

Classification Transfer for Constraint Satisfaction Problems

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Constraint Satisfaction Problems

(relational) structure $\mathfrak{B} = (B; R^{\mathfrak{B}} : R \in \tau)$; **finite** signature τ

Definition (CSP)

Constraint Satisfaction Problem for \mathfrak{B} ($\text{CSP}(\mathfrak{B})$):

Input: conjunction ϕ of atomic formulas

Question: Is ϕ satisfiable in \mathfrak{B} ?

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Example (3-SAT):

$\mathfrak{B} = (\{0, 1\}; R_{000}, R_{001}, R_{011}, R_{111})$, where $R_{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\}$

Rewrite input $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_2 \vee x_4) \wedge \dots$ as

$$R_{001}(x_1, x_3, x_2) \wedge R_{011}(x_4, x_3, x_2) \wedge \dots$$

$\text{CSP}(\mathfrak{B})$ is the same problem as 3-SAT.

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Example (**graph acyclicity**):

$\mathfrak{B} = (\mathbb{Q}; <) \rightsquigarrow \text{digraph } (\mathbb{Q}; E)$

Write the edges of an input digraph G in a conjunction

$$E(x_1, x_2) \wedge E(x_3, x_4) \dots$$

The formula is satisfiable in $(\mathbb{Q}; E)$ iff G has no directed cycle.

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Goal: **Classify** the **complexity** of $\text{CSP}(\mathfrak{B})$ depending on \mathfrak{B} .

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- **all finite structures** (Bulatov '17; Zhuk '17)

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\leftrightarrow many concrete classes where it is open

Cardinal Direction Calculus

- $\mathcal{C} = (\mathbb{Q}^2; N, E, S, W, NE, SE, SW, NW)$ (North, East, etc.)

N	E	S	W	NE	SE	SW	NW
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Conjecture: $\text{CSP}(\mathfrak{B})$ is in **P** iff all **relations** of \mathfrak{B} are **definable** by an **Ord-Horn formula**, i.e., a conjunction of clauses of the form

$$(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee x_{k+1} \geq y_{k+1}) \quad (\text{last disjunct is optional}).$$

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Algebraic products of $(\mathbb{Q}; <)$

Consider structures $(\mathbb{Q}^n; <_1, =_1, \dots, <_n, =_n)$, where

$(a_1, \dots, a_n) <_i (b_1, \dots, b_n)$ iff $a_i < b_i$ and

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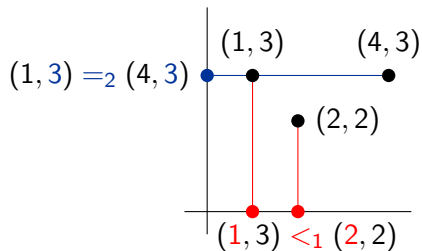
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Example: $n = 2$



Plan of attack

- **classify** the complexity of $\text{CSP}(\mathcal{D})$ where \mathcal{D} is a **fo-expansion** of $(\mathbb{Q}^n; <_1, =_1, \dots, <_n, =_n)$ using the results for **fo-expansions** of $(\mathbb{Q}; <)$

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Example (Interval Algebra): $\mathfrak{B} = (\mathbb{I}; \text{sUf})$

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Consequences:

- complexity *classification* for CDC_n and the *Block Algebra*
- *tractable* cases are definable by *Ord-Horn formulas*
- *solves* the *open problems* from '99 and '02

Verify the **infinite-domain CSP dichotomy conjecture** for:

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- structures **fo-interpretable** over $(\mathbb{Q}; <)$

Thank you for your attention

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