Gathering extensions into families

DANNY NEFTIN
Joint with P. Dèbes, J. König, D. Krashen, F. Legrand

Department of Mathematics
Technion - Israel Institute of Technology
Haifa, Israel

Summer school - Galois theory and number theory,
“Specializations in Inverse Galois Theory" minicourse,
Dresden, 2019

Essential dimension

K = number field; M/K finitely generated field extension; G = finite group.

Versal G-extensions

A G-extension N/M is versal if every G-extension of a field K_0 containing K is a specialization of N/M over K_0.

Example

Essential dimension

K = number field; M/K finitely generated field extension; G = finite group.

Versal G-extensions

A G-extension N/M is versal if every G-extension of a field K_0 containing K is a specialization of N/M over K_0.

Example

- $K(x_1,\ldots,x_n)/K(x_1,\ldots,x_n)^G$ is a versal G-extension if $G \le S_n$;

Essential dimension

K = number field; M/K finitely generated field extension; G = finite group.

Versal G-extensions

A G-extension N/M is versal if every G-extension of a field K_0 containing K is a specialization of N/M over K_0.

Example

- $K(x_1,\dots,x_n)/K(x_1,\dots,x_n)^G$ is a versal G-extension if $G \le S_n$;
- $K(\sqrt[n]{t})/K(t)$ is a versal C_n-extension if K contains the n-th roots of 1;

Essential dimension

K = number field; M/K finitely generated field extension; G = finite group.

Versal G-extensions

A G-extension N/M is versal if every G-extension of a field K_0 containing K is a specialization of N/M over K_0.

Example

- $K(x_1,\ldots,x_n)/K(x_1,\ldots,x_n)^G$ is a versal G-extension if $G \leq S_n$;
- $K(\sqrt[n]{t})/K(t)$ is a versal C_n-extension if K contains the n-th roots of 1;
- Let N be the splitting field of $f(t,x) := x^3 + tx + t \in \mathbb{Q}(t)[x]$ over $\mathbb{Q}(t)$. Then $N/\mathbb{Q}(t)$ is parametric for S_3.

Essential dimension

K = number field; M/K finitely generated field extension; G = finite group.

Versal G-extensions

A G-extension N/M is versal if every G-extension of a field K_0 containing K is a specialization of N/M over K_0.

Example

- $K(x_1,\ldots,x_n)/K(x_1,\ldots,x_n)^G$ is a versal G-extension if $G \leq S_n$;
- $K(\sqrt[n]{t})/K(t)$ is a versal C_n-extension if K contains the n-th roots of 1;
- Let N be the splitting field of $f(t,x) := x^3 + tx + t \in \mathbb{Q}(t)[x]$ over $\mathbb{Q}(t)$. Then $N/\mathbb{Q}(t)$ is parametric for S_3.

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

- Essential dimension: $\mathcal{F}$ is the set of all G extensions of fields K_0 containing K;

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

- Essential dimension: $\mathcal{F}$ is the set of all G extensions of fields K_0 containing K;
- Parametric dimension: $\mathcal{F}$ is the set of all G-extensions of K;

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

- Essential dimension: $\mathcal{F}$ is the set of all G extensions of fields K_0 containing K;
- Parametric dimension: $\mathcal{F}$ is the set of all G-extensions of K;
- Local dimension: $\mathcal{F}$ is the set of all G-extensions of K_v, over all places v of K;

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

- Essential dimension: $\mathcal{F}$ is the set of all G extensions of fields K_0 containing K;
- Parametric dimension: $\mathcal{F}$ is the set of all G-extensions of K;
- Local dimension: $\mathcal{F}$ is the set of all G-extensions of K_v, over all places v of K;
- Lifting dimension: $\mathcal{F}$ a set of finitely many G-extensions of K.

Parametrizing/lifting

General setup

Let $\mathcal{F} = \{L_i/K_i\}_{i\in I}$ be G-extensions of $K_i \supseteq K$. What is the minimal dimension of a G-extension N/M containing K such that L_i/K_i is a specialization of N/M over K_i for every $i \in I$?

Example

- Essential dimension: $\mathcal{F}$ is the set of all G extensions of fields K_0 containing K;
- Parametric dimension: $\mathcal{F}$ is the set of all G-extensions of K;
- Local dimension: $\mathcal{F}$ is the set of all G-extensions of K_v, over all places v of K;
- Lifting dimension: $\mathcal{F}$ a set of finitely many G-extensions of K.

