TECHNISCHE UNIVERSITÄT DRESDEN

Fakultät Mathematik und Naturwissenschaften Fachrichtung Mathematik, Institut für Analysis

Prof. Dr. Ralph Chill
Prof. Dr. Jürgen Voigt
E-Mail: ralph.chill@tu-dresden.de
juergen.voigt@tu-dresden.de

Dresden, June 4, 2012

Comments on some recent work by Shouryya Ray

Recently the news has spread in national and international newspapers about a 16 year old high school student from Dresden, Shouryya Ray, who "*has solved a mathematical problem which has stumped mathematicians for centuries*", a problem "*in fundamental particle dynamics posed by Sir Isaac Newton over 350 years ago*". The solution makes "*it possible to now calculate not only the flight path of a ball, but also to predict how it will hit and bounce off a wall. Previously it had only been possible to estimate this using a computer*" [1]. The news was discussed on the internet, but unfortunately without any information stating what the problem of Newton actually was, and what the work accomplished by the young student was. The work has now been shown to us, and with the consent of the young student, we are able to report more details of the work and to describe its relation with results in the literature.

Conducting an internship at the Chair of Fluid Mechanics at TU Dresden, Shouryya Ray encountered two ordinary differential equations which are special cases of Newton's law that the derivative of the momentum of a particle equals the forces acting on it. In the first one, which describes the motion of a particle in a gas or fluid, this force is the sum of a damping force, which depends quadratically on the velocity, and the (constant) gravitational force:

$$\begin{aligned} \dot{u} &= -u\sqrt{u^2+v^2}, \qquad & u(0) &= u_0 > 0 \\ \dot{v} &= -v\sqrt{u^2+v^2} - g, \qquad & v(0) &= v_0. \end{aligned} \tag{1}$$

Here, u and v are the horizontal and vertical velocity, respectively. The fact that the damping depends quadratically on the velocity, in a certain range of Reynolds numbers, is a classical assumption and goes back to Newton[2].

[1] See `http://www.thelocal.de/education/20120523-42687.html`. This article is basically a translation from an article in *Die WELT*; see `http://www.welt.de/vermischtes/article106354044/16-jaehriges-Mathegenie-loest-uraltes-Zahlenraetsel.html`

[2] See the discussion in Prandtl, Ludwig; Oswatitsch, Klaus; Wieghardt, Karl: *Führer durch die Strömungslehre*, Vieweg & Sohn, Korrigierter Nachdruck der 9. Auflage, 1993, p.254ff.

The second equation reads

$$\ddot{z} = -\dot{z} - z^{3/2}, \qquad z(0) = 0,\ \dot{z}(0) = z_1, \tag{2}$$

and describes the trajectory of the center point $z(t)$ of a spherical particle during a normal collision with a plane wall. The term $z^{3/2}$ has been proposed by Hertz[3] for purely elastic deformation of the particle. The linear term $\dot{z}$ accounts for additional damping. At $t = 0$ the particle hits the wall and the distance between the center of the particle and the wall is equal to the radius. This is identified with $z(0) = 0$, while $\dot{z}(0) > 0$ is the initial velocity. It should be mentioned that all our equations above are dimensionless and all physical constants (except for the gravitation) have been put $= 1$. This is not done in Shouryya Ray's work so that at the end of the analysis he can also consider limiting cases when certain constants tend to 0 or ∞.

We wish to emphasize that the amount of literature absorbed by the young student is impressive, and the same is true for for the variety of techniques for solving ordinary differential equations that he learned, applied, and sometimes developed by himself. The work is without doubt exceptional for a high school student and it merits the attention that it received in a national science competition for high school students[4]. However, it sometimes lacks the theoretical background of mathematical analysis.

Problem (1) is considered as a system of two "linear" equations for u and v – with unknown coefficient function $\sqrt{u^2 + v^2}$. By introducing the scalar functions

$$\begin{aligned} \Psi(t) &= \int_0^t \exp[\int_0^\tau \sqrt{u^2(s) + v^2(s)}\, ds]\, d\tau, \text{ and} \\ \psi(t) &= (v_0 - g\Psi(t))/u_0, \end{aligned}$$

Shouryya Ray observes on the one hand that the solutions can be represented as

$$\begin{aligned} u(t) &= \frac{u_0}{\dot{\Psi}(t)}, \\ v(t) &= \frac{v_0 - g\Psi(t)}{\dot{\Psi}(t)}, \end{aligned}$$

and on the other hand that ψ satisfies the second order ordinary differential equation

$$\ddot{\psi} = g\sqrt{1 + \psi^2}, \qquad \psi(0) = v_0/u_0,\ \dot{\psi}(0) = 1. \tag{3}$$

By multiplying this equation with $\dot{\psi}$, and by integrating the result, this equation is transformed into the first order ordinary differential equation

$$\begin{aligned} \dot{\psi} &= (g\psi\sqrt{1 + \psi^2} + g \operatorname{arsinh} \psi - C)^{1/2}, \\ \psi(0) &= v_0/u_0, \quad C = gv_0/u_0\sqrt{1 + (v_0/u_0)^2} - 1, \end{aligned} \tag{4}$$

[3]Hertz, H.: *Über die Berührung fester Körper.* Journal für die reine und angewandte Mathematik 92 (1882), 156–171.

