Constraint Satisfaction Problems of First-Order Expansions of Algebraic Products

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Constraint Satisfaction Problems

(relational) structure $\mathfrak{A} = (A; R^{\mathfrak{A}} : R \in \tau)$; **finite** signature τ

Definition (homomorphism of structures)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures, then a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h\colon A \to B$ that preserves all the relations, that is, if $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$, then $(h(a_1), \ldots, h(a_k)) \in R^{\mathfrak{B}}$.

Definition (CSP)

Let $\mathfrak{B}$ be a τ-structure. The constraint satisfaction problem for $\mathfrak{B}$, denoted by CSP($\mathfrak{B}$), is the computational problem of deciding for a given finite τ-structure $\mathfrak{A}$ whether $\mathfrak{A}$ has a homomorphism to $\mathfrak{B}$ or not.

Examples of CSPs

The complete graph on 3 vertices is the relational structure

$$K_3 = (\{0, 1, 2\}; \neq).$$

CSP(K_3) is the 3-colorability problem for graphs (and CSP(K_n) is the n-colorability problem).

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3-SAT is equivalent to

$$\mathrm{CSP}(\{0,1\}; R_{000}; R_{001}; R_{011}; R_{111}),$$

where $R_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$.
For example, the clause $x \vee \neg y \vee z$ is modelled by $R_{001}(x,z,y)$.

Primitive positive definitions

Definition (pp-formula)

An atomic formula is a formula of the form $x = y$, $R(x_1, \dots, x_n)$, or $\bot$.
A primitive positive formula is a formula $\phi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_l (\psi_1 \wedge \dots \wedge \psi_m)$$

where $\psi_1, \dots, \psi_k$ are atomic formulas.

Example: $\phi(x, y) = \exists z\, R(x, y, z) \wedge R(x, x, z)$ is a pp-formula.

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Proposition (Jeavons (1998))

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with the same domain. If every relation of $\mathfrak{A}$ has a pp-definition in $\mathfrak{B}$, then there is a poly-time reduction from $\mathrm{CSP}(\mathfrak{A})$ *to* $\mathrm{CSP}(\mathfrak{B})$.

Polymorphisms

Definition (polymorphism)

An operation $f : A^k \to A$ is a polymorphism of (or preserves) a structure $\mathfrak{A}$ if for every relation R of $\mathfrak{A}$ and for all tuples $\bar{r_1}, \dots, \bar{r_k} \in R$ also $f(\bar{r_1}, \dots, \bar{r_k}) \in R$ (computed row-wise).
The set of all polymorphisms of $\mathfrak{A}$ will be denoted by $\mathrm{Pol}(\mathfrak{A})$.

$$\begin{pmatrix} 1 \\ \wedge \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ \wedge \\ 3 \end{pmatrix} \begin{matrix} \to \\ \\ \to \end{matrix} \begin{pmatrix} 3 \\ \wedge \\ 8 \end{pmatrix}$$

Theorem (Bodirsky, Nešetřil (2006))

A relation $R \subseteq A^l$ is preserved by all polymorphisms of an ω-categorical structure $\mathfrak{A}$ if and only if R is has a pp-definition in $\mathfrak{A}$.

Model-theoretical definitions

Definition (first-order expansion)

A first-order expansion (fo-expansion) of $\mathfrak{A}$ is a structure $\mathfrak{A}'$ augmented by relations that are first-order definable in $\mathfrak{A}$.

Definition (ω-categorical structure, homogeneity)

A structure $\mathfrak{A}$ is

- ω-categorical if it is countable and all countable models of the first-order theory of $\mathfrak{A}$ are isomorphic;
- homogeneous if every isomorphism between finite substructures can be extended to an automorphism of the structure.

Example: $(\mathbb{Q}, <)$ is an ω-categorical homogeneous structure.

Complexity dichotomy

Theorem (Bulatov (2017), Zhuk (2017))

For every finite structure $\mathfrak{B}$ with finite signature $\mathrm{CSP}(\mathfrak{B})$ *is in P or NP-complete.*

Conjecture (Bodirsky, Pinsker (2011))

For reduct $\mathfrak{B}$ of a finitely bounded homogeneous structure $\mathrm{CSP}(\mathfrak{B})$ *is in P or NP-complete.*

Interesting examples of infinite structures that fall into the scope of the conjecture are e.g. fo-expansions of (algebraic powers of) $(\mathbb{Q}, <)$.

Algebraic products

Definition (algebraic product)

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures with signatures τ_1 and τ_2, respectively. The algebraic product $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is the structure with domain $A_1 \times A_2$ which has for every atomic τ_1-formula $\phi(x_1, \ldots, x_k)$ the relation

$$\{((u_1, v_1), \ldots, (u_k, v_k)) \mid \mathfrak{A}_1 \models \phi(u_1, \ldots, u_k)\}$$

and for every atomic τ_2-formula $\phi(x_1, \ldots, x_k)$ the relation

$$\{((u_1, v_1), \ldots, (u_k, v_k)) \mid \mathfrak{A}_2 \models \phi(v_1, \ldots, v_k)\}.$$

Example: $(\mathbb{Q}, <) \boxtimes (\mathbb{Q}, <)$ is the structure $(\mathbb{Q}^2; <_1, =_1, <_2, =_2)$, where e.g. $(1, 4) <_1 (2, 3)$ and $(-2, 5) =_2 (8/3, 5)$.