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

nBB Given G-extensions L_i/K, $i = 1, \ldots, n$, is there a regular G-extension $N/K(t)$ which specializes to L_i/K for all i?

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

nBB Given G-extensions L_i/K, $i = 1, \ldots, n$, is there a regular G-extension $N/K(t)$ which specializes to L_i/K for all i?

Example (1BB for abelian groups – Beckmann)

Let G be abelian, and L/K a G-extension.

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

nBB Given G-extensions L_i/K, $i = 1, \ldots, n$, is there a regular G-extension $N/K(t)$ which specializes to L_i/K for all i?

Example (1BB for abelian groups – Beckmann)

Let G be abelian, and L/K a G-extension.

- Form a regular G-extension $N/K(t)$ where 0 is not a branch point;

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

nBB Given G-extensions L_i/K, $i = 1, \ldots, n$, is there a regular G-extension $N/K(t)$ which specializes to L_i/K for all i?

Example (1BB for abelian groups – Beckmann)

Let G be abelian, and L/K a G-extension.

- Form a regular G-extension $N/K(t)$ where 0 is not a branch point;
- Twist it so that the specialization at 0 is L.

Lifting G-extensions

The Beckmann–Black problem

1BB (The Beckmann–Black conjecture) Every G-extension L/K is the specialization of a regular G-ext $N/K(t)$?

nBB Given G-extensions L_i/K, $i = 1, \ldots, n$, is there a regular G-extension $N/K(t)$ which specializes to L_i/K for all i?

Example (1BB for abelian groups – Beckmann)

Let G be abelian, and L/K a G-extension.

- Form a regular G-extension $N/K(t)$ where 0 is not a branch point;
- Twist it so that the specialization at 0 is L.

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t,x) \in K(t)[x]$ be a polynomial for which $p(i,x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t,x)$ is an S_m-extension that specializes to L_i, for all i.

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t, x) \in K(t)[x]$ be a polynomial for which $p(i, x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t, x)$ is an S_m-extension that specializes to L_i, for all i.
2. Similarly, $k(x_1, \ldots, x_m)^G$ rational implies nBB for all n.

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t,x) \in K(t)[x]$ be a polynomial for which $p(i,x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t,x)$ is an S_m-extension that specializes to L_i, for all i.
2. Similarly, $k(x_1, \ldots, x_m)^G$ rational implies nBB for all n.

Example

nBB therefore holds for:

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t,x) \in K(t)[x]$ be a polynomial for which $p(i,x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t,x)$ is an S_m-extension that specializes to L_i, for all i.
2. Similarly, $k(x_1, \ldots, x_m)^G$ rational implies nBB for all n.

Example

nBB therefore holds for:

- “Small": cyclic groups of order not divisible by 8, dihedral groups of odd degree, Heizenberg groups, etc.

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t, x) \in K(t)[x]$ be a polynomial for which $p(i, x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t, x)$ is an S_m-extension that specializes to L_i, for all i.
2. Similarly, $k(x_1, \ldots, x_m)^G$ rational implies nBB for all n.

Example

nBB therefore holds for:

- “Small": cyclic groups of order not divisible by 8, dihedral groups of odd degree, Heizenberg groups, etc.
- Reflection groups.

Examples - continued

S_m-example + generalization

1. Given S_m-extensions L_i/K, we may pick monic polynomials $p_i(x) \in K[x]$ of degree m whose splitting field is L_i, for $i = 1, \ldots, n$. Let $p(t,x) \in K(t)[x]$ be a polynomial for which $p(i,x) = p_i$ for all i. Then the splitting field $N/K(t)$ of $p(t,x)$ is an S_m-extension that specializes to L_i, for all i.
2. Similarly, $k(x_1, \ldots, x_m)^G$ rational implies nBB for all n.

Example

nBB therefore holds for:

- “Small": cyclic groups of order not divisible by 8, dihedral groups of odd degree, Heizenberg groups, etc.
- Reflection groups.

