Valued Constraint Satisfaction Problem and Resilience in Database Theory

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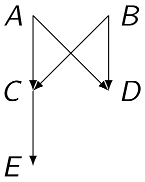
Queries and databases

Database: a relational structure $\mathfrak{A}$

parent	of
Alice	Cecilia
Bob	Cecilia
Alice	Daniel
Bob	Daniel
Cecilia	Eva



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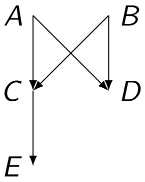

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$$\exists y_1, \dots, y_l (\psi_1 \wedge \cdots \wedge \psi_m),$$

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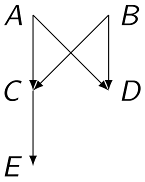

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Example: $\mathfrak{A}$ as above, $q := \exists x, y, z(\text{parent}(x, y) \wedge \text{parent}(y, z))$, then $\mathfrak{A} \models q$ with $x = A$, $y = C$ and $z = B$.

Resilience

Definition (Resilience)

Fixed conjunctive query q.
Input: a database $\mathfrak{A}$
Output: minimum number of tuples to be removed from relations of $\mathfrak{A}$ so that $\mathfrak{A} \not\models q$

Appears first in the paper of Meliou, Gatterbauer, Moore, Suciu ('10).

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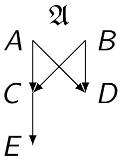
Example: The resilience of $\mathfrak{A}$ with respect to

$$q = \exists x, y, z(\text{parent}(x, y) \wedge \text{parent}(y, z))$$

is 1 – remove either (A, C) or (C, E).



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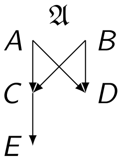

Research Goal: Classify complexity of resilience for all conjunctive queries.

Constraint satisfaction

Fixed τ-structure $\mathfrak{A}$ (τ – finite relational signature)
Input: list of constraints in τ
Output:

- **CSP**: Decide whether there is a solution that satisfies all constraints.
- **MaxCSP**: Find the maximal number of constraints that can be satisfied at once.
- **VCSP**: Find the minimal cost with which the constraints can be satisfied (each constraint comes with a cost depending on the chosen values).

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Observation: VCSP generalizes CSP and MaxCSP.
Proof: Model the tuples in relations with the cost 0 and the tuples outside with the cost

- 1 and the same threshold (for MaxCSP);
- ∞ and threshold 0 (for CSP).

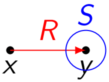
Different formulations of constraint satisfaction

Satisfying a list of constraints can be viewed alternatively as:

- satisfying a primitive positive formula
- being able to map the corresponding structure homomorphically
- a sum of the constraints (in the valued setting) being below a threshold

Example: Consider the list $R(x,y)$, $S(y)$. A structure $\mathfrak{B}$ satisfies these constraints iff:

- $\mathfrak{B} \models \exists x, y (R(x,y) \wedge S(y))$
- the canonical structure



$\rightarrow \mathfrak{B}$ (maps homomorphically)

- the sum $R(x,y) + S(y)$ in $\mathfrak{B}$ is 0

Focus on VCSP

A valued structure Γ, consists of:

- (countable) domain D
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^\Gamma : D^k \to \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is of the form $R(x_{i_1}, \ldots, x_{i_k})$ for $R \in \tau$
Question: Is

$$\inf_{\bar{a} \in D^n} \phi(\bar{a}) \leq u \text{ in } \Gamma?$$

Max-Cut as a VCSP

Example:
Input: $G = (V, E)$ – finite directed graph
Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.
Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ be a valued structure where:

- $D = \{0, 1\}$
- $\tau = \{E\}$, E binary

$$E(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

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every instance of VCSP(Γ) corresponds to a digraph $\rightsquigarrow$ VCSP(Γ) is the Max-Cut problem (NP-hard)

Homomorphism duality

For a query q, take its canonical structure $\mathfrak{Q}$.
Search for a structure $\mathfrak{B}_q$ such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \Leftrightarrow \mathfrak{Q} \not\to \mathfrak{A} \Leftrightarrow \mathfrak{A} \to \mathfrak{B}_q$$

$\rightsquigarrow$ corresponds to the CSP($\mathfrak{B}_q$) (if we represent the constraints by their canonical structure)

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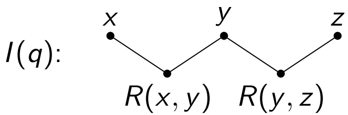
$\rightsquigarrow$ existence of the structure $\mathfrak{B}_q$ enables studying resilience of q using the results about (valued) constraint satisfaction problems

Existence of dual structures

Example (incidence graph): $q := \exists x, y, z (R(x, y) \wedge R(y, z))$



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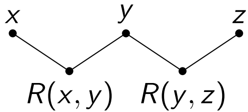
$I(q)$:



Theorem (Nešetřil, Tardiff ('00); Larose, Loten, Tardiff ('07); Foniok ('07))

A conjunctive query q has a finite dual if and only if $I(q)$ is a tree.

