Constraint Satisfaction Problems of First-Order Expansions of Algebraic Products

Žaneta Semanišinová
with Manuel Bodirsky, Peter Jonsson, Barnaby Martin, Antoine Mottet

Institute of Algebra
TU Dresden

Spring School of Algebra, 14.5.2022

Constraint Satisfaction Problems

(relational) structure $\mathfrak{A} = (A; R^{\mathfrak{A}} : R \in \tau)$; **finite** signature τ

Definition (homomorphism of structures)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures, then a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h\colon A \to B$ that preserves all the relations, that is, if $(a_1, \dots, a_k) \in R^{\mathfrak{A}}$, then $(h(a_1), \dots, h(a_k)) \in R^{\mathfrak{B}}$.

Definition (CSP)

Let $\mathfrak{B}$ be a τ-structure. The constraint satisfaction problem for $\mathfrak{B}$, denoted by CSP($\mathfrak{B}$), is the computational problem of deciding for a given finite τ-structure $\mathfrak{A}$ whether $\mathfrak{A}$ has a homomorphism to $\mathfrak{B}$ or not.

Examples of CSPs

The complete graph on 3 vertices is the relational structure

$$K_3 = (\{0, 1, 2\}; \neq).$$

CSP(K_3) is the 3-colorability problem for graphs (and CSP(K_n) is the n-colorability problem).

Examples of CSPs

The complete graph on 3 vertices is the relational structure

$$K_3 = (\{0,1,2\}; \neq).$$

CSP(K_3) is the 3-colorability problem for graphs (and CSP(K_n) is the n-colorability problem).

3-SAT is equivalent to

$$\mathrm{CSP}(\{0,1\}; R_{000}; R_{001}; R_{011}; R_{111}),$$

where $R_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$.
For example, the clause $x \vee \neg y \vee z$ is modelled by $R_{001}(x,z,y)$.

Primitive positive definitions

Definition (pp-formula)

An atomic formula is a formula of the form $x = y$, $R(x_1, \dots, x_n)$, or $\bot$.
A primitive positive formula is a formula $\phi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_l (\psi_1 \wedge \cdots \wedge \psi_m)$$

where $\psi_1, \dots, \psi_k$ are atomic formulas.

Example: $\phi(x, y) = \exists z \, R(x, y, z) \wedge R(x, x, z)$ is a pp-formula.

Primitive positive definitions

Definition (pp-formula)

An atomic formula is a formula of the form $x = y$, $R(x_1, \ldots, x_n)$, or $\bot$.
A primitive positive formula is a formula $\phi(x_1, \ldots, x_n)$ of the form

$$\exists y_1, \ldots, y_l (\psi_1 \wedge \cdots \wedge \psi_m)$$

where $\psi_1, \ldots, \psi_k$ are atomic formulas.

Example: $\phi(x, y) = \exists z\, R(x, y, z) \wedge R(x, x, z)$ is a pp-formula.

Proposition (Jeavons (1998))

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with the same domain. If every relation of $\mathfrak{A}$ has a pp-definition in $\mathfrak{B}$, then there is a poly-time reduction from $\mathrm{CSP}(\mathfrak{A})$ *to* $\mathrm{CSP}(\mathfrak{B})$.

Polymorphisms

Definition (polymorphism)

An operation $f : A^k \to A$ is a polymorphism of (or preserves) a structure $\mathfrak{A}$ if for every relation R of $\mathfrak{A}$ and for all tuples $\bar{r_1}, \ldots, \bar{r_k} \in R$ also $f(\bar{r_1}, \ldots, \bar{r_k}) \in R$ (computed row-wise).
The set of all polymorphisms of $\mathfrak{A}$ will be denoted by $\mathrm{Pol}(\mathfrak{A})$.

$$\begin{pmatrix} 1 \\ \wedge \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ \wedge \\ 3 \end{pmatrix} \begin{matrix} \to \\ \\ \to \end{matrix} \begin{pmatrix} 3 \\ \wedge \\ 8 \end{pmatrix}$$

Theorem (Bodirsky, Nešetřil (2006))

A relation $R \subseteq A^l$ is preserved by all polymorphisms of an ω-categorical structure $\mathfrak{A}$ if and only if R is has a pp-definition in $\mathfrak{A}$.