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over $\mathbb{Q}_p$ where $(p, \#G) = 1$;*

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over* $\mathbb{Q}_p$ *where* $(p, \#G) = 1$;
3. *nBB for R-equivalent extensions over an ample field K.*

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over* $\mathbb{Q}_p$ *where* $(p, \#G) = 1$;
3. *nBB for R-equivalent extensions over an ample field K.*

Example (PAC and ample)

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over $\mathbb{Q}_p$ where $(p, \#G) = 1$;*
3. *nBB for R-equivalent extensions over an ample field K.*

Example (PAC and ample)

- PAC fields: $\mathbb{Q}^{tr}(\sqrt{-1})$; $\overline{\mathbb{Q}}^{\sigma}$;

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over $\mathbb{Q}_p$ where $(p, \#G) = 1$;*
3. *nBB for R-equivalent extensions over an ample field K.*

Example (PAC and ample)

- PAC fields: $\mathbb{Q}^{tr}(\sqrt{-1})$; $\overline{\mathbb{Q}}^{\sigma}$;
- Ample fields: PAC fields; Complete fields ($\mathbb{Q}_p$, $\mathbb{R}$, $k((x))$); $\mathbb{Q}^{tot,p}$.

Other fields

Theorem (Dèbes, Colliot–Thélène, Moret-Bailley)

1. *nBB holds over PAC fields;*
2. *nBB holds over* $\mathbb{Q}_p$ *where* $(p, \#G) = 1$*;*
3. *nBB for R-equivalent extensions over an ample field* K.

Example (PAC and ample)

- PAC fields: $\mathbb{Q}^{tr}(\sqrt{-1})$; $\overline{\mathbb{Q}}^{\sigma}$;
- Ample fields: PAC fields; Complete fields ($\mathbb{Q}_p$, $\mathbb{R}$, $k((x))$); $\mathbb{Q}^{tot,p}$.

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \ldots, m$ the following holds:

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \ldots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \ldots, b$ such that every L_i, $i \in \{1, \ldots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \dots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \dots, b$ such that every L_i, $i \in \{1, \dots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Main evidence

BBb holds if:

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \ldots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \ldots, b$ such that every L_i, $i \in \{1, \ldots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Main evidence

BBb holds if:

- $K(x_1, \ldots, x_n)^G$ is rational, with $b = 1$;

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \ldots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \ldots, b$ such that every L_i, $i \in \{1, \ldots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Main evidence

BBb holds if:

- $K(x_1, \ldots, x_n)^G$ is rational, with $b = 1$;
- G is a abelian; Note $b > 1$ is needed, e.g. for $G = C_8$!

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \ldots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \ldots, b$ such that every L_i, $i \in \{1, \ldots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Main evidence

BBb holds if:

- $K(x_1, \ldots, x_n)^G$ is rational, with $b = 1$;
- G is a abelian; Note $b > 1$ is needed, e.g. for $G = C_8$!
- G is of nilpotency class 2, and K contains the $\exp(G)$ roots of unity.

Recent advancements (work in progress)

Example

2BB fails for $G = C_8$ over $K = \mathbb{Q}(\sqrt{17})$.

Question/Conjecture

There exists a finite number $b = b_{G,K}$ such that for every (arbitrary) finitely many G-extensions L_i/K, $i = 1, \dots, m$ the following holds:

BBb There exist b regular G-extensions $N_j/K(t)$, $j = 1, \dots, b$ such that every L_i, $i \in \{1, \dots, m\}$ is a specialization of some $N_{j(i)}$ for $j(i) \leq b$.

Main evidence

BBb holds if:

- $K(x_1, \dots, x_n)^G$ is rational, with $b = 1$;
- G is a abelian; Note $b > 1$ is needed, e.g. for $G = C_8$!
- G is of nilpotency class 2, and K contains the $\exp(G)$ roots of unity.

Parametrization problems

A parametric extension

A G-extension N/M of $M \supseteq K$ is (strongly) parametric if every G-extension of K is a specialization of N/M.

Basic examples

Parametrization problems

A parametric extension

A G-extension N/M of $M \supseteq K$ is (strongly) parametric if every G-extension of K is a specialization of N/M.

Basic examples

- $K(x_1, \ldots, x_n)/K(x_1, \ldots, x_n)^G$ is always parametric.

Parametrization problems

A parametric extension

A G-extension N/M of $M \supseteq K$ is (strongly) parametric if every G-extension of K is a specialization of N/M.

Basic examples

- $K(x_1, \ldots, x_n)/K(x_1, \ldots, x_n)^G$ is always parametric.
- $K(\sqrt[n]{t})/K(t)$ is parametric for C_n when K contains the n-th roots of unity.