Algebraic powers of $(\mathbb{Q}, <)$ in spatial reasoning

Cardinal Direction Calculus:

- $\mathfrak{C} = (\mathbb{Q}^2; \mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}, \mathsf{NE}, \mathsf{SE}, \mathsf{SW}, \mathsf{NW})$ (North, East, etc.)

N	E	S	W
$(=,>)$	$(>,=)$	$(=,<)$	$(<,=)$

NE	SE	SW	NW
$(>,>)$	$(>,<)$	$(<,<)$	$(<,>)$

- the relations $<_i$ and $=_i$ are pp-definable in $\mathfrak{C}$
- fo-expansions of $\mathfrak{C}$ can be then viewed as fo-expansions of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$
- we may then classify complexity of CSPs of fo-expansions of $\mathfrak{C}$

Algebraic powers of $(\mathbb{Q}, <)$ in temporal reasoning

Allen's Interval Algebra:

- $\mathbb{I}$ is the set of all pairs $(x, y) \in \mathbb{Q}^2$ such that $x < y$
- we view $\mathbb{I}$ as the set of all closed intervals $[a, b]$ of rational numbers
- basic relations defined on $\mathbb{I}$ correspond to relative positions of the intervals (e. g. meets, starts, finishes)
- e.g., $\mathrm{s}(X, Y)$ corresponds to X starts Y and $\mathrm{f}(X, Y)$ to X finishes Y

```
s(X,Y):   XXX              f(X,Y):      XXX
          YYYYYY                     YYYYYY
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- one may prove that $(\mathbb{I}, \mathrm{s}, \mathrm{f})$ is isomorphic to a structure that is pp-interdefinable with $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$
- complexity classification of CSPs of fo-expansions of $(\mathbb{I}, \mathrm{s}, \mathrm{f})$ then follows from the classification for $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$

Complexity of CSPs of (fo-expansions) of alg. products

$\mathfrak{A}_1, \mathfrak{A}_2$ – countable ω-categorical structures
$\mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) = \mathrm{Pol}(\mathfrak{A}_1) \times \mathrm{Pol}(\mathfrak{A}_2) \Rightarrow$ the complexity of the CSP (of an fo-expansion) of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is related to "the complexity in each dimension"

Proposition

If $\mathrm{CSP}(\mathfrak{A}_1)$ *is in P and* $\mathrm{CSP}(\mathfrak{A}_2)$ *is in P, then* $\mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$ *is in P.*

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Proposition

If CSP($\mathfrak{A}_1$) *is in P and* CSP($\mathfrak{A}_2$) *is in P, then* CSP($\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$) *is in P.*

$\theta_i : \mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) \to \mathrm{Pol}(\mathfrak{A}_i)$ (projects on the i-th coordinate) Follows

from the results by Barto, Opršal, Pinsker (2018):

Proposition

Let $\mathfrak{D}$ *be an fo-expansion of* $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$. *Let* i *be such that* $\theta_i(\mathrm{Pol}(\mathfrak{D}))$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$. *Then* $\mathrm{Pol}(\mathfrak{D})$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$ *as well and* CSP($\mathfrak{D}$) *is NP-complete.*

CSPs of fo-expansions of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$

Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ be an fo-expansion of $(\mathbb{Q}; <)$. If Pol($\mathfrak{B}$) *contains a* min-, mx-, mi-, *or* ll-*operation, or the dual of such an operation, then* CSP($\mathfrak{B}$) *is in P. Otherwise,* CSP($\mathfrak{B}$) *is NP-complete.*

Theorem (Bodirsky, Jonsson, Martin, Mottet, S. (2022))

Let $\mathfrak{D}$ be an fo-expansion of $(\mathbb{Q}^2, <_1, =_1, <_2, =_2)$. Exactly one of the following two cases applies.

- θ_i(Pol($\mathfrak{D}$)) *contains a* min-, mx-, mi-, *or* ll-*operation, or the dual of such an operation, for each $i \in \{1, 2\}$ and* CSP($\mathfrak{D}$) *is in P.*
- *There is $i \in \{1, 2\}$ such that* θ_i(Pol($\mathfrak{D}$)) *has a uniformly continuous minor-preserving map to* Pol(K_3) *and* CSP($\mathfrak{D}$) *is NP-complete.*

Proof idea

NP-complete:

- follows directly from the previous proposition

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P:

- relations of $\mathfrak{D}$ are defined by fo-formulas in $<_i$ and $=_i$
- we may assume quantifier-free definitions in conjunctive normal form
- the key is to have a conjunctions of clauses which are (almost) i-determined (contains literals only with index i)
- we aim to run first the poly-time algorithm to decide satisfiability of the 1-determined constraints and then the poly-time algorithm to decide satisfiability of the (possibly modified) 2-determined constraints
- existence of such poly-time algorithms follows from the theorem for $(\mathbb{Q}, <)$

What is next

Classify the complexity of:

- CSPs of (reducts) of fo-expansions of

$$\underbrace{(\{0,1\};\{0\},\{1\}) \boxtimes \cdots \boxtimes (\{0,1\};\{0\},\{1\})}_{n} \boxtimes (\mathbb{Q},<)$$

  for $n = 1$ and general n
- more generally: CSPs of fo-expansions of $\mathfrak{B} \boxtimes (\mathbb{Q},<)$, where $\mathfrak{B}$ is a finite structure
- challenge: CSPs of structures fo-interpretable over $(\mathbb{Q},<)$

All of the above fall into the scope of the infinite-domain dichotomy conjecture.

Thank you for your attention