Model-theoretical definitions

Definition (first-order expansion)

A first-order expansion (fo-expansion) of $\mathfrak{A}$ is a structure $\mathfrak{A}'$ augmented by relations that are first-order definable in $\mathfrak{A}$.

Definition (ω-categorical structure, homogeneity)

A structure $\mathfrak{A}$ is

- ω-categorical if it is countable and all countable models of the first-order theory of $\mathfrak{A}$ are isomorphic;
- homogeneous if every isomorphism between finite substructures can be extended to an automorphism of the structure.

Example: $(\mathbb{Q}, <)$ is an ω-categorical homogeneous structure.

Complexity dichotomy

Theorem (Bulatov (2017), Zhuk (2017))

For every finite structure $\mathfrak{B}$ with finite signature CSP($\mathfrak{B}$) *is in P or NP-complete.*

Conjecture (Bodirsky, Pinsker (2011))

For reduct $\mathfrak{B}$ of a finitely bounded homogeneous structure CSP($\mathfrak{B}$) *is in P or NP-complete.*

Interesting examples of infinite structures that fall into the scope of the conjecture are e.g. fo-expansions of (algebraic powers of) $(\mathbb{Q}, <)$.

Algebraic products

Definition (algebraic product)

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures with signatures τ_1 and τ_2, respectively. The algebraic product $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is the structure with domain $A_1 \times A_2$ which has for every atomic τ_1-formula $\phi(x_1, \dots, x_k)$ the relation

$$\{((u_1, v_1), \dots, (u_k, v_k)) \mid \mathfrak{A}_1 \models \phi(u_1, \dots, u_k)\}$$

and for every atomic τ_2-formula $\phi(x_1, \dots, x_k)$ the relation

$$\{((u_1, v_1), \dots, (u_k, v_k)) \mid \mathfrak{A}_2 \models \phi(v_1, \dots, v_k)\}.$$

Example: $(\mathbb{Q}, <) \boxtimes (\mathbb{Q}, <)$ is the structure $(\mathbb{Q}^2; <_1, =_1, <_2, =_2)$, where e.g. $(1, 4) <_1 (2, 3)$ and $(-2, 5) =_2 (8/3, 5)$.

Algebraic powers of $(\mathbb{Q}, <)$ in spatial reasoning

Cardinal Direction Calculus:

- $\mathfrak{C} = (\mathbb{Q}^2; \mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}, \mathsf{NE}, \mathsf{SE}, \mathsf{SW}, \mathsf{NW})$ (North, East, etc.)

N	E	S	W
$(=, >)$	$(>, =)$	$(=, <)$	$(<, =)$

NE	SE	SW	NW
$(>, >)$	$(>, <)$	$(<, <)$	$(<, >)$

- the relations $<_i$ and $=_i$ are pp-definable in $\mathfrak{C}$
- fo-expansions of $\mathfrak{C}$ can be then viewed as fo-expansions of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$
- we may then classify complexity of CSPs of fo-expansions of $\mathfrak{C}$

Algebraic powers of $(\mathbb{Q}, <)$ in temporal reasoning

Allen's Interval Algebra:

- $\mathbb{I}$ is the set of all pairs $(x, y) \in \mathbb{Q}^2$ such that $x < y$
- we view $\mathbb{I}$ as the set of all closed intervals $[a, b]$ of rational numbers
- basic relations defined on $\mathbb{I}$ correspond to relative positions of the intervals (e. g. meets, starts, finishes)
- e.g., $\mathrm{s}(X, Y)$ corresponds to X starts Y and $\mathrm{f}(X, Y)$ to X finishes Y

```
s(X,Y):   XXX              f(X,Y):      XXX
          YYYYYY                     YYYYYY
```

- one may prove that $(\mathbb{I}, \mathrm{s}, \mathrm{f})$ is isomorphic to a structure that is pp-interdefinable with $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$
- complexity classification of CSPs of fo-expansions of $(\mathbb{I}, \mathrm{s}, \mathrm{f})$ then follows from the classification for $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$