Parametrization problems

A parametric extension

A G-extension N/M of $M \supseteq K$ is (strongly) parametric if every G-extension of K is a specialization of N/M.

Basic examples

- $K(x_1,\dots,x_n)/K(x_1,\dots,x_n)^G$ is always parametric.
- $K(\sqrt[n]{t})/K(t)$ is parametric for C_n when K contains the n-th roots of unity.
- Let N be the splitting field of $f(t,x) := x^3+tx+t \in \mathbb{Q}(t)[x]$ over $\mathbb{Q}(t)$. Then $N/\mathbb{Q}(t)$ is parametric for S_3.

Parametrization problems

A parametric extension

A G-extension N/M of $M \supseteq K$ is (strongly) parametric if every G-extension of K is a specialization of N/M.

Basic examples

- $K(x_1, \ldots, x_n)/K(x_1, \ldots, x_n)^G$ is always parametric.
- $K(\sqrt[n]{t})/K(t)$ is parametric for C_n when K contains the n-th roots of unity.
- Let N be the splitting field of $f(t,x) := x^3 + tx + t \in \mathbb{Q}(t)[x]$ over $\mathbb{Q}(t)$. Then $N/\mathbb{Q}(t)$ is parametric for S_3.

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

- (Legrand) $G = C_2 \times C_2$;

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

- (Legrand) $G = C_2 \times C_2$;
- (Dèbes) Evidence towards: there exists a parametric G-extension $N/K(t)$ if and only if $G \leq PGL_2(\mathbb{Q})$ (that is, $G = 1, C_2, C_3$, or S_3);

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

- (Legrand) $G = C_2 \times C_2$;
- (Dèbes) Evidence towards: there exists a parametric G-extension $N/K(t)$ if and only if $G \leq PGL_2(\mathbb{Q})$ (that is, $G = 1, C_2, C_3$, or S_3);
- (König–Legrand) G is of order prime to 6 and is not of prime order.

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

- (Legrand) $G = C_2 \times C_2$;
- (Dèbes) Evidence towards: there exists a parametric G-extension $N/K(t)$ if and only if $G \leq PGL_2(\mathbb{Q})$ (that is, $G = 1, C_2, C_3$, or S_3);
- (König–Legrand) G is of order prime to 6 and is not of prime order.

Most extensions are not specializations (König–Legrand)

Given a G-extension $N/K(t)$, almost all G-extensions L/K are not specializations of $N/K(t)$.

Non-examples

Example

The extension $\mathbb{Q}(t, \sqrt{t^2+1})$ is not C_2-parametric.

Groups with no parametric extension

In the following cases, no G-extension $N/\mathbb{Q}(t)$ is parametric:

- (Legrand) $G = C_2 \times C_2$;
- (Dèbes) Evidence towards: there exists a parametric G-extension $N/K(t)$ if and only if $G \leq PGL_2(\mathbb{Q})$ (that is, $G = 1, C_2, C_3$, or S_3);
- (König–Legrand) G is of order prime to 6 and is not of prime order.

Most extensions are not specializations (König–Legrand)

Given a G-extension $N/K(t)$, almost all G-extensions L/K are not specializations of $N/K(t)$.

Local parametrization

Locally parametric extensions

A G-extension N/M for $M \supseteq K$ is called *locally parametric* if every G-extension $L^{(v)}/K_v$ is a specialization of N/M over K_v for every place v of K.

Problem (Grunwald- problems)

Let S be a finite set of places of K. Given G-extensions $L^{(v)}/K_v$, $v \in S$, is there a G-extension L/K such that $L_v \cong L^{(v)}$ for every $v \in S$.

Local parametrization

Locally parametric extensions

A G-extension N/M for $M \supseteq K$ is called *locally parametric* if every G-extension $L^{(v)}/K_v$ is a specialization of N/M over K_v for every place v of K.

Problem (Grunwald- problems)

Let S be a finite set of places of K. Given G-extensions $L^{(v)}/K_v$, $v \in S$, is there a G-extension L/K such that $L_v \cong L^{(v)}$ for every $v \in S$.

Remark

Suppose N/M is parametric, and all Grunwald problems are solvable for G, then N/M is locally parametric.

Local parametrization

Locally parametric extensions

A G-extension N/M for $M \supseteq K$ is called *locally parametric* if every G-extension $L^{(v)}/K_v$ is a specialization of N/M over K_v for every place v of K.

