INTERNATIONAL SEMINAR
OPEN PROBLEMS SESSION
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Henri Mühle

We present permutations $\pi \in S_n$ by strings $\pi_1\pi_2\dots\pi_n$ where $\pi_i = \pi(i)$ for all $i \in \{1,\dots,n\}$. Let $\pi = \pi_1\pi_2\dots\pi_n \in S_n$, $\sigma = \sigma_1\sigma_2\dots\sigma_m \in S_m$, $n \leq m$. We say that σ *contains the pattern* π, and we write $\pi \leq \sigma$, if there exist indices $1 \leq i_1 \leq i_2 \leq \dots i_n \leq m$ such that $\pi_1\pi_2\dots\pi_n$ is order-isomorphic to $\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_n}$. For example, $4213 \leq 5741263$, because the subsequence 7416 of 5741263 is order-isomorphic to 4213. Write $\mathrm{Av}^{(n)}(\pi_1,\pi_2,\dots,\pi_k) := \{\sigma \in S_n \mid \forall i \in [k]\colon \pi_i \not\leq \sigma\}$.

Problem. Does the equality

$$|\mathrm{Av}^{(n)}(12345, 12354, 21345, 21354)| = |\mathrm{Av}^{(n)}(41352, 42351, 51342, 52341)|$$

hold for all $n \in \mathbb{N}_+$?

Antoine Mottet

A subgroup G of $\mathrm{Sym}(\mathbb{Q}\times\mathbb{Z})$ is *closed* if for all $h \in \mathrm{Sym}(\mathbb{Q}\times\mathbb{Z})$ it holds that $h \in G$ whenever for every finite subset A of $\mathbb{Q}\times\mathbb{Z}$ there exists $g \in G$ such that $g|_A = h_A$.

Problem. Find the closed subgroups G of $\mathrm{Sym}(\mathbb{Q}\times\mathbb{Z})$ containing $\mathrm{Aut}(\mathbb{Z},<) \wr_{\mathbb{Q}} \mathrm{Aut}(\mathbb{Q},<)$ with the property that the elements of G act on the copies of $\mathbb{Z}$ by isometries, i.e., for all $\alpha \in G$, $x \sim y$ implies $|\alpha(x) - \alpha(y)| = x - y$.

Jens Zumbrägel

Denote the set of all k-element subsets of $[n]$ by $\binom{[n]}{k}$. A *Steiner system* $S(t,k,n)$ is a collection $\mathcal{B} \subseteq \binom{[n]}{k}$ such that for every $A \in \binom{[n]}{t}$, there exists a unique $B \in \mathcal{B}$ such that $A \subseteq B$. If we modify the above definition and replace $\binom{[n]}{k}$ by the Grassmannian $\begin{bmatrix}\mathbb{F}_q^n\\k\end{bmatrix} := \{U \leq \mathbb{F}_q^n \mid \dim U = k\}$, we obtain the definition of a *q-analog of a Steiner system* or a *q-Steiner system* $S[t,k,n]_q$.

Problem. Does there exist $S[2,3,7]_2$?

Remark. It has been shown that q-Steiner systems $S[2,3,13]_2$ exist; see [1].

[1] M. Braun, T. Etzion, P. R. J. Östergård, A. Vardy, A. Wassermann, Existence of q-analogs of Steiner systems, *Forum Math. Pi* **4** (2016) e7, 14 pp.

Manuel Bodirsky

A structure A has an *interpretation* in a structure B if there exists a surjective partial map $I\colon B^n \to A$ such that preimages of relations defined by atomic formulas are first-order definable in B.

Fact. *All finite structures are interpretable over* $(\mathbb{N},=)$.

Fact. *For every structure interpretable over* $(\mathbb{N},=)$*, there exists a homomorphically equivalent structure* B *such that* $\overline{\mathrm{Aut}(B)} = \mathrm{End}(B)$*. Moreover,* B *is unique up to isomorphism.*

Problem. Is B interpretable over $(\mathbb{N},=)$?

Szymon Toruńczyk

Graphs interpretable over $(\mathbb{N},=)$ are precisely those which can be described by finite expressions involving possibly nested set-builder expressions with first-order formulas in the language of $(\mathbb{N},=)$, such as for instance the one below:

$$V = \{\{a,b\} : a,b \in \mathbb{N},\ a \neq b\},$$
$$E = \{\{\{a,b\},\{b,c\}\} : a,b,c \in \mathbb{N},\ a \neq b,\ b \neq c,\ a \neq c\}.$$

We are interested in the decidability of the following decision problem.

Problem. Given two graphs that are interpretable over $(\mathbb{N},=)$, decide if they are isomorphic. Equivalently, given a graph $G=(V,E)$ and $v,w \in V$, decide whether v and w are in the same orbit of $\mathrm{Aut}(G)$.

Bertalan Bodor

A structure B is a *reduct* of a structure A if B has the same domain as A and all relations of B are first-order definable in A. A and B are *interdefinable* if they are reducts of each other.

Problem. Let A be a structure interpretable over $(\mathbb{N},=)$? Does A have finitely many reducts up to interdefinability?

Caterina Viola

Consider a structure $(D,\leq)$, where $\leq$ is a total order. A function $f\colon D^k \to \mathbb{Q}$ is *submodular* if for all $\mathbf{x},\mathbf{y} \in D^k$,

$$f(\mathbf{x}) + f(\mathbf{y}) \geq f(\max(\mathbf{x},\mathbf{y})) + f(\min(\mathbf{x},\mathbf{y})).$$

Theorem. *A function* $f\colon D^2 \to \mathbb{Q}$ *is submodular if and only if for all* $\alpha,\beta,\gamma,\delta \in D$ *such that* $\alpha < \beta$ *and* $\gamma < \delta$*, it holds that* $f(\alpha,\delta) + f(\beta,\gamma) \geq f(\alpha,\gamma) + f(\beta,\delta)$*.*

Theorem. *Let* $f\colon D^k \to \mathbb{Q}$*, and assume that* $k > 2$*. Then* f *is submodular if and only if for all* $\alpha,\beta,\gamma,\delta \in D$ *such that* $\alpha < \beta$ *and* $\gamma < \delta$*, for all* $i,j \in \{1,\dots,k\}$ *with* $i \neq j$*, and for all* $x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_k \in D$*, it holds that*

$$f(x_1,\dots,\alpha,\dots,\delta,\dots,x_k) + f(x_1,\dots,\beta,\dots,\gamma,\dots,x_k) \geq$$
$$f(x_1,\dots,\alpha,\dots,\gamma,\dots,x_k) + f(x_1,\dots,\beta,\dots,\delta,\dots,x_k).$$

Problem. If $f\colon D^n \to \mathbb{Q}$ is of the form

$$f(x_1,\dots,x_n) = \sum_i \max(g_{i1}(x_{i1}),\dots,g_{iq_i}(x_{iq_i})) + \sum_j \min(f_{j1}(x_{j1}), f_{j2}(x_{j2})) + \sum_k h_k(x_{k1}),$$

where the $g_{i\ell}$ and f_{j1} are non-decreasing and the f_{j2} are non-increasing, then f is submodular. Does the converse implication hold if D is finite?