A}$ be a temporal relational structure. Then exactly one of the following holds:

- *At least one of the operations* const, min, mx, mi, ll, *or one of their duals lies in* $\mathrm{Pol}(\mathfrak{A})$ *and* $\mathrm{CSP}(\mathfrak{A})$ *is P.*
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$\hookrightarrow$ const is the unary constant 0 operation

$\hookrightarrow$ the remaining polymorphisms are tailored to the structure $(\mathbb{Q}; <)$

Temporal valued structures

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Corollary (of the proof): Given a temporal valued structure Γ, it is decidable whether VCSP(Γ) is in P or NP-complete.

Outline

1. Introduction to VCSPs
2. Tools for VCSPs
3. Temporal VCSPs
4. Resilience problems
5. Outlook to the future

Resilience of queries

database – a relational structure $\mathfrak{A}$

conjunctive query – a formula q of the form $\exists y_1, \dots, y_l \, (\psi_1 \wedge \cdots \wedge \psi_m)$, where ψ_i are atomic

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Fixed conjunctive query q.
Input: a finite database $\mathfrak{A}$, $u \in \mathbb{N}$
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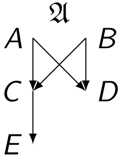
Example: The resilience of

$$q = \exists x, y, z(R(x, y) \wedge R(y, z))$$

with respect to $\mathfrak{A}$ is 1 – remove (C, E).



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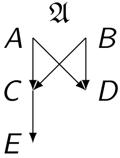
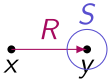


Goal: Classify complexity of resilience for all q.

Homomorphism duality

Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

For a query q, take its canonical structure $\mathfrak{Q}$.
Search for a structure $\mathfrak{B}_q$ such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \Leftrightarrow \mathfrak{Q} \not\to \mathfrak{A} \Leftrightarrow \mathfrak{A} \to \mathfrak{B}_q$$

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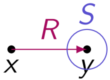
Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

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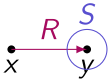
Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

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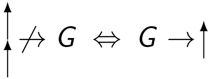
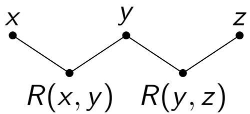


$\rightsquigarrow$ existence of $\mathfrak{B}_q$ enables studying resilience of q using the results about (valued) constraint satisfaction problems

Existence of dual structures

Example (incidence graph): $q := \exists x, y, z(R(x,y) \wedge R(y,z))$

$I(q)$:



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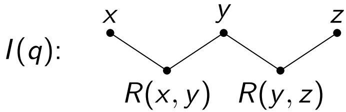

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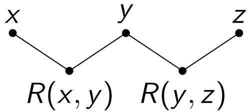

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Theorem (Cherlin, Shelah, Shi '99)

If $I(q)$ is connected, then q has a countable dual $\mathfrak{B}_q$, which can be chosen so that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic.

Connection of resilience and VCSPs

query q with $I(q)$ connected (WLOG) $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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Remark: We have to consider bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1 (differs from the original setting).

Example: Input $R(x, y) + R(x, y)$ for $\mathrm{VCSP}(\Gamma)$ corresponds to a database with multiplicity 2 for $R(x, y)$.

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$\mathfrak{B}_q \rightsquigarrow \Gamma_{\mathrm{MC}} = (\{0, 1\}; R)$
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Combined with the theorem on finite duals and the complexity dichotomy for finite-domain VCSPs this yields:

Corollary (Bodirsky, S., Lutz '24)

Let q be a conjunctive query such that I(q) is acyclic. Then the resilience problem for q in bag semantics is in P or NP-complete.

Sufficient condition for tractability

A more concrete version of the finite-domain VCSP dichotomy:

Theorem

Γ – *a finite-domain valued structure*

- *If* Γ *does not pp-construct* K_3*, then* Γ *has cyclic fractional polymorphism (essentially [Kozik, Ochremiak '15]).*
- *If* Γ *has a cyclic fractional polymorphism, then* $\mathrm{VCSP}(\Gamma)$ *is in P [Kolmogorov, Krokhin, Rolínek '15].*

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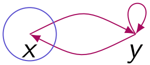
Theorem (Bodirsky, S., Lutz '24)

If Γ_q has a fractional polymorphism which is canonical and pseudo cyclic with respect to $\mathrm{Aut}(\Gamma_q)$, then $\mathrm{VCSP}(\Gamma_q)$ and hence resilience of q is in P.

Tractability conjecture

Example:

$$q := \exists x, y \big(S(x) \wedge R(x, y) \wedge R(y, x) \wedge R(y, y) \big)$$



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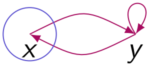

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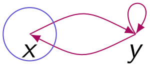

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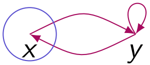

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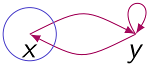

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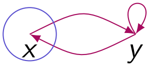

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- the conjecture is true for all queries with finite duals
- verified also for a lot of examples with cycles

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Resilience:

- Classify the complexity of resilience problems depending on q.
- Prove or disprove the conjecture.

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Graph VCSPs:

- Classify the complexity of VCSPs of valued structures Γ such that $\mathrm{Aut}(\Gamma)$ contains the automorphism group of the countable random graph.
- Is $\mathrm{VCSP}(\Gamma)$ in P whenever Γ does not pp-construct K_3?

Algebraic properties

Questions:

- If $\mathrm{Aut}(\Gamma)$ is oligomorphic, is it true that if a valued relation R on the domain of Γ is improved by $\mathrm{fPol}(\Gamma)$, then $R \in \langle \Gamma \rangle$?

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- Is it necessary to consider arbitrary probability distributions for fractional polymorphisms? Can we restrict to discrete (i.e., countably additive) ones?

Thank you for your attention

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Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.