Problem (Grunwald- problems)

Let S be a finite set of places of K. Given G-extensions $L^{(v)}/K_v$, $v \in S$, is there a G-extension L/K such that $L_v \cong L^{(v)}$ for every $v \in S$.

Remark

Suppose N/M is parametric, and all Grunwald problems are solvable for G, then N/M is locally parametric.

Colliot–Thélène's conjecture

Example (Wang)

Let $G = C_8$, and $L^{(2)}/\mathbb{Q}_2$ the unramified G-extension. Then there is no G-extension $L/\mathbb{Q}$ whose completion at 2 is $L^{(2)}$.

Conjecture (Colliot-Thélène)

There exists a finite set $T = T(G, K)$ of "bad" primes, such that all Grunwald problems for G, K and a set S that is disjoint from T are solvable.

Colliot–Thélène's conjecture

Example (Wang)

Let $G = C_8$, and $L^{(2)}/\mathbb{Q}_2$ the unramified G-extension. Then there is no G-extension $L/\mathbb{Q}$ whose completion at 2 is $L^{(2)}$.

Conjecture (Colliot-Thélène)

There exists a finite set $T = T(G, K)$ of “bad" primes, such that all Grunwald problems for G, K and a set S that is disjoint from T are solvable.

Evidence

The conjecture holds when:

Colliot–Thélène's conjecture

Example (Wang)

Let $G = C_8$, and $L^{(2)}/\mathbb{Q}_2$ the unramified G-extension. Then there is no G-extension $L/\mathbb{Q}$ whose completion at 2 is $L^{(2)}$.

Conjecture (Colliot-Thélène)

There exists a finite set $T = T(G, K)$ of “bad" primes, such that all Grunwald problems for G, K and a set S that is disjoint from T are solvable.

Evidence

The conjecture holds when:

- G is of odd order (Neukrich);

Colliot–Thélène's conjecture

Example (Wang)

Let $G = C_8$, and $L^{(2)}/\mathbb{Q}_2$ the unramified G-extension. Then there is no G-extension $L/\mathbb{Q}$ whose completion at 2 is $L^{(2)}$.

Conjecture (Colliot-Thélène)

There exists a finite set $T = T(G, K)$ of “bad" primes, such that all Grunwald problems for G, K and a set S that is disjoint from T are solvable.

Evidence

The conjecture holds when:

- G is of odd order (Neukrich);
- G is supersolvable (Harpaz–Wittenberg).

Colliot–Thélène's conjecture

Example (Wang)

Let $G = C_8$, and $L^{(2)}/\mathbb{Q}_2$ the unramified G-extension. Then there is no G-extension $L/\mathbb{Q}$ whose completion at 2 is $L^{(2)}$.

Conjecture (Colliot-Thélène)

There exists a finite set $T = T(G, K)$ of "bad" primes, such that all Grunwald problems for G, K and a set S that is disjoint from T are solvable.

Evidence

The conjecture holds when:

- G is of odd order (Neukrich);
- G is supersolvable (Harpaz–Wittenberg).

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

- Twist N by the extension $L^{(p)}/\mathbb{Q}_p$ to obtain $N^p/\mathbb{Q}_p(t)$;

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

- Twist N by the extension $L^{(p)}/\mathbb{Q}_p$ to obtain $N^p/\mathbb{Q}_p(t)$;
- Reduce mod p to obtain a G-extension $\overline{N}^p/\mathbb{F}_p(t)$;

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

- Twist N by the extension $L^{(p)}/\mathbb{Q}_p$ to obtain $N^p/\mathbb{Q}_p(t)$;
- Reduce mod p to obtain a G-extension $\overline{N}^p/\mathbb{F}_p(t)$;
- By Lang–Weil if p is large, $\overline{N}^p$ has an $\mathbb{F}_p$-point;

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

- Twist N by the extension $L^{(p)}/\mathbb{Q}_p$ to obtain $N^p/\mathbb{Q}_p(t)$;
- Reduce mod p to obtain a G-extension $\overline{N}^p/\mathbb{F}_p(t)$;
- By Lang–Weil if p is large, $\overline{N}^p$ has an $\mathbb{F}_p$-point;
- Lift to a $\mathbb{Q}_p$-point of N^p. Approximate resulting point by a $\mathbb{Q}$-point.

