Valued Constraints over Infinite Domains

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Outline

1. Introduction to VCSPs
2. Tools for VCSPs
3. Temporal VCSPs
4. Resilience problems
5. Outlook to the future

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  Output: Can we assign values to the variables violating at most u constraints?

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  Input: a database $\mathfrak{A}$, threshold u

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P = class of efficiently solvable problems
NP = class of problems with efficiently verifiable solution
NP-complete problems = hardest problems in NP

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Observation: VCSP generalizes CSP and MinCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MinCSP) or ∞ (for CSP).

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Output: Is

$$\inf_{t \in D^n} \phi(t) \leq u \text{ in } \Gamma?$$

Max-Cut as a VCSP

Example:

Input: $G = (V, E)$ – finite directed (multi)graph

Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.

Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $D = \{0, 1\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

$\min_{x \in D^n} \sum_{(x_i, x_j) \in E} R(x_i, x_j)$. The partition of V is given by the values 0 and 1.

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every instance of VCSP(Γ_{MC}) corresponds to a directed multigraph

$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Revisiting problems from the beginning

- **least correlation clustering** $= \mathrm{VCSP}(\mathbb{N}; (=)_0^1, (\neq)_0^1)$

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  $\hookrightarrow$ not obvious how to model as a VCSP

Complexity of VCSPs

Theorem (Kozik, Ochremiak '15; Kolmogorov, Rolínek, Krokhin '15; Bulatov '17; Zhuk '17)

Let Γ *be a valued structure with a finite domain. Then* $\mathrm{VCSP}(\Gamma)$ *is in P or NP-complete.*

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- automorphism of Γ – permutation α of C such that for $R \in \tau$ of arity k and every $t \in C^k$, $R(\alpha(t)) = R(t)$

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Example: $\mathrm{Aut}(\mathbb{Q}; (<)_0^1) = \mathrm{Aut}(\mathbb{Q}; <)$ is oligomorphic.

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$\text{Feas}(R) := \{t \mid R(t) < \infty\}$
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Fact (Bodirsky, S., Lutz '24): If Aut(Γ) is oligomorphic and $R \in \langle \Gamma \rangle$, VCSP($\Gamma$; R) reduces to VCSP(Γ) in poly-time.

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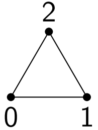
K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ \infty \text{ if } x = y \end{cases}$$



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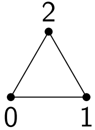

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Corollary (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\Gamma)$ *is oligomorphic and* Γ *pp-constructs* K_3*, then* $\mathrm{VCSP}(\Gamma)$ *is NP-hard.*

Fractional polymorphisms

polymorphism of a relational structure $\mathfrak{A}$ – $f : A^n \to A$ such that for all relations R of $\mathfrak{A}$ and $t^1, \ldots, t^n \in R$, $f(t^1, \ldots, t^n) \in R$ (applied row-wise)

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Example: The operation min is a polymorphism of $(\mathbb{Q}; <)$.

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Definition (fractional polymorphism)

A fractional polymorphism of Γ of arity n is a probability distribution ω on the maps $f : C^n \to C$ such that for every k-ary $R \in \tau$ and $t^1, \ldots, t^n \in C^k$

$$E_\omega[f \mapsto R(f(t^1, \ldots, t^n))] \leq \frac{1}{n} \sum_{j=1}^{n} R(t^j) \quad (\omega \text{ improves } R).$$

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Proposition (Bodirsky, S., Lutz '24)

If Aut(Γ) *is oligomorphic and* $R \in \langle \Gamma \rangle$, *then* fPol($\Gamma$) *improves* R.

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A relational structure $\mathfrak{A}$ is

- an equality structure if $\mathfrak{A}$ is fo-definable in $(\mathbb{Q}; =)$ $\Leftrightarrow$ $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q}; =) = \mathrm{Sym}(\mathbb{Q})$;

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- temporal: $(\mathbb{Q}; (<)_0^1)$ (models minimum feedback arc set problem)

Classification of equality VCSPs

Known for CSPs:

Theorem (Bodirsky, Kára '08)

If $\mathfrak{A}$ is an equality relational structure, then exactly one of the following:

- $\mathrm{Pol}(\mathfrak{A})$ *contains a unary constant operation or a binary injection and* $\mathrm{CSP}(\mathfrak{A})$ *is in P.*
- $\mathfrak{A}$ *pp-constructs* K_3 *and* $\mathrm{CSP}(\mathfrak{A})$ *is NP-complete.*

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Theorem (Bodirsky, Bonnet, S. '24)

If Γ is an equality valued structure, then exactly one of the following:

- fPol(Γ) *contains a unary constant operation or a binary injection and* VCSP(Γ) *is in P.*
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$\hookrightarrow$ the considered probability distributions put all weight on one operation

Classification of temporal CSPs

Theorem (Bodirsky, Kára '10)

Let $\mathfrak{A}$ be a temporal relational structure. Then exactly one of the following holds:

- *At least one of the operations* const, min, mx, mi, ll, *or one of their duals lies in* Pol($\mathfrak{A}$) *and* CSP($\mathfrak{A}$) *is P.*
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$\hookrightarrow$ const is the unary constant 0 operation

$\hookrightarrow$ the remaining polymorphisms are tailored to the structure $(\mathbb{Q}; <)$

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lex : $\mathbb{Q}^2 \to \mathbb{Q}$ is an operation satisfying

$$\text{lex}(a,b) < \text{lex}(c,d) \text{ iff } a < c \text{ or } (a = c) \wedge b < d$$

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Corollary (of the proof): Given a temporal valued structure Γ, it is decidable whether VCSP(Γ) is in P or NP-complete.