[4]This competition is called *Jugend forscht* `https://www.jugend-forscht.de/`. See in particular `http://www.jufo-dresden.de/projekt/teilnehmer/matheinfo/m1` or `http://jugend-forscht-sachsen.de/2012/teilnehmer/fachgebiet/id/5`

the solution of which is given implicitly by

$$\int_{\psi(0)}^{\psi(t)} \frac{1}{(gx\sqrt{1+x^2} + g\,\mathrm{arsinh}\, x - C)^{1/2}}\, dx = t. \tag{5}$$

Equations (3)-(5) appear in the literature[5], but it seems not possible to compute the integral in (5) and to invert the resulting function in order to obtain ψ explicitly. At this point, Shouryya Ray applies a recent result of D. Dominici[6] in order to obtain a power series representation for ψ,

$$\psi(t) = \sum_{n=0}^{\infty} d_n\, t^n, \tag{6}$$

with a recursion formula for the coefficients d_n. He thus obtains an analytic solution of the problem (3), and hence of problem (1). In addition, he validated his result by comparing approximate solutions obtained by taking partial sums in (6) with approximate solutions obtained with a Runge-Kutta scheme.

While the above derivation is mathematically correct, we have to say that the classical theory of ordinary differential equations provides two fundamental theorems which are of relevance in this context[7]. The theory yields

(i) existence and uniqueness of classical (C^1) solutions of equations of the type as considered above, and

(ii) that these solutions are analytic, that is, they admit a power series representation, if the data in the equation are analytic (for example, the function $\psi \mapsto \sqrt{1+\psi^2}$ is analytic).

Both points are classical, but it is true that point (ii) may not be part of every undergraduate course on ordinary differential equations, and not even of every textbook on the subject. Having these existence theorems at hand, the coefficients $d_n = \psi^{(n)}(0)/n!$ may also be obtained from equation (3) or (4) by successively differentiating the equations and thus obtaining a recursion formula for the higher order derivatives of ψ at 0.

The situation is a little bit different for problem (2) since the function $z^{3/2}$ is not analytic at $z = 0$ while the initial condition is $z(0) = 0$. Shouryya Ray tried to give an analytic solution also for this problem, but the arguments given in the original work were partly erroneous and would need to be revised. In a recent discussion, however, he has shown to us a new ansatz towards a solution which is analytic for positive times. This ansatz again reveals the impressive talent as far as computational techniques are concerned.

[5] See Parker, G. W.: *Projectile motion with air resistance quadratic in the speed.* American J. Phys. 45 (1977), no. 7, 606–610, where the equation is deduced in a different way. In the blog `http://www.reddit.com/user/vaporism` we found also a reference to Didion, Isidore: *Traité de balistique*, Paris, 1860, pp. 200–, and especially page 211, where the function under the integral appears, or page 218, where power series solutions are discussed.

[6] Dominici, Diego: *Nested derivatives: a simple method for computing series expansions of inverse functions.* Int. J. Math. Math. Sci. 58 (2003), 3699–3715.

[7] Kamke, Erich: Differentialgleichungen. (German) Lösungsmethoden und Lösungen. I: Gewöhnliche Differentialgleichungen. Neunte Auflage. Mit einem Vorwort von Detlef Kamke. B. G. Teubner, Stuttgart, 1977.

Let us come back to problem (1) which was the starting point of the media stories. In the context of Shouryya Ray's work it was an unfortunate circumstance, that a recent article from 2007[8] claims that no analytic solution of problem (1) was known, or that it was known only in special cases, namely for falling objects[9] This might have misled Shouryya Ray who was not aware of the classical theory of ordinary differential equations. Actually, many mathematicians have considered the problem of projectile motion in air over a long time. An additional approach, for example, was proposed by Johann Bernoulli who transformed the problem (1) in a different way in order to obtain the equation

$$\frac{dw}{d\varphi} = \tan\varphi \, w + \alpha^2 \sec\varphi \, w^3, \tag{7}$$

which today is known as a Bernoulli differential equation. Here, the velocity w is parametrized over the angle φ. The points (i) and (ii) apply also to this ordinary differential equation, and there is an analytic solution. The coefficients of the power series of w can again be computed by successively differentiating equation (7).

To conclude, Shouryya Ray has obtained analytic solutions of the problem (1), by transforming it successively to the problems (3)-(5), and by applying a recent result of D. Dominici in order to obtain a recursion formula for the coefficients of the power series representation of ψ. He then validated his results numerically. Given the level of prerequisites that he had, he made great progress. Nevertheless all his steps are basically known to experts, and we emphasize that he did not solve an open problem posed by Newton.

We do not know how this regrettable claim entered several newspapers. Apparently, this claim was not endorsed by experts in the field who should have been involved in the evaluation of the work. We hope that this small text gives the necessary information to the mathematical community, and that it allows the community to both put in context and appreciate the work of Shouryya Ray who plans to start a career in mathematics and physics.

Prof. Dr. Ralph Chill and Prof. Dr. Jürgen Voigt

[8] Yabushita, Kazuki; Yamashita, Mariko; Tsuboi, Kazuhiro. *An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method.* J. Phys. A 40 (2007), no. 29, 8403–8416,

[9] We cite from this article, page 8404: "*The motion equation of this case (*that is, the case of quadratic resistance*) is unsolvable analytically, although the problem is fundamental and practical in elementary dynamics. ... The analytic solutions of falling motion problems with linear air resistance and with quadratic air resistance have already been obtained. Also, the case of projectile motion with linear air resistance has already been solved. As mentioned before, the case of quadratic air resistance has previously been unsolvable without adding some specific conditions.*"