The unramified case - dimension 1

Theorem (Dèbes–Ghazi)

For every regular G-extension $N/K(t)$, there exists a finite set T_N satisfying: For every finite S disjoint from T_N, and unramified G-extensions $L^{(v)}/K_v$, $v \in S$, there exists a specialization L/K of $N/K(t)$ such that $L_v \cong L^{(v)}$, $v \in S$.

Idea of proof for $K = \mathbb{Q}$

- Twist N by the extension $L^{(p)}/\mathbb{Q}_p$ to obtain $N^p/\mathbb{Q}_p(t)$;
- Reduce mod p to obtain a G-extension $\overline{N}^p/\mathbb{F}_p(t)$;
- By Lang–Weil if p is large, $\overline{N}^p$ has an $\mathbb{F}_p$-point;
- Lift to a $\mathbb{Q}_p$-point of N^p. Approximate resulting point by a $\mathbb{Q}$-point.

The ramified case

Let $N/K(t)$ be regular a G-extension and S a finite set. Assume the branch points $t_1, \ldots, t_r \in K$ of $N/K(t)$ are K-rational, and $v(t_i) > 0$ for $v \in S$. Denote by D_i, I_i, L_i the decomposition gp, inertia gp, and residue field of $N/K(t)$ at t_i, resp.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that:

1. *Let L/K denote the residue of N at $t_0 \in K$ with $v(t_0 - t_i) > 0$ coprime to $\#I_i$ for $v \notin T_N$. Then the Galois group $D(i, v) \leq G$ of L_v/K_v is determined by I_i, and the residue field of L_i at v.*

The ramified case

Let $N/K(t)$ be regular a G-extension and S a finite set. Assume the branch points $t_1, \ldots, t_r \in K$ of $N/K(t)$ are K-rational, and $v(t_i) > 0$ for $v \in S$. Denote by D_i, I_i, L_i the decomposition gp, inertia gp, and residue field of $N/K(t)$ at t_i, resp.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that:

1. *Let L/K denote the residue of N at $t_0 \in K$ with $v(t_0 - t_i) > 0$ coprime to $\#I_i$ for $v \notin T_N$. Then the Galois group $D(i, v) \leq G$ of L_v/K_v is determined by I_i, and the residue field of L_i at v.*
2. *For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.*

The ramified case

Let $N/K(t)$ be regular a G-extension and S a finite set. Assume the branch points $t_1, \ldots, t_r \in K$ of $N/K(t)$ are K-rational, and $v(t_i) > 0$ for $v \in S$. Denote by D_i, I_i, L_i the decomposition gp, inertia gp, and residue field of $N/K(t)$ at t_i, resp.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that:

1. *Let L/K denote the residue of N at $t_0 \in K$ with $v(t_0 - t_i) > 0$ coprime to $\#I_i$ for $v \notin T_N$. Then the Galois group $D(i,v) \leq G$ of L_v/K_v is determined by I_i, and the residue field of L_i at v.*
2. *For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.*

The ramified case - continued

Let $N/K(t)$ be a G-extension and S a finite set.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that: For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.

Corollary

Let G be a group containing a noncyclic abelian group. Then

1. $N/K(t)$ is not locally parametric.

The ramified case - continued

Let $N/K(t)$ be a G-extension and S a finite set.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that: For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.

Corollary

Let G be a group containing a noncyclic abelian group. Then

1. $N/K(t)$ is not locally parametric.
2. If furthermore CT-conjecture holds for G, then $N/K(t)$ is not parametric.

The ramified case - continued

Let $N/K(t)$ be a G-extension and S a finite set.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that: For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.

Corollary

Let G be a group containing a noncyclic abelian group. Then

1. $N/K(t)$ is not locally parametric.
2. If furthermore CT-conjecture holds for G, then $N/K(t)$ is not parametric.

Note, $K(t)$ can be replaced by function fields of curves over K.

The ramified case - continued

Let $N/K(t)$ be a G-extension and S a finite set.

Theorem (Legrand–König–N)

There is a finite set T_N of bad places such that: For every finite S disjoint from T_N, and Galois field extensions $L^{(v)}/K_v$ with groups $D(i_v, v)$, $v \in S$ and some $i_v \leq r$, there exist residue fields L/K of N over $t_0 \in K$ such that $L_v \cong L^{(v)}$ for every $v \in S$.