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Resilience of queries

database – a relational structure $\mathfrak{A}$

conjunctive query – a formula q of the form $\exists y_1, \ldots, y_l \, (\psi_1 \wedge \cdots \wedge \psi_m)$, where ψ_i are atomic

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Fixed conjunctive query q.
Input: a finite database $\mathfrak{A}$, $u \in \mathbb{N}$
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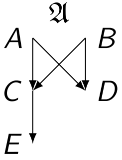
Example: The resilience of

$$q = \exists x, y, z (R(x, y) \wedge R(y, z))$$

with respect to $\mathfrak{A}$ is 1 – remove (C, E).



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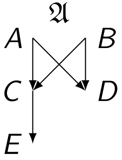
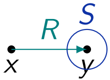


Goal: Classify complexity of resilience for all q.

Homomorphism duality

Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

For a query q, take its canonical structure $\mathfrak{Q}$.
Search for a structure $\mathfrak{B}_q$ such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \Leftrightarrow \mathfrak{Q} \not\to \mathfrak{A} \Leftrightarrow \mathfrak{A} \to \mathfrak{B}_q$$

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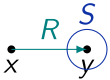
Example (canonical structure): $\exists x, y (R(x,y) \wedge S(y)) \rightsquigarrow$ 

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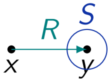
Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

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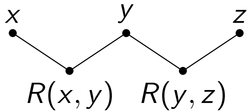
$\rightsquigarrow$ existence of $\mathfrak{B}_q$ enables studying resilience of q using the results about (valued) constraint satisfaction problems

Existence of dual structures

Example (incidence graph): $q := \exists x, y, z(R(x, y) \land R(y, z))$

$I(q)$:



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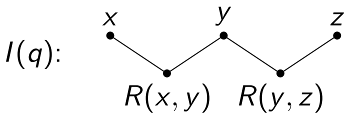

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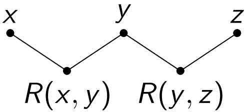
$I(q)$: 

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Theorem (Cherlin, Shelah, Shi '99)

If $I(q)$ is connected, then q has a countable dual $\mathfrak{B}_q$, which can be chosen so that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic.

Connection of resilience and VCSPs

query q with $I(q)$ connected (WLOG) $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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Remark: We have to consider bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1 (differs from the original setting).

Example: Input $R(x, y) + R(x, y)$ for VCSP(Γ) corresponds to a database with multiplicity 2 for $R(x, y)$.

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$\mathfrak{B}_q \rightsquigarrow \Gamma_{\mathrm{MC}} = (\{0, 1\}; R)$
Resilience of $q = \mathrm{VCSP}(\Gamma_{\mathrm{MC}}) =$ Max-Cut is NP-hard

Complexity dichotomy for resilience of acyclic queries

Corollary (Bodirsky, S., Lutz '24)

Let q be a conjunctive query such that $I(q)$ is acyclic. Then the resilience problem for q in bag semantics is in P or NP-complete.

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- VCSP(Γ_q) is in P or NP-complete by the dichotomy theorem for finite-domain VCSPs.

Sufficient condition for tractability

A more concrete version of the finite-domain VCSP dichotomy:

Theorem

Γ – *a finite-domain valued structure*

- *If* Γ *does not pp-construct* K_3, *then* Γ *has cyclic fractional polymorphism (essentially [Kozik, Ochremiak '15]).*
- *If* Γ *has a cyclic fractional polymorphism, then* VCSP(Γ) *is in P [Kolmogorov, Krokhin, Rolínek '15].*

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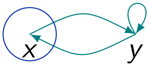
Theorem (Bodirsky, S., Lutz '24)

If Γ_q has a fractional polymorphism which is canonical and pseudo cyclic with respect to $\mathrm{Aut}(\Gamma_q)$, then $\mathrm{VCSP}(\Gamma_q)$ and hence resilience of q is in P.

Tractability conjecture

Example:

$$q := \exists x, y \big(S(x) \wedge R(x,y) \wedge R(y,x) \wedge R(y,y) \big)$$



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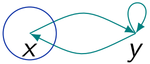

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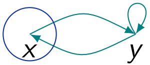

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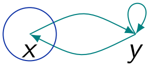

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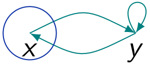

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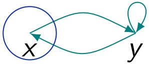

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- the conjecture is true for all queries with finite duals
- verified also for a lot of examples with cycles

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Resilience:

- Classify the complexity of resilience problems depending on q.
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Graph VCSPs:

- Classify the complexity of VCSPs of valued structures Γ such that $\mathrm{Aut}(\Gamma)$ contains the automorphism group of the countable random graph.
- Is $\mathrm{VCSP}(\Gamma)$ in P whenever Γ does not pp-construct K_3?

Algebraic properties

Questions:

- If $\mathrm{Aut}(\Gamma)$ is oligomorphic, is it true that if a valued relation R on the domain of Γ is improved by $\mathrm{fPol}(\Gamma)$, then $R \in \langle \Gamma \rangle$?

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- Is the union of the conditions for tractability in the temporal VCSP classification disjoint from the hardness condition (regardless of $\mathrm{P} \neq \mathrm{NP}$)?
- Is it necessary to consider arbitrary probability distributions for fractional polymorphisms? Can we restrict to discrete (i.e., countably additive) ones?

Thank you for your attention

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