Complexity of CSPs of (fo-expansions) of alg. products

$\mathfrak{A}_1, \mathfrak{A}_2$ – countable ω-categorical structures
$\mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) = \mathrm{Pol}(\mathfrak{A}_1) \times \mathrm{Pol}(\mathfrak{A}_2) \Rightarrow$ the complexity of the CSP (of an fo-expansion) of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is related to "the complexity in each dimension"

Proposition

If $\mathrm{CSP}(\mathfrak{A}_1)$ *is in P and* $\mathrm{CSP}(\mathfrak{A}_2)$ *is in P, then* $\mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$ *is in P.*

Complexity of CSPs of (fo-expansions) of alg. products

$\mathfrak{A}_1, \mathfrak{A}_2$ – countable ω-categorical structures
$\mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) = \mathrm{Pol}(\mathfrak{A}_1) \times \mathrm{Pol}(\mathfrak{A}_2) \Rightarrow$ the complexity of the CSP (of an fo-expansion) of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is related to "the complexity in each dimension"

Proposition

If $\mathrm{CSP}(\mathfrak{A}_1)$ *is in P and* $\mathrm{CSP}(\mathfrak{A}_2)$ *is in P, then* $\mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$ *is in P.*

$\theta_i : \mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) \to \mathrm{Pol}(\mathfrak{A}_i)$ (projects on the i-th coordinate) Follows

from the results by Barto, Opršal, Pinsker (2018):

Proposition

Let $\mathfrak{D}$ *be an fo-expansion of* $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$. *Let* i *be such that* $\theta_i(\mathrm{Pol}(\mathfrak{D}))$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$. *Then* $\mathrm{Pol}(\mathfrak{D})$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$ *as well and* $\mathrm{CSP}(\mathfrak{D})$ *is NP-complete.*

CSPs of fo-expansions of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$

Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ be an fo-expansion of $(\mathbb{Q}; <)$. If Pol($\mathfrak{B}$) *contains a* min-, mx-, mi-, *or* ll-*operation, or the dual of such an operation, then* CSP($\mathfrak{B}$) *is in P. Otherwise,* CSP($\mathfrak{B}$) *is NP-complete.*

Theorem (Bodirsky, Jonsson, Martin, Mottet, S. (2022))

Let $\mathfrak{D}$ be an fo-expansion of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$. Exactly one of the following two cases applies.

- θ_i(Pol($\mathfrak{D}$)) *contains a* min-, mx-, mi-, *or* ll-*operation, or the dual of such an operation, for each $i \in \{1, 2\}$ and* CSP($\mathfrak{D}$) *is in P.*
- *There is $i \in \{1, 2\}$ such that* θ_i(Pol($\mathfrak{D}$)) *has a uniformly continuous minor-preserving map to* Pol(K_3) *and* CSP($\mathfrak{D}$) *is NP-complete.*

Proof idea

NP-complete:

- follows directly from the previous proposition

Proof idea

NP-complete:

- follows directly from the previous proposition

P:

- relations of $\mathfrak{D}$ are defined by fo-formulas in $<_i$ and $=_i$
- we may assume quantifier-free definitions in conjunctive normal form
- the key is to have a conjunctions of clauses which are (almost) i-determined (contains literals only with index i)
- we aim to run first the poly-time algorithm to decide satisfiability of the 1-determined constraints and then the poly-time algorithm to decide satisfiability of the (possibly modified) 2-determined constraints
- existence of such poly-time algorithms follows from the theorem for $(\mathbb{Q}, <)$

What is next

Classify the complexity of:

- CSPs of (reducts) of fo-expansions of

$$\underbrace{(\{0,1\};\{0\},\{1\}) \boxtimes \cdots \boxtimes (\{0,1\};\{0\},\{1\})}_{n} \boxtimes (\mathbb{Q},<)$$

  for $n = 1$ and general n
- more generally: CSPs of fo-expansions of $\mathfrak{B} \boxtimes (\mathbb{Q},<)$, where $\mathfrak{B}$ is a finite structure
- challenge: CSPs of structures fo-interpretable over $(\mathbb{Q},<)$

All of the above fall into the scope of the infinite-domain dichotomy conjecture.

Thank you for your attention