Corollary

Let G be a group containing a noncyclic abelian group. Then

1. $N/K(t)$ is not locally parametric.
2. If furthermore CT-conjecture holds for G, then $N/K(t)$ is not parametric.

Note, $K(t)$ can be replaced by function fields of curves over K.

On the proof of KLN

Let $N/K(t)$ be a regular G-extension. Assume the branch points $t_1, \ldots, t_r \in K$ of π are K-rational, and $v(t_i) > 0$ for v in a finite S. Denote by I_i, L_i the inertia gp, and residue field of π at t_i, resp.

Theorem (Inertia specialization – Beckmann)

Assume v is not in a finite set T_N of "bad" primes, and let L/K be the residue field extension of N at $t_0 \in K$. If $v(t_0 - t_i) > 0$ is coprime to v, then the inertia group of L/K at v is I_i.

Idea of proof of KLN

On the proof of KLN

Let $N/K(t)$ be a regular G-extension. Assume the branch points $t_1, \ldots, t_r \in K$ of π are K-rational, and $v(t_i) > 0$ for v in a finite S. Denote by I_i, L_i the inertia gp, and residue field of π at t_i, resp.

Theorem (Inertia specialization – Beckmann)

Assume v is not in a finite set T_N of "bad" primes, and let L/K be the residue field extension of N at $t_0 \in K$. If $v(t_0 - t_i) > 0$ is coprime to v, then the inertia group of L/K at v is I_i.

Idea of proof of KLN

- Show that the residue field of L at v is the same as that of L_i at v;

On the proof of KLN

Let $N/K(t)$ be a regular G-extension. Assume the branch points $t_1, \ldots, t_r \in K$ of π are K-rational, and $v(t_i) > 0$ for v in a finite S. Denote by I_i, L_i the inertia gp, and residue field of π at t_i, resp.

Theorem (Inertia specialization – Beckmann)

Assume v is not in a finite set T_N of "bad" primes, and let L/K be the residue field extension of N at $t_0 \in K$. If $v(t_0 - t_i) > 0$ is coprime to v, then the inertia group of L/K at v is I_i.

Idea of proof of KLN

- Show that the residue field of L at v is the same as that of L_i at v;
- Use twisting (Brauer embedding problems) to change a given residue extension L/K whose v-completion has group $D(i, v)$ to arbitrary extensions whose completion have Galois group $D(i, v)$.

On the proof of KLN

Let $N/K(t)$ be a regular G-extension. Assume the branch points $t_1, \ldots, t_r \in K$ of π are K-rational, and $v(t_i) > 0$ for v in a finite S. Denote by I_i, L_i the inertia gp, and residue field of π at t_i, resp.

Theorem (Inertia specialization – Beckmann)

Assume v is not in a finite set T_N of "bad" primes, and let L/K be the residue field extension of N at $t_0 \in K$. If $v(t_0 - t_i) > 0$ is coprime to v, then the inertia group of L/K at v is I_i.

Idea of proof of KLN

- Show that the residue field of L at v is the same as that of L_i at v;
- Use twisting (Brauer embedding problems) to change a given residue extension L/K whose v-completion has group $D(i, v)$ to arbitrary extensions whose completion have Galois group $D(i, v)$.

Recent advancements

Example (2BB as an obstruction)

For $K = \mathbb{Q}(\sqrt{17})$, there is no parametric C_8-extension $N/K(t_1, \ldots, t_n)$ for any n!

Local dimension (Work in progress)

Suppose $K(x_1, \ldots, x_n)^G$ is rational. Then there exists $N/K(t,s)$ which specializes to every G-extension $L^{(v)}/K_v$ over almost every place v of K, in the following cases:

Recent advancements

Example (2BB as an obstruction)

For $K = \mathbb{Q}(\sqrt{17})$, there is no parametric C_8-extension $N/K(t_1, \ldots, t_n)$ for any n!

Local dimension (Work in progress)

Suppose $K(x_1, \ldots, x_n)^G$ is rational. Then there exists $N/K(t, s)$ which specializes to every G-extension $L^{(v)}/K_v$ over almost every place v of K, in the following cases:

1. G is a p-group, and $\sqrt{-1} \in K$ if $p = 2$.

Recent advancements

Example (2BB as an obstruction)

For $K=\mathbb{Q}(\sqrt{17})$, there is no parametric C_8-extension $N/K(t_1,\dots,t_n)$ for any n!