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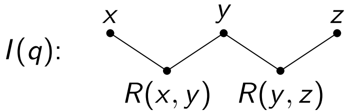

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Theorem (Cherlin, Shelah, Shi ('99))

If $I(q)$ is connected, then q has a countable dual $\mathfrak{B}_q$. $\mathfrak{B}_q$ can be chosen so that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic.

oligomorphic – countable domain B_q and the action of $\mathrm{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

Connection of resilience and VCSPs

query q with $I(q)$ connected $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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Generalisations:

- presence of exogenous tuples – specified tuples may not be removed $\rightsquigarrow$ use cost ∞ instead of 1 in Γ_q
- holds for finite unions of conjunctive queries
- the assumption on connectivity could be made WLOG

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Remark: Correspondence between VCSPs and resilience problems in both directions requires considering bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1.

Example: Input $R(x, y) + R(x, y)$ for $\mathrm{VCSP}(\Gamma)$ corresponds to a database with multiplicity 2 for $R(x, y)$.

Complexity with finite duals

Follows from results of: Kozik, Ochremiak ('15); Kolmogorov, Rolínek, Krokhin ('15); Bulatov ('17); Zhuk ('17)

Theorem

For a finite-domain valued structure Γ, $\mathrm{VCSP}(\Gamma)$ *is in P or NP-complete.*

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Question: For general queries, choose $\mathfrak{B}_q$ such that $\text{Aut}(\mathfrak{B}_q)$ is oligomorphic $\Rightarrow$ finitely many orbits of n-tuples for every n.
Can we use some results for finite domains?

Hard resilience problems

pp-construction – a notion of 'expressing' one valued structure in another (generalizes pp-constructions for classical structures)

Fact: If $\mathrm{Aut}(\Gamma)$ is oligomorphic and Γ pp-constructs Δ, then $\mathrm{VCSP}(\Delta)$ reduces to $\mathrm{VCSP}(\Gamma)$ in poly-time.

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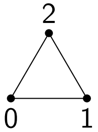
Fact: If Aut(Γ) is oligomorphic and Γ pp-constructs Δ, then VCSP(Δ) reduces to VCSP(Γ) in poly-time.

K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ \infty \text{ if } x = y \end{cases}$$



Observation: VCSP(K_3) is the 3-colorability problem and hence NP-hard.

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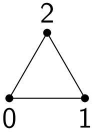

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Corollary

If $\mathrm{Aut}(\Gamma)$ is oligomorphic and Γ pp-constructs K_3, then $\mathrm{VCSP}(\Gamma)$ is NP-hard.

Fractional polymorphisms

polymorphism f of $\mathfrak{B}$ – $f : B^n \to B$ that preserves all relations of $\mathfrak{B}$
Idea: CSP($\mathfrak{B}$) is tractable iff it has nice polymorphisms

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Definition (fractional polymorphism)

Γ – valued τ-structure with domain D

Fractional polymorphism of Γ of arity n is a probablity distribution ω on operations $D^n \to D$ such that for every k-ary $R \in \tau$ and $a^1, \ldots, a^n \in D^k$

$$\underbrace{E_\omega[f \mapsto R(f(a^1, \ldots, a^n))]}_{\text{expected value}} \leq \underbrace{\frac{1}{n}\sum_{j=1}^{n} R(a^j)}_{\text{arithmetic mean}}$$

Tractability conjecture

Known for finite-domain VCSPs:

Theorem

Γ – *a finite-domain valued structure*

- *If* Γ *does not pp-construct* K_3*, then* Γ *has cyclic fractional polymorphism of arity* ≥ 2 *(essentially Kozik, Ochremiak ('15)).*
- *If* Γ *has a cyclic fractional polymorphism of arity* ≥ 2*, then* VCSP(Γ) *is in P (Kolmogorov, Krokhin, Rolínek ('15)).*

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Theorem (Bodirsky, Lutz, S.)

q – *conjunctive query*
If Γ_q *has a fractional polymorphism which is canonical and pseudo-cyclic with respect to* $\mathrm{Aut}(\Gamma_q)$, *then* $\mathrm{VCSP}(\Gamma_q)$ *is in P.*

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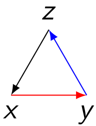
Conjecture: If Γ_q does not pp-construct K_3, then the tractability theorem applies and $\mathrm{VCSP}(\Gamma_q)$ and hence resilience of q is in P.

Examples

Example (hardness):

$q := \exists x, y, z(R(x, y) \wedge S(y, z) \wedge T(z, x))$

- resilience of q is known to be NP-hard (Freire, Gatterbauer, Immerman, Meliou ('15))
- Γ_q pp-constructs K_3



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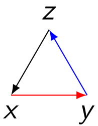

Example (tractability):

$q := \exists x, y(R(x,y) \wedge R(y,y) \wedge R(y,x) \wedge S(x))$

- complexity left open in Freire, Gatterbauer, Immerman, Meliou ('20)
- Γ_q has a canonical and pseudo-cyclic fractional polymorphism

Thank you for your attention

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