Local dimension (Work in progress)

Suppose $K(x_1,\dots,x_n)^G$ is rational. Then there exists $N/K(t,s)$ which specializes to every G-extension $L^{(v)}/K_v$ over almost every place v of K, in the following cases:

1. G is a p-group, and $\sqrt{-1}\in K$ if $p=2$.
2. the roots of unity of order $\exp(G)$ are in K.

Recent advancements

Example (2BB as an obstruction)

For $K = \mathbb{Q}(\sqrt{17})$, there is no parametric C_8-extension $N/K(t_1, \ldots, t_n)$ for any n!

Local dimension (Work in progress)

Suppose $K(x_1, \ldots, x_n)^G$ is rational. Then there exists $N/K(t, s)$ which specializes to every G-extension $L^{(v)}/K_v$ over almost every place v of K, in the following cases:

1. G is a p-group, and $\sqrt{-1} \in K$ if $p = 2$.
2. the roots of unity of order $\exp(G)$ are in K.

Question/Conjecture

There are G-extensions $N_j/K, j = 1, \ldots, b = b_{G,K}$ such that every G-extension $L^{(v)}/K_v$ is a specialization of some N_j/K.

Recent advancements

Example (2BB as an obstruction)

For $K = \mathbb{Q}(\sqrt{17})$, there is no parametric C_8-extension $N/K(t_1, \ldots, t_n)$ for any n!

Local dimension (Work in progress)

Suppose $K(x_1, \ldots, x_n)^G$ is rational. Then there exists $N/K(t, s)$ which specializes to every G-extension $L^{(v)}/K_v$ over almost every place v of K, in the following cases:

1. G is a p-group, and $\sqrt{-1} \in K$ if $p = 2$.
2. the roots of unity of order $\exp(G)$ are in K.

Question/Conjecture

There are G-extensions $N_j/K, j = 1, \ldots, b = b_{G,K}$ such that every G-extension $L^{(v)}/K_v$ is a specialization of some N_j/K.

Summary

Question/Conjecture I: Lifting dimension should be 1

Every finitely many G-extensions $L_i/\mathbb{Q}$, $i = 1, \ldots, m$ are specializations of b regular G-extensions $N_j/K(t)$, for some b depending only on G!

Question/Conjecture I.5: Lifting as an obstruction to parametrization

If there are b R-equivalence classes, then the number of parametrizing extensions $N_j/K(t)$ should be at least b.

Summary

Question/Conjecture I: Lifting dimension should be 1

Every finitely many G-extensions $L_i/\mathbb{Q}$, $i = 1, \ldots, m$ are specializations of b regular G-extensions $N_j/K(t)$, for some b depending only on G!

Question/Conjecture I.5: Lifting as an obstruction to parametrization

If there are b R-equivalence classes, then the number of parametrizing extensions $N_j/K(t)$ should be at least b.

Question II: What is the parametric dimension?

When are there finitely many G-extensions $N_j/K(t,s)$, $j = 1, \ldots, b$ such that every G-extension is a specialization of $N_j/K(t,s)$ for some j!

Summary

Question/Conjecture I: Lifting dimension should be 1

Every finitely many G-extensions $L_i/\mathbb{Q}$, $i = 1, \ldots, m$ are specializations of b regular G-extensions $N_j/K(t)$, for some b depending only on G!

Question/Conjecture I.5: Lifting as an obstruction to parametrization

If there are b R-equivalence classes, then the number of parametrizing extensions $N_j/K(t)$ should be at least b.

Question II: What is the parametric dimension?

When are there finitely many G-extensions $N_j/K(t,s)$, $j = 1, \ldots, b$ such that every G-extension is a specialization of $N_j/K(t,s)$ for some j!

Thank you!

Questions?

Summary

Question/Conjecture I: Lifting dimension should be 1

Every finitely many G-extensions $L_i/\mathbb{Q}$, $i = 1, \ldots, m$ are specializations of b regular G-extensions $N_j/K(t)$, for some b depending only on G!

Question/Conjecture I.5: Lifting as an obstruction to parametrization

If there are b R-equivalence classes, then the number of parametrizing extensions $N_j/K(t)$ should be at least b.

Question II: What is the parametric dimension?

When are there finitely many G-extensions $N_j/K(t,s)$, $j = 1, \ldots, b$ such that every G-extension is a specialization of $N_j/K(t,s)$ for some j!

Thank you!

Questions?